MAGMA PROOF OF STRICT INEQUALITIES FOR MINIMAL DEGREES OF FINITE GROUPS

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Abstract. The minimal faithful permutation degree of a finite group G, denote by $\mu(G)$ is the least non-negative integer n such that G embeds inside the symmetric group $\mathrm{Sym}(n)$. In this paper, we outline a Magma proof that 10 is the smallest degree for which there are groups G and H such that $\mu(G \times H) < \mu(G) + \mu(H)$.

1. Introduction

The study of this topic dates back to Johnson [2] and Wright [5], who among other things investigated the inequality

$$\mu(G \times H) \leq \mu(G) + \mu(H), \tag{1}$$

which clearly holds.

Johnson first showed that equality holds when G and H have coprime orders or are abelian. Wright went further to show that equality holds when G and H are p-groups and hence extended this to nilpotent groups. In that same paper, he constructs a class of groups $\mathscr{C}$ with the defining property that for every G in $\mathscr{C}$, there exists a nilpotent subgroup G_1 in G such that $\mu(G_1) = \mu(G)$. It is clear that equality in (1) holds for any two groups in $\mathscr{C}$ and that $\mathscr{C}$ is closed under taking direct products.

Wright [5], asked the question: does $\mu(G \times H) = \mu(G) + \mu(H)$ for all finite groups G and H? The referee to [5] provided an example of strict inequality of degree 15 and attached it as an addendum to that paper. The second author of this article recognised that the example quoted in that paper involved the complex reflection group $G(5, 5, 3)$ and its centraliser in $\mathrm{Sym}(15)$.

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This led to the investigation in [4] where the second author proved that a similar result occurs with the complex reflection groups $G(4,4,3)$ and $G(2,2,5)$, which are of degree 12 and 10 respectively. That is, these groups have non-trivial centralisers in their minimal embedding that intersect trivially their embedded image.

In [3], the second author extended this idea exhibiting that for p and q distinct odd primes, with $q \geq 5$ or $q = 3$ and $p \not\equiv 1 \bmod 3$, the groups $G(p,p,q)$ and their centralisers in $\mathrm{Sym}(pq)$ have the same property that

$$\mu(G(p,p,q)) = \mu(G(p,p,q) \times C_{\mathrm{Sym}(pq)}(G(p,p,q))) = pq,$$

and so give examples of strict inequality in (1).

The authors do not know whether there are groups G and H such that

$$\max\{\mu(G), \mu(H)\} < \mu(G \times H) < \mu(G) + \mu(H).$$

In the following section, we prove using the computation algebra system Magma [1], that 10 is the smallest degree for the scenario that $\mu(G) = \mu(G \times C)$ where G is minimally embedded group in $\mathrm{Sym}(\mu(G))$ and C is its centraliser which intersects trivially with it. This is done by a brute-force search of the subgroups of $\mathrm{Sym}(m)$ for $m \leq 9$ and examining their centralisers.

2. The Magma Code

The following code was implemented in magma for $m \leq 9$

```
n:=m;
S:=Sym(m);
num:=NumberOfTransitiveGroups(m);
subs:=Subgroups(Sym(m));
subs:=[s`subgroup: s in Subgroups(Sym(m))];
smaller:=[[s`subgroup: s in Subgroups(Sym(i))] : i in [1..m-1]];
minemb:=[ G : G in subs | forall{H : H in smaller[i],
i in [1..m-1]| not IsIsomorphic(G,H)}];
Ind:=[Index(sub<S|Centraliser(S,G),G>,G) : G in minemb];
indices_min:=[i : i in [1..#minemb]| Ind[i] ne 1];
```

Thus the code constructs the entire subgroup lattice of the symmetric group, isolates the subgroups which are minimally embedded inside the symmetric group and then computes their centralisers. For G a minimally embedded group in $\mathrm{Sym}(m)$ and C the centraliser of G in this minimal embedding, the Ind sequence returns the index of G in the group generated G and C. Once this index is known, one can either determine that the centraliser is contained inside the group, or there is a possibility of a subgroup in C which intersects trivially with G by searching for an element in x in C, such that the intersection of $\langle x \rangle$ with G is trivial.

3. Results

Since in the cases $m = 2, 3, 4$ are easily dealt with by hand we only give the Magma output for the higher cases.

$\underline{m = 5}$

```
> Ind;
[ 1, 1, 1, 1, 1, 1, 1 ]
```

$\underline{m = 6}$

```
> Ind;
[ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 ]
```

$\underline{m = 7}$

```
> Ind;
[ 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 1, 1 ]
```

$\underline{m = 8}$

```
> Ind;
[ 1, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 2, 1,
 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1,1, 1, 1, 1 ]
```

$\underline{m = 9}$

```
Ind;
[ 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 ]
```

An inspection of the numbers above shows that they are either 1 or divisible by 2. This means that any subgroup of the centraliser of G in

Sym(n) which intersects trivially with G must have order divisible by 2. Therefore, to search for such a subgroup, we implement the following function;

```
Comp:= [ G : G in minemb |exists{g : g in Centraliser(S,G) |
 Order(g) eq 2 and Order(G meet sub<S|g>) eq 1} ];
```

In each case, we find that

```
> Comp;
[]
```

Thus for every minimally embedded group of degree at most 9, there does not exist a subgroup of its centraliser which intersects it trivially. Therefore we cannot obtain a strict inequality in (1) by this method.

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