Pfaffian-type Sugawara operators

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Abstract

We show that the Pfaffian of a generator matrix for the affine Kac–Moody algebra $\widehat{\mathfrak{o}}_{2n}$ is a Segal–Sugawara vector. Together with our earlier construction involving the symmetrizer in the Brauer algebra, this gives a complete set of Segal–Sugawara vectors in type D.

1 Introduction

For each affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ associated with a simple Lie algebra $\mathfrak{g}$, the corresponding vacuum module $V(\mathfrak{g})$ at the critical level is a vertex algebra. The structure of the center $\mathfrak{z}(\widehat{\mathfrak{g}})$ of this vertex algebra was described by a remarkable theorem of Feigin and Frenkel in [3], which states that $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the algebra of polynomials in infinitely many variables which are associated with generators of the algebra of $\mathfrak{g}$-invariants in the symmetric algebra $\mathrm{S}(\mathfrak{g})$. For a detailed proof of the theorem, its extensions and significance for the representation theory of the affine Kac–Moody algebras see [4].

In a recent paper [5] we used the symmetrizer in the Brauer algebra to construct families of elements of $\mathfrak{z}(\widehat{\mathfrak{g}})$ (Segal–Sugawara vectors) for the Lie algebras $\mathfrak{g}$ of types B, C and D in an explicit form. In types B and C they include complete sets of Segal–Sugawara vectors generating the center $\mathfrak{z}(\widehat{\mathfrak{g}})$, while in type D one vector in the complete set of [5] was not given explicitly. The aim of this note is to produce this Segal–Sugawara vector in $\mathfrak{z}(\widehat{\mathfrak{o}}_{2n})$ which is associated with the Pfaffian invariant in $\mathrm{S}(\mathfrak{o}_{2n})$.

Simple explicit formulas for generators of the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ were given recently in [1], [2], following Talalaev's construction of higher Gaudin Hamiltonians [6]. So, together with the results of [5] we get a construction of generators of the Feigin–Frenkel centers for all classical types.

2 Pfaffian-type generators

Denote by E_{ij}, $1 \leqslant i,j \leqslant 2n$, the standard basis vectors of the Lie algebra $\mathfrak{gl}_{2n}$. Introduce the elements F_{ij} of $\mathfrak{gl}_{2n}$ by the formulas

$$F_{ij} = E_{ij} - E_{ji}. \tag{2.1}$$

The Lie subalgebra of $\mathfrak{gl}_{2n}$ spanned by the elements F_{ij} is isomorphic to the even orthogonal Lie algebra $\mathfrak{o}_{2n}$. The elements of $\mathfrak{o}_{2n}$ are skew-symmetric matrices. Introduce the standard normalized invariant bilinear form on $\mathfrak{o}_{2n}$ by

$$\langle X, Y\rangle = \frac{1}{2}\,\mathrm{tr}\, XY, \qquad X, Y \in \mathfrak{o}_{2n}.$$

Now consider the affine Kac–Moody algebra $\widehat{\mathfrak{o}}_{2n} = \mathfrak{o}_{2n}[t, t^{-1}] \oplus \mathbb{C}K$ and set $X[r] = Xt^r$ for any $r \in \mathbb{Z}$ and $X \in \mathfrak{o}_{2n}$. The element K is central in $\widehat{\mathfrak{o}}_{2n}$ and

$$\big[X[r], Y[s]\big] = [X, Y][r+s] + r\,\delta_{r,-s}\langle X, Y\rangle\, K.$$

Therefore, for the generators we have

$$\begin{aligned}\big[F_{ij}[r], F_{kl}[s]\big] = \delta_{kj}\, F_{il}[r+s] - \delta_{il}\, F_{kj}[r+s] - \delta_{ki}\, F_{jl}[r+s] + \delta_{jl}\, F_{ki}[r+s]\\ + r\,\delta_{r,-s}\big(\delta_{kj}\,\delta_{il} - \delta_{ki}\,\delta_{jl}\big)\, K.\end{aligned}$$

The *vacuum module at the critical level* $V(\widehat{\mathfrak{o}}_{2n})$ can be defined as the quotient of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{o}}_{2n})$ by the left ideal generated by $\mathfrak{o}_{2n}[t]$ and $K + 2n - 2$ (note that the dual Coxeter number in type D_n is $h^\vee = 2n - 2$). The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{o}}_{2n})$ is defined by

$$\mathfrak{z}(\widehat{\mathfrak{o}}_{2n}) = \{v \in V(\widehat{\mathfrak{o}}_{2n}) \mid \mathfrak{o}_{2n}[t]\, v = 0\}.$$

Any element of $\mathfrak{z}(\widehat{\mathfrak{o}}_{2n})$ is called a *Segal–Sugawara vector*. A *complete set of Segal–Sugawara vectors* $\phi_{2\,2}, \phi_{4\,4}, \dots, \phi_{2n-2\,2n-2}, \phi'_n$ was produced in [5], where all of them, except for ϕ'_n, were given explicitly. We will produce ϕ'_n in Theorem 2.1 below.

Combine the generators $F_{ij}[-1]$ into the skew-symmetric matrix $F[-1] = \big[F_{ij}[-1]\big]$ and define its *Pfaffian* by

$$\mathrm{Pf}\, F[-1] = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \mathrm{sgn}\,\sigma \cdot F_{\sigma(1)\,\sigma(2)}[-1] \dots F_{\sigma(2n-1)\,\sigma(2n)}[-1].$$

Note that the elements $F_{ij}[-1]$ and $F_{kl}[-1]$ of $\widehat{\mathfrak{o}}_{2n}$ commute, if the indices i, j, k, l are distinct. Therefore, we can write the formula for the Pfaffian in the form

$$\mathrm{Pf}\, F[-1] = \sum_{\sigma} \mathrm{sgn}\,\sigma \cdot F_{\sigma(1)\,\sigma(2)}[-1] \dots F_{\sigma(2n-1)\,\sigma(2n)}[-1], \tag{2.2}$$

summed over the elements σ of the subset $\mathcal{B}_{2n} \subset \mathfrak{S}_{2n}$ which consists of the permutations with the properties $\sigma(2k-1) < \sigma(2k)$ for all $k = 1, \dots, n$ and $\sigma(1) < \sigma(3) < \dots < \sigma(2n-1)$.

Theorem 2.1. *The element* $\phi'_n = \mathrm{Pf}\, F[-1]$ *is a Segal–Sugawara vector for* $\widehat{\mathfrak{o}}_{2n}$.

Proof. We need to show that $\mathfrak{o}_{2n}[t]\,\phi'_n = 0$ in the vacuum module $V(\widehat{\mathfrak{o}}_{2n})$. It suffices to verify that for all i, j,

$$F_{ij}[0]\,\mathrm{Pf}\, F[-1] = F_{ij}[1]\,\mathrm{Pf}\, F[-1] = 0. \tag{2.3}$$

Note that for any permutation $\pi \in \mathfrak{S}_{2n}$ the mapping

$$F_{ij}[r] \mapsto F_{\pi(i)\,\pi(j)}[r], \qquad K \mapsto K$$

defines an automorphism of the Lie algebra $\widehat{\mathfrak{o}}_{2n}$. Moreover, the image of $\mathrm{Pf}\, F[-1]$ under its extension to $\mathrm{U}(\widehat{\mathfrak{o}}_{2n})$ coincides with $\mathrm{sgn}\,\pi \cdot \mathrm{Pf}\, F[-1]$. Hence, it is enough to verify (2.3) for $i = 1$ and $j = 2$.

Observe that $F_{12}[0]$ commutes with all summands in (2.2) with $\sigma(1) = 1$ and $\sigma(2) = 2$. Suppose now that $\sigma \in \mathcal{B}_{2n}$ is such that $\sigma(2) > 2$. Then $\sigma(3) = 2$ and $\sigma(4) > 2$. In $V(\widehat{\mathfrak{o}}_{2n})$ we have

$$\begin{aligned} F_{12}[0]\, F_{1\,\sigma(2)}[-1]\, F_{2\,\sigma(4)}[-1] \dots F_{\sigma(2n-1)\,\sigma(2n)}[-1] \\ = -\, F_{2\,\sigma(2)}[-1]\, F_{2\,\sigma(4)}[-1] \dots F_{\sigma(2n-1)\,\sigma(2n)}[-1] \\ +\, F_{1\,\sigma(2)}[-1]\, F_{1\,\sigma(4)}[-1] \dots F_{\sigma(2n-1)\,\sigma(2n)}[-1]. \end{aligned}$$

Set $i = \sigma(2)$ and $j = \sigma(4)$. Note that the permutation $\sigma' = \sigma\,(2\,4)$ also belongs to the subset $\mathcal{B}_{2n}$, and $\mathrm{sgn}\,\sigma' = -\mathrm{sgn}\,\sigma$. We have

$$\begin{aligned} -\, F_{2\,i}[-1]\, F_{2\,j}[-1] + F_{1\,i}[-1]\, F_{1\,j}[-1] + F_{2\,j}[-1]\, F_{2\,i}[-1] - F_{1\,j}[-1]\, F_{1\,i}[-1] \\ = F_{ij}[-2] - F_{ij}[-2] = 0. \end{aligned}$$

This implies that the terms in the expansion of $F_{12}[0]\,\mathrm{Pf}\, F[-1]$ corresponding to pairs of the form (σ, σ') cancel pairwise. Thus, $F_{12}[0]\,\mathrm{Pf}\, F[-1] = 0$.

Now we verify that

$$F_{12}[1]\,\mathrm{Pf}\, F[-1] = 0. \tag{2.4}$$

Consider first the summands in (2.2) with $\sigma(1) = 1$ and $\sigma(2) = 2$. In $V(\widehat{\mathfrak{o}}_{2n})$ we have

$$F_{12}[1]\, F_{1\,2}[-1]\, F_{\sigma(3)\,\sigma(4)}[-1] \dots F_{\sigma(2n-1)\,\sigma(2n)}[-1] = -K\, F_{\sigma(3)\,\sigma(4)}[-1] \dots F_{\sigma(2n-1)\,\sigma(2n)}[-1].$$

Furthermore, let $\tau \in \mathcal{B}_{2n}$ with $\tau(2) > 2$. Then $\tau(3) = 2$ and $\tau(4) > 2$. We have

$$\begin{aligned} F_{12}[1]\, F_{1\,\tau(2)}[-1]\, F_{2\,\tau(4)}[-1] \dots F_{\tau(2n-1)\,\tau(2n)}[-1] \\ = -F_{2\,\tau(2)}[0]\, F_{2\,\tau(4)}[-1] \dots F_{\tau(2n-1)\,\tau(2n)}[-1] \\ = F_{\tau(2)\,\tau(4)}[-1] \dots F_{\tau(2n-1)\,\tau(2n)}[-1]. \end{aligned}$$

Note that for any given σ, the number of elements τ such that the product

$$F_{\tau(2)\,\tau(4)}[-1] \dots F_{\tau(2n-1)\,\tau(2n)}[-1]$$

coincides, up to a sign, with the product

$$F_{\sigma(3)\,\sigma(4)}[-1]\dots F_{\sigma(2n-1)\,\sigma(2n)}[-1]$$

equals $2n-2$. Indeed, for each $k=2,\dots,n$ the unordered pair $\{\tau(2),\tau(4)\}$ can coincide with the pair $\{\sigma(2k-1),\sigma(2k)\}$. Hence, taking the signs of permutations into account, we find that

$$F_{12}[1]\,\mathrm{Pf}\,F[-1]=(-K-2n+2)\sum_{\sigma}\mathrm{sgn}\,\sigma\cdot F_{\sigma(3)\,\sigma(4)}[-1]\dots F_{\sigma(2n-1)\,\sigma(2n)}[-1],$$

summed over $\sigma\in\mathcal{B}_{2n}$ with $\sigma(1)=1$ and $\sigma(2)=2$. Since $-K-2n+2=0$ at the critical level, we get (2.4). □

Introduce formal Laurent series

$$F_{ij}(z)=\sum_{r\in\mathbb{Z}}F_{ij}[r]\,z^{-r-1}\qquad\text{and}\qquad F_{ij}(z)_+=\sum_{r<0}F_{ij}[r]\,z^{-r-1}$$

and expand the Pfaffians of the matrices $F(z)=[F_{ij}(z)]$ and $F(z)_+=[F_{ij}(z)_+]$ by

$$\mathrm{Pf}\,F(z)=\sum_{p\in\mathbb{Z}}S_p\,z^{-p-1}\qquad\text{and}\qquad \mathrm{Pf}\,F(z)_+=\sum_{p<0}S^+_p\,z^{-p-1}.$$

Invoking the vertex algebra structure on the vacuum module $V(\mathfrak{o}_{2n})$ (see [4]), we derive from Theorem 2.1 that the coefficients S_p are *Sugawara operators* for $\widehat{\mathfrak{o}}_{2n}$; they commute with the elements of $\widehat{\mathfrak{o}}_{2n}$ (note that normal ordering is irrelevant here, as the coefficients of the series pairwise commute). Moreover, the coefficients S^+_p are elements of the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{o}}_{2n})$.

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