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Local behaviour of singular solutions for nonlinear elliptic equations in divergence form

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Abstract We consider the following class of nonlinear elliptic equations

$$-\mathrm{div}(\mathcal{A}(|x|)\nabla u) + u^q = 0 \quad \text{in } B_1(0) \setminus \{0\},$$

where $q > 1$ and $\mathcal{A}$ is a positive $C^1(0,1]$ function which is regularly varying at zero with index ϑ in $(2-N,2)$. We prove that all isolated singularities at zero for the positive solutions are removable if and only if $\Phi \notin L^q(B_1(0))$, where Φ denotes the fundamental solution of $-\mathrm{div}(\mathcal{A}(|x|)\nabla u) = \delta_0$ in $\mathcal{D}'(B_1(0))$ and δ_0 is the Dirac mass at 0. Moreover, we give a complete classification of the behaviour near zero of all positive solutions in the more delicate case that $\Phi \in L^q(B_1(0))$. We also establish the existence of positive solutions in all the categories of such a classification. Our results apply in particular to the model case $\mathcal{A}(|x|) = |x|^\vartheta$ with $\vartheta \in (2-N,2)$.

Keywords Nonlinear elliptic equations · Isolated singularities · Removable singularities · Regular variation theory

Mathematics Subject Classification (2000) 35J60 · 35B40 · 35J25

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1 Introduction

In the celebrated paper [4], Brezis and Véron considered the equation

$$-\Delta u+|u|^{q-1}u=0 \quad \text{in } B_1^*:=B_1\setminus\{0\}, \tag{1.1}$$

where B_1 is the unit ball centered at the origin in $\mathbb{R}^N$ with $N\geq 3$. They proved that if $q\geq N/(N-2)$, then 0 is a removable singularity for all positive solutions of (1.1), i.e., if $u\in C^2(B_1^*)$ is a positive solution of (1.1), then u can be extended as a classical solution of (1.1) in the whole ball B_1. The restriction $q\geq N/(N-2)$ is essential, since if $1<q<N/(N-2)$ there are solutions of (1.1) with isolated singularities. The following complete classification of the behaviour near zero of all positive solutions of (1.1) was first given by Véron [26,27] and later proved, with different techniques, by Brezis and Oswald [3]:

Theorem 1 *If $1<q<N/(N-2)$ and u is a positive solution of (1.1) in $C^2(B_1^*)$, then one of the following holds:*

1) *u can be extended as a positive C^2 solution of (1.1) in B_1;*
2) *$\lim_{|x|\to 0} u(x)/E(x)=\lambda\in(0,\infty)$ and u satisfies*

$$-\Delta u+u^q=\lambda\delta_0 \quad \textit{in } \mathcal{D}'(B_1);$$

3) $\lim_{|x|\to 0}|x|^{2/(q-1)}u(x)=\left[\frac{2}{q-1}\left(\frac{2q}{q-1}-N\right)\right]^{1/(q-1)}$.

In this classification, $E(x)=\frac{1}{N(N-2)\omega_N}|x|^{2-N}$ is the fundamental solution of $(-\Delta)$ in $\mathbb{R}^N$, where here and throughout, ω_N and δ_0 denote the volume of B_1 and the Dirac mass at 0, respectively.

Our main aim is to extend this kind of results to the following nonlinear elliptic equation in divergence form

$$-\mathrm{div}\,(\mathcal{A}(|x|)\nabla u)+|u|^{q-1}u=0 \quad \text{in } B_1^*. \tag{1.2}$$

We reveal how the function $\mathcal{A}$ affects the afore-mentioned classification under the following assumption made throughout the paper.
Assumption A. *The function $\mathcal{A}$ is a positive $C^1(0,1]$-function such that*

$$\lim_{t\to 0}\frac{t\mathcal{A}'(t)}{\mathcal{A}(t)}=\vartheta \quad \textit{for some } \vartheta\in(2-N,2). \tag{1.3}$$

This means that $L(t)=\mathcal{A}(t)/t^{\vartheta}$ is a positive $C^1(0,1]$ function satisfying $\lim_{t\to 0}tL'(t)/L(t)=0$. In particular L is slowly varying at 0 (see Definition 3 in Appendix A). Non-trivial examples of such functions L are given below for small $t>0$:

(a) the logarithm $\log(1/t)$, its m iterates $\log_m(1/t)$ defined as $\log\log_{m-1}(1/t)$ and powers of $\log_m(1/t)$ for any integer $m\geq 1$;
(b) $\exp((\log(1/t))^{\alpha})$ with $\alpha\in(0,1)$.
(c) $\exp\left(-(\log t)^{1/3}\cos((\log t)^{1/3})\right)$.

We stress that, in the last example, L exhibits infinite oscillations at 0 since

$$\liminf_{t\to 0} L(t) = 0, \qquad \limsup_{t\to 0} L(t) = +\infty.$$

Moreover, a solution of (1.2) is understood in the following sense.

Definition 1 A function u is said to be a *solution* (*sub-solution, super-solution*) of (1.2) if $u \in C^1(B_1^*)$ and satisfies

$$\int_{B_1} \mathcal{A}(|x|)\nabla u \cdot \nabla\varphi \, dx + \int_{B_1} |u|^{q-1} u\varphi \, dx = 0 \qquad (\leq 0, \geq 0) \tag{1.4}$$

for all functions (non-negative functions) $\varphi(x)$ in $C_c^1(B_1^*)$.

Definition 2 We say that a positive solution u of (1.2) *can be extended as a positive continuous solution* of (1.2) in the whole ball B_1 if $u(x)$ converges to a positive number as $|x| \to 0$ and u satisfies (1.2) in $\mathcal{D}'(B_1)$, that is

$$\int_{B_1} \mathcal{A}(|x|)\nabla u \cdot \nabla\varphi \, dx + \int_{B_1} u^q \varphi \, dx = 0 \quad \text{for every } \varphi \in C_c^1(B_1). \tag{1.5}$$

In our analysis, instead of the fundamental solution of the Laplacian, a crucial role is played by the function Φ given by

$$\Phi(x) = \Phi(|x|) := \frac{1}{N\omega_N} \int_{|x|}^1 \frac{t^{1-N}}{\mathcal{A}(t)} \, dt \quad \text{for every } x \in B_1(0), \tag{1.6}$$

which can be seen as the fundamental solution of

$$-\mathrm{div}\,(\mathcal{A}(|x|)\nabla\Phi) = \delta_0 \ \text{ in } \mathcal{D}'(B_1(0)) \tag{1.7}$$

with homogeneous Dirichlet boundary condition.

Now we can state our main results.

Theorem 2 *Let $q > 1$. The following assertions are true.*

(i) *There always exist positive continuous solutions of* (1.2) *in B_1.*
(ii) *Every positive solution of* (1.2) *can be extended as a positive continuous solution of* (1.2) *in B_1 if and only if $\Phi \notin L^q(B_1)$.*
(iii) *There exist positive solutions of* (1.2) *such that* $\lim_{|x|\to 0} u(x)/\Phi(x) \in (0, \infty]$ *if and only if $\Phi \in L^q(B_1)$.*

Using that Φ is regularly varying at zero with index $2 - N - \vartheta$ (see Appendix A), we find that if $q \neq N/(N - 2 + \vartheta)$, then $\Phi \in L^q(B_1)$ if and only if $q < N/(N-2+\vartheta)$. If $q = N/(N-2+\vartheta)$, then since $\mathcal{A}(r) = r^\vartheta L(r)$ is smooth away from the origin, we obtain that

$$\Phi \in L^q(B_1) \ \text{ if and only if } \int_{0^+} \frac{dr}{rL^q(r)} < \infty,$$

which means that $\Phi \notin L^q(B_1)$ for some examples of $\mathcal{A}$ (e.g., $\mathcal{A}(r) = r^\vartheta$), while $\Phi \in L^q(B_1)$ for other choices of $\mathcal{A}$ (e.g., $\mathcal{A}(r) \sim r^\vartheta \log^2(1/r)$ as $r \to 0$).

Theorem 3 *Let $q > 1$. Assume that $\Phi \in L^q(B_1)$. Then for every positive solution u of (1.2), exactly one of the following cases occurs:*

(I) *u can be extended as a positive continuous solution of (1.2) in B_1.*
(II) $\lim_{|x|\to 0} u(x)/\Phi(x) = \lambda \in (0,\infty)$ *and u satisfies*

$$-\mathrm{div}\,(\mathcal{A}(|x|)\nabla u) + u^q = \lambda\delta_0 \quad \text{in } \mathcal{D}'(B_1). \tag{1.8}$$

(III) $\lim_{|x|\to 0} u(x)/\Phi(x) = \infty$, *in which case we have*[1] $u(x) \sim \tilde{u}(|x|)$ *as* $|x| \to 0$, *where we define $\tilde{u}(r)$ for $r \in (0,1)$ as follows*

$$\tilde{u}(r) = \begin{cases} \left[\dfrac{(q-1)^2}{N-(N-2+\vartheta)q}\displaystyle\int_0^r \frac{t}{\mathcal{A}(t)}\,dt\right]^{\frac{-1}{q-1}} & \text{if } q < \dfrac{N}{N-2+\vartheta}, \\[2ex] (q-1)^{\frac{-1}{q-1}}\Phi(r)\|\Phi\|_{L^q(B_r)}^{\frac{-q}{q-1}} & \text{if } q = \dfrac{N}{N-2+\vartheta}. \end{cases} \tag{1.9}$$

Remark 1 We want to emphasize the role of ϑ in the above classification. It is easy to verify that

$$\lim_{|x|\to 0}\frac{E(x)}{\Phi(x)} = \begin{cases} \infty & \text{if } 2-N < \vartheta < 0, \\ 0 & \text{if } 0 < \vartheta < 2. \end{cases} \tag{1.10}$$

For $q > 1$ fixed, we denote by u and v a positive solution of (1.1) and (1.2), which is comparable to E and Φ near zero, respectively. As expected, from (1.10) we deduce that the blow-up rate of u at zero is greater than that of v when the operator in (1.2) is singular (i.e., $\vartheta < 0$), while it is lower when the equation is degenerate (i.e., $\vartheta > 0$). A similar observation applies when u and v grows faster than E and Φ near zero, respectively.

Theorems 2 and 3 can be extended to more general classes of nonlinear elliptic equations in the framework of regular variation theory by adapting some arguments from [6]. For this reason, we focus on power nonlinearities in (1.2) to underline the role of the coefficient $\mathcal{A}$ appearing in our operator. In this paper we completely solve both questions of removability and classification of singular solutions of (1.2) under optimal conditions.

The removability question for nonlinear elliptic equations has been addressed by many authors. For example, Vázquez and Véron [25] considered equations involving the p-Laplacian operator, while Labutin [15] studied fully nonlinear uniformly elliptic equations of the form $F(D^2u) + f(u) = 0$, under some sharp conditions on f depending on F. In [9] Felmer and Quaas extended the removability results obtained in [15] to a wide class of nonlinear elliptic equations for which a "fundamental" solution can be constructed. We also refer to [16], [17] for other removability results.

[1] Here and throughout, $f(x) \sim g(x)$ as $|x| \to 0$ means that $\lim_{|x|\to 0}\dfrac{f(x)}{g(x)} = 1$.

In general it is much more difficult to classify the behaviour of all solutions near an isolated singularity (see the monograph by Véron [28] and the references therein). For recent generalization of Theorem 1 in the context of regular variation theory we refer, for example, to [5], [6], [7]. The paper [6] extends results in [12], [7] and [5] by providing a complete classification of the isolated singularities for equations such as $\Delta u + \lambda |x|^{-2} u = b(x) h(u)$ in B_1^*, where $-\infty < \lambda \le (N-2)^2/4$.

Further research is devoted to understanding the behaviour of singular solutions for more general elliptic differential operators. The classification of all isolated singularities was recently obtained by Cîrstea and Du [8] for weighted quasilinear elliptic equations of the form $\Delta_p u = b(x) h(u)$ in B_1^*, where $\Delta_p u = \mathrm{div}\,(|\nabla u|^{p-2} \nabla u)$ and $1 < p \le N$. Extending previous results of Friedman–Véron [10] and Vázquez–Véron [24] for $h(u) = u^q$ and $b = 1$, the approach in [8] is based on the regular variation theory and a new perturbation method for constructing sub- and super-solutions. Other recent progress includes the classification of singularities for non-negative viscosity solutions for the infinite Laplace equation $\Delta_\infty u =: \sum_{i,j=1}^N u_{x_i} u_{x_j} u_{x_i x_j} = 0$ (see [20]) and, more generally, for the Aronsson equation (see [13]).

The paper is organised as follows. In Section 2 we consider the radial case and, by performing a change of variable to shift the singularity from 0 to ∞ (see (2.2)), we can apply ODE's results by Taliaferro [22]. Then in Section 3 we provide an upper bound for any positive solution of (1.2) generalising a boundary blow-up argument contained in [2]. Moreover, we prove a Harnack type inequality and a regularity result (see, e.g., [26], [10] and [8]). Theorems 2 and 3 are proved in Section 4 and 5, respectively. The existence assertions (i) and (iii) in Theorem 2 are proved in Propositions 4 and 5, where we actually prove the existence and uniqueness of Dirichlet boundary value problems. If $\Phi \notin L^q(B_1)$, then by reduction to the radial case we prove that every positive solution of (1.2) is dominated by Φ near 0. In Proposition 3 we complete the proof of Theorem 2 by showing that 0 is a removable singularity for a positive solution u if and only if $\lim_{|x|\to 0} u(x)/\Phi(x) = 0$. Proposition 3 is also used to show Case (I) in Theorem 3. Case (II) is proved in Proposition 6. Both cases are addressed by extending blow-up techniques of Friedman-Véron [10] (see also [8], [6]). Case (III) in Theorem 3 is analysed in Proposition 7 using again a reduction to the radial case. Finally, in Appendix A we recall properties of regularly varying functions.

2 The case of radial solutions

In this section, we consider the positive $C^2(0,1]$ solutions of the equation

$$u''(r) + \left(N - 1 + \frac{r\mathcal{A}'(r)}{\mathcal{A}(r)} \right) \frac{u'(r)}{r} = \frac{u^q(r)}{\mathcal{A}(r)} \quad \text{for } 0 < r < 1. \tag{2.1}$$

In Propositions 1 and 2 we give the asymptotics near zero of the positive solutions of (2.1) when $q>1$. We use the following change of variable

$$y(s)=u(r) \quad \text{with } s=\Phi(r), \quad \text{where } \Phi(r) \text{ is given by (1.6).} \tag{2.2}$$

It follows that y satisfies the differential equation

$$\frac{d^2y}{ds^2}=(N\omega_N)^2\, r^{2N-2}\mathcal{A}(r)\, y^q(s) \quad \text{for } 0<s<\infty. \tag{2.3}$$

The asymptotic profile of $y(s)$ as $s\to\infty$ is obtained from the results of Taliaferro [22]. We explicitly observe that $\Phi\in L^q(B_1)$ is equivalent to

$$\int_0^1 r^{N-1}\Phi^q(r)\, dr<\infty. \tag{2.4}$$

Proposition 1 *Let $q>1$ and let γ be any positive number. The following assertions are true:*

(a) *There exists a unique positive solution of* (2.1)*, subject to $u(1)=\gamma$ and $\lim_{r\to 0} u(r)/\Phi(r)=0$. This solution satisfies $\lim_{r\to 0} u(r)\in(0,\infty)$.*
(b) *There exist positive solutions for* (2.1) *with $\lim_{r\to 0} u(r)/\Phi(r)\in(0,\infty)$ if and only if* (2.4) *holds. Assuming that* (2.4) *holds, then we have:*
  (b1) *For any positive number λ, there is a unique positive solution $u_{\lambda,\gamma}$ of*

$$\begin{cases} u''(r)+\left(N-1+\dfrac{r\mathcal{A}'(r)}{\mathcal{A}(r)}\right)\dfrac{u'(r)}{r}=\dfrac{u^q(r)}{\mathcal{A}(r)} \; \textit{for } 0<r<1, \\ \displaystyle\lim_{r\to 0}\frac{u(r)}{\Phi(r)}=\lambda, \qquad u(1)=\gamma. \end{cases} \tag{2.5}$$

  (b2) *There also exist positive solutions for the problem* (2.5) *with $\lambda=\infty$.*

Proof (a) The assumption A implies that $\int_0^1 r^{N-1}\mathcal{A}(r)\, dr<\infty$. By applying Theorem 1.1 in [22] to (2.3) and using (2.2), we conclude the assertion of (a).

(b) By Theorem 2.4 in [22], we find that (2.3) has positive solutions with $\lim_{s\to\infty}\frac{dy}{ds}\in(0,\infty)$ if and only if (2.4) holds. If $y(s)$ is such a solution, then $\lim_{s\to\infty} y(s)/s=c_0$ for some positive constant c_0. This proves the first part of the assertion of (b). If (2.4) holds, then by Corollary 2.5 in [22] (respectively, Theorem 3.2 in [22]), we obtain that (2.5) has positive solutions for any positive number λ (respectively, $\lambda=\infty$). The uniqueness of the positive solution $u_{\lambda,\gamma}$ of (2.5) with $\lambda\in(0,\infty)$ follows from a standard comparison principle.

The conclusion of Proposition 1(b2) can be refined if we assume that

$$\begin{cases} r^{2N-2}\Phi^{q+3}(r)\mathcal{A}(r)\sim h_1(r) \quad \text{as } r\to 0, \\ \text{where } h_1 \text{ is a positive increasing } C(0,1]\text{-function.} \end{cases} \tag{2.6}$$

More precisely, if (2.6) holds (which automatically implies (2.4)), then there is a *unique* positive solution $u_{\infty,\gamma}$ of (2.5) with $\lambda=\infty$ (see Proposition 2(a)).

Remark 2 If (2.4) is not satisfied, then by Theorem 3.2 in [22], we infer that

(i) **either** for each $\gamma > 0$, there are infinitely many positive solutions $u_{\infty,\gamma}$ of (2.5) with $\lambda = \infty$;
(ii) **or** there are no positive solutions of (2.1) with $\lim_{r\to 0} u(r)/\Phi(r) = \infty$.

When (2.4) is not satisfied, Taliaferro [22] shows that Case (i) in Remark 2 may occur (see Example 3.14 in [22]); however, a sufficient condition guaranteeing Case (ii) is the following:

$$\begin{cases} r^{2N-2}\Phi^{q+3}(r)\mathcal{A}(r) \sim h_2(r) \quad \text{as } r \to 0, \\ \text{where } h_2 \text{ is a positive non-increasing } C^1(0,1]\text{-function.} \end{cases} \tag{2.7}$$

This situation is included in our next result.

Proposition 2 *Let $q > 1$ and let γ be any positive number.*

(a) *If (2.6) holds, then there is a unique positive solution $u_{\infty,\gamma}$ of (2.5) with $\lambda = \infty$.*
(b) *Assume that (2.7) is fulfilled.*
 (b1) *If (2.4) holds, then there is a unique positive solution $u_{\infty,\gamma}$ of (2.5) with $\lambda = \infty$.*
 (b2) *If (2.4) fails, then there exists exactly one positive solution of (2.1) with $u(1) = \gamma$, namely the positive solution whose behaviour is given by Proposition 1(a).*

Proof The assertion of (a) follows from Theorem 3.10 in [22]. By applying Theorem 3.12 in [22] to (2.3), jointly with (2.2), we conclude (b1) and (b2).

Remark 3 In the framework of either Proposition 2(a) or (b1), the unique positive solution $u_{\infty,\gamma}$ of (2.5) with $\lambda = \infty$ is asymptotic at zero to any positive C^2 function $\mathcal{U}$ satisfying

$$\begin{cases} \mathcal{U}''(r) + \left(N - 1 + \dfrac{r\mathcal{A}'(r)}{\mathcal{A}(r)}\right)\dfrac{\mathcal{U}'(r)}{r} \sim \dfrac{\mathcal{U}^q(r)}{\mathcal{A}(r)} \quad \text{as } r \to 0, \\ \lim\limits_{r\to 0} \dfrac{\mathcal{U}(r)}{\Phi(r)} = \infty. \end{cases} \tag{2.8}$$

This follows by applying Theorem 3.7 in [22] to (2.3), then using (2.2).

To prove Case (III) in Theorem 3, we rely on Corollaries 1 and 2, which are consequences of Proposition 2.

Corollary 1 *Let $1 < q < N/(N - 2 + \vartheta)$ and let γ be any positive number. Assume that*

$$\rho \neq 0, \quad \text{where } \rho := 4 - N - 2\vartheta - q(N - 2 + \vartheta).$$

Then (2.5) with $\lambda = \infty$ has a unique positive solution $u_{\infty,\gamma}$. Moreover, we have $u_{\infty,\gamma}(r) \sim \tilde{u}(r)$ as $r \to 0$, where $\tilde{u}$ is defined by (1.9).

Proof Observe that $r \longmapsto r^{N-1}\Phi^q(r)$ is regularly varying at zero with index $N-1-q(N-2+\vartheta)$, which is greater than -1. Thus condition (2.4) is automatically verified. On the other hand, $r \longmapsto r^{2N-2}\Phi^{q+3}(r)\mathcal{A}(r)$ is regularly varying at zero of index ρ, which is different from zero. Hence, if $\rho > 0$ (respectively, $\rho < 0$), then (2.6) (respectively, (2.7)) holds. Then the existence and uniqueness of the positive solution $u_{\infty,\gamma}$ of (2.5) with $\lambda = \infty$ is given by Proposition 2. We finish the proof by using Remark 3 after checking (2.8) for $\mathcal{U}(r) = \tilde{u}(r)$. We skip the details of this straightforward calculation.

Corollary 2 *Let $q = N/(N-2+\vartheta)$. If (2.4) holds, then for every positive number γ, there exists a unique positive solution $u_{\infty,\gamma}$ of (2.5) with $\lambda = \infty$. Moreover, we have $u_{\infty,\gamma} \sim \tilde{u}(r)$ as $r \to 0$, where $\tilde{u}$ is given by (1.9).*

Proof When $q = N/(N-2+\vartheta)$, we find that $r^{2N-2}\Phi^q(r)\mathcal{A}(r)$ is regularly varying at zero with index $-2N+4-2\vartheta$. This being a negative number, we infer that (2.7) is satisfied. Then by Proposition 2(b1), we conclude that (2.5) with $\lambda = \infty$ has a unique positive solution $u_{\infty,\gamma}$. In view of Remark 3, we conclude the proof once we show that (2.8) holds with $\mathcal{U}(r) = \tilde{u}(r)$. This simple calculation is left to the reader.

3 Auxiliary tools

In this section, we only require $q > 1$ in (1.2). Let Υ be defined by

$$\Upsilon(r) := \left(\int_0^r \frac{t}{\mathcal{A}(t)}\, dt \right)^{-\frac{1}{q-1}} \quad \text{for every } r \in (0,1). \tag{3.1}$$

We denote by B_r the ball of $\mathbb{R}^n$ centered at the origin with radius r. We set $B_r^* := B_r \setminus \{0\}$ for any $r > 0$.

3.1 A priori estimates

Lemma 1 *Let $q > 1$. For every $r_0 \in (0, 1/2)$, there exists a positive constant C_0, which depends on r_0, such that for any positive sub-solution of (1.2), it holds*

$$u(x) \le C_0 \Upsilon(|x|) \quad \text{for every } x \in \mathbb{R}^N \text{ with } 0 < |x| \le r_0. \tag{3.2}$$

Proof Let $x_0 \in \mathbb{R}^N$ be fixed with $0 < |x_0| \le r_0$. We denote $\omega := B_{|x_0|/2}(x_0)$ and define a positive function S on ω such that $S = \infty$ on $\partial\omega$. More precisely, for some constant $C > 0$, we set

$$S(x) = C \left[\frac{1}{|x_0|\sqrt{\mathcal{A}(|x_0|)}} \left(|x_0|^2 - 4|x - x_0|^2 \right) \right]^{-\frac{2}{q-1}} \quad \text{for } x \in \omega. \tag{3.3}$$

Claim: *There exists a positive constant C, which is independent of x_0, such that S in* (3.3) *is a super-solution of* (1.2) *in ω, that is*

$$-\mathrm{div}(\mathcal{A}(x)\nabla S(x)) + S^q(x) \geq 0 \quad \text{in } \omega. \tag{3.4}$$

From $\mathcal{A} \in RV_\vartheta(0+)$, we can write $\mathcal{A}(t) = t^\vartheta L(t)$, where L is a slowly varying function at zero. Since $|x_0|/2 \leq |x| \leq 3|x_0|/2$ for $x \in \omega$, Proposition 8 in Appendix A shows that

$$c_0 \leq \frac{\mathcal{A}(|x|)}{\mathcal{A}(|x_0|)} = \frac{|x|^\vartheta}{|x_0|^\vartheta}\frac{L(|x|)}{L(|x_0|)} \leq c_1 \tag{3.5}$$

for some positive constants c_0 and c_1, which depend on r_0, but are independent of x_0. Direct computations imply that (3.4) holds if and only if

$$\frac{16}{q-1}\frac{\mathcal{A}(|x|)}{\mathcal{A}(|x_0|)}\left[N + \frac{8|x-x_0|^2}{|x_0|^2}\left(\frac{q+1}{q-1} - \frac{N}{2}\right) + \frac{\mathcal{A}'(|x|)}{|x|\mathcal{A}(|x|)}\left(1 - \frac{4|x-x_0|^2}{|x_0|^2}\right) x\cdot(x-x_0)\right] \leq C^{q-1}$$

for every $x \in \omega$. Using (3.5) and Assumption A, we obtain that for every $x \in \omega$, the left-hand side of the above inequality is bounded from above by a positive constant independent of x_0. Hence, we can choose $C > 0$ large enough and independent of x_0 such that (3.4) holds.

We now conclude (3.2) by the comparison principle. Indeed, $u(x) \leq S(x)$ for every $x \in \omega$ and, in particular, $u(x_0) \leq S(x_0)$. Thus we have

$$u(x_0) \leq C\left(\frac{|x_0|^2}{\mathcal{A}(|x_0|)}\right)^{-\frac{1}{q-1}} \quad \text{for every } x_0 \in \mathbb{R}^N \ \text{ with } 0 < |x_0| \leq r_0. \tag{3.6}$$

Since $\mathcal{A} \in RV_\vartheta(0+)$ with $\vartheta < 2$, by Theorem 5 in Appendix A, we find that

$$\lim_{r\to 0}\frac{r^2}{\mathcal{A}(r)\int_0^r \frac{t}{\mathcal{A}(t)}\,dt} = 2 - \vartheta \ \text{ so that } \alpha := \inf_{0<r\leq r_0}\frac{r^2}{\mathcal{A}(r)\int_0^r \frac{t}{\mathcal{A}(t)}\,dt} > 0. \tag{3.7}$$

Then using (3.6) and the definition of Υ in (3.1), we obtain that

$$u(x_0) \leq C\alpha^{-1/(q-1)}\Upsilon(|x_0|) \ \text{ for every } x_0 \in \mathbb{R}^N \ \text{ with } 0 < |x_0| \leq r_0.$$

This completes the proof of Lemma 1.

Since $\Upsilon \in RV_{\frac{\vartheta-2}{q-1}}(0+)$, we have $\lim_{r\to 0}\Upsilon(r)/f(r) = 0$ for any $f \in RV_\sigma(0+)$ with $\sigma < (\vartheta-2)/(q-1)$. As a consequence of Lemma 1, we obtain the following.

Corollary 3 *If $q > 1$, then any positive sub-solution of* (1.2) *satisfies*

$$\lim_{|x|\to 0}\frac{u(x)}{f(|x|)} = 0 \quad \text{for every } f \in RV_\sigma(0+) \text{ with } \sigma < \frac{\vartheta-2}{q-1}.$$

3.2 A Harnack-type inequality

Using Lemma 1 and the Harnack inequality (Theorem 8.20 in [11]), we derive the following.

Lemma 2 *Let $q>1$. For every $r_0\in(0,1/4)$, there exists a positive constant C, which depends on r_0, such that for any positive solution u of* (1.2)*, we have*

$$\max_{|x|=r} u(x)\le C\min_{|x|=r} u(x)\quad \textit{for all } 0<r\le r_0. \tag{3.8}$$

Proof The argument is standard, following ideas similar to Lemma 1.5 in [26]. However, some changes appear here due to the divergence form of (1.2). Thus for the reader's convenience, we provide the details.

We first observe that equation (1.2) is equivalent in $\mathcal{D}'(B_1^*)$ to the following

$$-\Delta u-\nabla u\cdot\nabla\log\mathcal{A}(|x|)+\frac{|u|^{q-1}u}{\mathcal{A}(|x|)}=0\ \text{ in } B_1^*, \tag{3.9}$$

meaning that

$$\int_{B_1}\nabla u\cdot\nabla\varphi\,dx-\int_{B_1}\varphi\frac{\mathcal{A}'(|x|)}{\mathcal{A}(|x|)}\frac{x}{|x|}\cdot\nabla u(x)\,dx+\int_{B_1}\frac{|u|^{q-1}u\varphi}{\mathcal{A}(|x|)}\,dx=0$$

for every $\varphi\in C_c^1(B_1^*)$.

Let $y\in\mathbb{R}^N$ be such that $0<|y|\le r_0$. We have $\Omega_y\subset B_{2r_0}^*$, where we define

$$\Omega_y:=B_{\frac{2|y|}{3}}(y).$$

We apply the Harnack inequality of Theorem 8.20 in [11] for the operator

$$Lu:=\Delta u+\sum_{i=1}^N c^i(x)\frac{\partial u}{\partial x_i}+d(x)u\quad\text{in }\Omega_y, \tag{3.10}$$

where $c^i(x)$ (for $i=1,\dots,N$) and $d(x)$ are defined by

$$c^i(x):=\frac{x_i}{|x|}\frac{\mathcal{A}'(|x|)}{\mathcal{A}(|x|)}\ \text{ and }\ d(x):=-\frac{u^{q-1}(x)}{\mathcal{A}(|x|)}\ \text{ for } x=(x_1,\dots,x_N)\in B_{2r_0}^*.$$

The hypotheses (8.5) and (8.6) in Gilbarg–Trudinger [11, p. 178] are satisfied here with

$$\lambda=1,\quad \Lambda=\sqrt{N}\ \text{ and }\ \nu(y):=\sup_{x\in\Omega_y}\sqrt{\left[\frac{\mathcal{A}'(|x|)}{\mathcal{A}(|x|)}\right]^2+|d(x)|}. \tag{3.11}$$

Since $B_{4\eta}(y)\subset\Omega_y$ with $\eta(y):=|y|/8$, by Theorem 8.20 in [11] we have

$$\sup_{B_\eta(y)} u\le\mathcal{C}\inf_{B_\eta(y)} u, \tag{3.12}$$

where $\mathcal{C} = \mathcal{C}(N, \eta(y)\nu(y))$ is a positive constant that can be estimated by

$$\mathcal{C} \leq \mathcal{C}_0^{\sqrt{N}+\eta(y)\nu(y)} \quad \text{with } \mathcal{C}_0 = \mathcal{C}_0(N). \tag{3.13}$$

We now show that $\eta(y)\nu(y)$ is bounded above by a constant independent of y with $0 < |y| \leq r_0$. From (1.3) and (3.7), we infer that

$$\sup_{0<r\leq 2r_0} \frac{r|\mathcal{A}'(r)|}{\mathcal{A}(r)} := M_1(r_0) < \infty \text{ and } \sup_{0<r\leq 2r_0} \frac{r^2}{\mathcal{A}(r)\int_0^r \frac{t}{\mathcal{A}(t)}\,dt} := M_2(r_0) < \infty.$$

Then by Lemma 1, there exists a positive constant $C_1 = C_1(r_0)$ such that

$$|x|^2|d(x)| = \frac{|x|^2 u^{q-1}(x)}{\mathcal{A}(|x|)} \leq C_1 \frac{|x|^2}{\mathcal{A}(|x|)\int_0^{|x|} \frac{t}{\mathcal{A}(t)}\,dt} \leq C_1 M_2 < \infty \tag{3.14}$$

for every $0 < |x| \leq 2r_0$. From $x \in \Omega_y$, we have $2r_0 > |x| > |y|/3$. Hence, using the definition of $\nu(y)$ in (3.11) and $\eta(y) = |y|/8$, jointly with (3.14), we get

$$\begin{aligned}
8\eta(y)\nu(y) &= |y| \sup_{x\in\Omega_y} \sqrt{\left[\frac{\mathcal{A}'(|x|)}{\mathcal{A}(|x|)}\right]^2 + |d(x)|} \\
&\leq 3 \sup_{0<|x|\leq 2r_0} \sqrt{\left[\frac{|x|\mathcal{A}'(|x|)}{\mathcal{A}(|x|)}\right]^2 + |x|^2|d(x)|} \\
&\leq 3 \sup_{0<|x|\leq 2r_0} \left(\frac{|x||\mathcal{A}'(|x|)|}{\mathcal{A}(|x|)} + |x|\sqrt{|d(x)|}\right) \leq 3(M_1 + \sqrt{C_1 M_2}) < \infty.
\end{aligned}$$

This means that $\eta(y)\nu(y)$ is bounded above by a positive constant that is independent of y for any $0 < |y| \leq r_0$. From (3.12) and (3.13), we conclude (3.8) using a standard covering argument.

Using the Harnack-type inequality in Lemma 2, we prove the following.

Corollary 4 *Let u be a positive solution of* (1.2) *with $q > 1$. If we have* $\limsup_{|x|\to 0} u(x)/\Phi(x) = \infty$, *then* $\lim_{|x|\to 0} u(x)/\Phi(x) = \infty$.

Proof Suppose by contradiction that $l := \liminf_{|x|\to 0} u(x)/\Phi(x) < \infty$. Then there exists a sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}^N$ that converges to zero as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{u(x_n)}{\Phi(x_n)} = l.$$

Fix $r_0 \in (0, 1/4)$. We can assume that $|x_n|$ decreases to zero with $0 < |x_n| \leq r_0$. Let n_0 be a large positive integer such that

$$\frac{u(x_n)}{\Phi(x_n)} \leq l + 1 \quad \text{for every } n \geq n_0.$$

By Lemma 2, there is a constant $C > 0$ such that (3.8) holds. Thus we have

$$\max_{|x|=|x_n|} u(x) \leq C \min_{|x|=|x_n|} u(x) \leq C u(x_n) \leq C(l+1)\Phi(x_n) \tag{3.15}$$

for all $n \geq n_0$. By the comparison principle on each annulus $\{x \in \mathbb{R}^N : |x_n| < |x| < |x_{n_0}|\}$ with $n > n_0$, we have

$$u(x) \leq C(l+1)\Phi(x) \quad \text{for all } 0 < |x| \leq |x_{n_0}|, \tag{3.16}$$

which is in contradiction with $\limsup_{|x|\to 0} u(x)/\Phi(x) = \infty$.

3.3 A regularity result

We fix $r_0 \in (0, 1/4)$ and let g be a positive continuous function defined on $(0, 4r_0]$. The following regularity result will be used many times in the paper.

Lemma 3 *Let $q > 1$ and $0 \leq \delta \leq (2-\vartheta)/(q-1)$. Let $g \in RV_{-\delta}(0+)$ satisfy*

$$\limsup_{r\to 0} \frac{g(r)}{\Upsilon(r)} < \infty, \tag{3.17}$$

where Υ is defined by (3.1). If u is a positive solution of (1.2) and $C_1 > 0$ is a constant such that

$$0 < u(x) \leq C_1 g(|x|) \quad \textit{for } 0 < |x| < 2r_0, \tag{3.18}$$

then there exist positive constants C and $\alpha \in (0,1)$ such that

$$|\nabla u(x)| \leq C\frac{g(|x|)}{|x|} \quad \textit{and} \quad |\nabla u(x) - \nabla u(x')| \leq C\frac{g(|x|)}{|x|^{1+\alpha}}|x - x'|^{\alpha} \tag{3.19}$$

for any x, x' in $\mathbb{R}^N$ satisfying $0 < |x| \leq |x'| < r_0$.

Remark 4 If (3.18) holds for $g \in RV_{-\delta}(0+)$ and $0 \leq \delta < (2-\vartheta)/(q-1)$, then we have $g(r)/\Upsilon(r) \to 0$ as $r \to 0$ since g/Υ varies regularly at zero with positive index $(2-\vartheta)/(q-1) - \delta$ (see Proposition 9 in Appendix A). On the other hand, Lemma 1 shows that there exists a constant $C_1 > 0$ such that (3.18) holds with $g \equiv \Upsilon$ for *every* positive solution u of (1.2).

Proof We modify an argument in [8, Lemma 4.1], which is similar to [10, Lemma 1.1]. However, in our case here, we need to take extra care due to the additional gradient term in (3.9). We set

$$\Gamma := \{y \in \mathbb{R}^N : 1 < |y| < 7\} \quad \text{and} \quad \Gamma^* := \{y \in \mathbb{R}^N : 2 < |y| < 6\}.$$

Fix $\beta \in (0, r_0/6)$ and define Ψ_β on $\overline{\Gamma}$ as follows

$$\Psi_\beta(\xi) := \frac{u(\beta\xi)}{g(\beta)} \quad \text{for every } \xi \in \overline{\Gamma}. \tag{3.20}$$

Since $u \in C^1(B_1^*)$ is a weak solution of (3.9), we obtain that $\Psi_\beta \in C^1(\overline{\Gamma})$ is a weak solution of

$$L\Psi_\beta(\xi) := \Delta\Psi_\beta(\xi) + \sum_{i=1}^{N} c_\beta^i(\xi)\frac{\partial}{\partial\xi_i}\Psi_\beta(\xi) = f_\beta(\xi) \quad \text{in } \Gamma, \tag{3.21}$$

where for convenience, we set

$$\begin{cases} c^i_\beta(\xi) := \dfrac{\xi_i}{|\xi|}\beta\dfrac{\mathcal{A}'(\beta|\xi|)}{\mathcal{A}(\beta|\xi|)} & \text{for } i = 1,\ldots,N; \\ f_\beta(\xi) := \dfrac{\beta^2[u(\beta\xi)]^q}{\mathcal{A}(\beta|\xi|)g(\beta)} & \text{for } \xi = (\xi_1,\ldots,\xi_N) \in \overline{\Gamma}. \end{cases} \tag{3.22}$$

Notice that c^i_β, Ψ_β and f_β are in $C(\overline{\Gamma})$ and thus they belong to $L^\infty(\Gamma)$.

Claim: *The L^∞-norms of c^i_β, Ψ_β and f_β are bounded from above by constants that are independent of β in $(0, r_0/6)$.*

Indeed, in view of (1.3), we see that

$$|c^i_\beta(\xi)| \le \beta\frac{|\mathcal{A}'(\beta|\xi|)|}{\mathcal{A}(\beta|\xi|)} \le \beta|\xi|\frac{|\mathcal{A}'(\beta|\xi|)|}{\mathcal{A}(\beta|\xi|)} \le \sup_{0<r\le 2r_0}\frac{r|\mathcal{A}'(r)|}{\mathcal{A}(r)} < \infty$$

for every $\xi \in \Gamma$ and $i = 1,\ldots,N$. From the continuity assumption on $g \in RV_{-\delta}(0+)$, we have that $\mathcal{L}(t) := t^\delta g(t)$ is a positive continuous function on $(0, 4r_0]$ that is slowly varying at zero. By Proposition 8, we have

$$\lim_{\beta\to 0}\frac{\mathcal{L}(\beta|\xi|)}{\mathcal{L}(\beta)} = 1 \quad \text{uniformly with respect to } \xi \in \Gamma.$$

So, there exist positive constants c_1 and c_2, which are independent of β, such that

$$c_1\mathcal{L}(\beta) \le \mathcal{L}(\beta|\xi|) \le c_2\mathcal{L}(\beta) \quad \text{for every } \beta \in (0, r_0/6) \text{ and every } \xi \in \Gamma.$$

In relation to Ψ_β given by (3.20), we use (3.18) to obtain that

$$|\Psi_\beta(\xi)| \le C_1\frac{g(\beta|\xi|)}{g(\beta)} = C_1|\xi|^{-\delta}\frac{\mathcal{L}(\beta|\xi|)}{\mathcal{L}(\beta)} \le C_1\frac{\mathcal{L}(\beta|\xi|)}{\mathcal{L}(\beta)} \le C_1 c_2 \tag{3.23}$$

for every $\xi \in \Gamma$. This proves that $\|\Psi_\beta\|_{L^\infty(\Gamma)} \le C_1 c_2$. From (3.18), (3.22) and Lemma 1, we find that

$$\begin{aligned} |f_\beta(\xi)| &\le C_1(C_0)^{q-1}\frac{g(\beta|\xi|)}{g(\beta)}\frac{(\beta|\xi|)^2}{\mathcal{A}(\beta|\xi|)\int_0^{\beta|\xi|}\frac{t}{\mathcal{A}(t)}\,dt} \\ &\le C_1(C_0)^{q-1}c_2\sup_{0<r\le 2r_0}\frac{r^2}{\mathcal{A}(r)\int_0^r\frac{t}{\mathcal{A}(t)}\,dt} \end{aligned}$$

for every $\xi \in \Gamma$ and every $\beta \in (0, r_0/6)$. Hence, the L^∞-norm of f_β is bounded from above by a constant independent of β. This concludes the above claim.

By Theorem 8.8 in [11], we infer that $\Psi_\beta \in W^{2,2}_{\rm loc}(\Gamma)$ and $L\Psi_\beta = f_\beta$ a.e. in Γ, where $L\Psi_\beta$ is given by (3.21). Furthermore, by Corollary 9.18 in [11, p. 243], we have $\Psi_\beta \in W^{2,p}_{\rm loc}(\Gamma)$ for $p > N$. Hence, by Theorem 7.26 in [11, p.

171], we obtain that $\Psi_\beta \in C^{1,\alpha}_{\rm loc}(\Gamma)$ with $\alpha = 1 - N/p \in (0,1)$ and there exists a positive constant $\mathcal{C}$ such that

$$\|\nabla \Psi_\beta\|_{C^{0,\alpha}(\Gamma^*)} \leq \mathcal{C} \|\Psi_\beta\|_{W^{2,p}(\Gamma^*)}. \tag{3.24}$$

Since $\Psi_\beta \in W^{2,p}_{\rm loc}(\Gamma) \cap L^p(\Gamma)$ is a strong solution of $L\Psi_\beta = f_\beta$ in Γ, by Theorem 9.11 in [11, p. 235], we get the estimate

$$\|\Psi_\beta\|_{W^{2,p}(\Gamma^*)} \leq C_* \left(\|\Psi_\beta\|_{L^p(\Gamma)} + \|f_\beta\|_{L^p(\Gamma)} \right),$$

where C_* is a positive constant depending on N, p and $\max_{1\leq i\leq N} \|c^i_\beta\|_{L^\infty(\Gamma)}$. In particular, C_* is independent of β. Using that Γ is a bounded set, there exists a positive constant C^* such that $\|v\|_{L^\infty(\Gamma)} \leq C^* \|v\|_{L^p(\Gamma)}$ for every $v \in L^\infty(\Gamma)$. It follows that

$$\|\Psi_\beta\|_{W^{2,p}(\Gamma^*)} \leq C_* C^* (\|\Psi_\beta\|_{L^\infty(\Gamma)} + \|f_\beta\|_{L^\infty(\Gamma)}).$$

Thus $\|\Psi_\beta\|_{W^{2,p}(\Gamma^*)}$ is bounded from above by a constant independent of β. This, jointly with (3.24), shows that there exists a positive constant $\widetilde{C}$ independent of β such that

$$\|\nabla \Psi_\beta\|_{C^{0,\alpha}(\Gamma^*)} \leq \widetilde{C}, \quad \text{where } \alpha = 1 - \frac{N}{p} \in (0,1). \tag{3.25}$$

The proof of (3.19), which relies on (3.25), follows now exactly in the same manner as that of (4.2) in [8, Lemma 4.1]. Thus we skip the details.

Lemma 4 *Let $q > 1$ and let u be a positive solution of* (1.2). *Then there exist two positive radial solutions of* (1.2) *in $B^*_{1/2}$, say u_* and u^*, such that*

$$\frac{1}{K} u \leq u_* \leq u \leq u^* \leq Ku \quad \text{in } B^*_{1/2}, \tag{3.26}$$

where $K > 1$ is a sufficiently large constant.

Proof We first construct u^*. For any integer $n \geq 3$, we define

$$D_n := \{x \in \mathbb{R}^N : \ 1/n < |x| < 1/2\}.$$

We consider the problem

$$\begin{cases} -\operatorname{div}(\mathcal{A}(|x|)\nabla w) + w^q = 0 & \text{in } D_n, \\ w(x) = \max\limits_{|y|=|x|} u(y) & \text{for } |x| = 1/n \text{ and } |x| = 1/2. \end{cases} \tag{3.27}$$

Let w_n denote the unique positive C^2 solution of (3.27). The uniqueness follows from the comparison principle. From the invariance of the operator under rotation, the symmetry of the domain and the boundary data, we have that w_n is radially symmetric in D_n. The comparison principle yields that $u \leq w_n \leq w_{n+1}$ in D_n. By the Harnack inequality (Lemma 2), there exists a large constant $K > 1$ such that for every $n \geq 3$, we have $w_n(x) \leq Ku(x)$ for

$|x| = 1/n$ and $|x| = 1/2$. Since Ku is a super-solution of (1.2), the comparison principle gives that $w_n \le Ku$ in D_n. Using Lemma 1 and Lemma 3, we find that, up to a subsequence, w_n converges to u^* in $C^1_{\rm loc}(B^*_{1/2})$ and u^* is a positive radial solution of (1.2) in $B^*_{1/2}$ such that

$$u \le u^* \le Ku \quad \text{in } B^*_{1/2}.$$

To show the existence of u_*, we proceed in a somehow similar fashion. For $n \ge 3$, let v_n denote the unique positive solution of the problem

$$\begin{cases} -\operatorname{div}\left(\mathcal{A}(|x|)\nabla v\right) + v^q = 0 \quad \text{in } D_n, \\ v(x) = \min\limits_{|y|=|x|} u(y) \quad \text{for } |x| = 1/n \text{ and } |x| = 1/2. \end{cases} \tag{3.28}$$

Note that we changed only the boundary condition in (3.27). As before, v_n is radially symmetric in D_n. From the comparison principle, we find that $v_{n+1} \le v_n \le u$ in D_n. In light of Lemma 2, we have $v_n \ge u/K$ on ∂D_n. Since u/K is a sub-solution of (1.2), we infer that $v_n \ge u/K$ in D_n. Using again Lemma 1 and Lemma 3, we conclude that, up to a subsequence, $v_n \to u_*$ in $C^1_{\rm loc}(B^*_{1/2})$ and u_* is a positive radial solution of (1.2) in $B^*_{1/2}$ such that

$$\frac{1}{K} u \le u_* \le u \quad \text{in } B^*_{1/2}.$$

This completes the proof of Lemma 4.

4 Proof of Theorem 2

We first prove that every positive solution of (1.2) which is dominated by Φ near zero can be extended as a positive continuous solution of (1.2) in B_1 (see Proposition 3). To conclude the assertion of (i), we show that (1.2) has a unique positive solution satisfying $\lim_{|x|\to 0} u(x)/\Phi(x) = 0$ and a Dirichlet boundary condition on ∂B_1 (see Proposition 4). Proposition 5 completely proves the assertion of (iii) and the direct implication of (ii). The converse implication of (ii) follows by Proposition 3.

Proposition 3 *Let $q > 1$. The following hold:*

(a) *If u is a positive solution of (1.2) such that $\lim_{|x|\to 0} u(x)/\Phi(x) = 0$, then*

$$\lim_{|x|\to 0} u(x) = \theta \in (0, \infty) \quad \text{and} \quad \lim_{|x|\to 0} |x|\, |\nabla u(x)| = 0. \tag{4.1}$$

Moreover, $u \in C^{1,\alpha}_{\rm loc}(B^*_1)$ for some $0 < \alpha < 1$ and satisfies

$$\int_{B_1} \mathcal{A}(|x|)\nabla u \cdot \nabla \varphi \, dx + \int_{B_1} u^q \varphi \, dx = 0 \quad \text{for all } \varphi \in C^1_c(B_1). \tag{4.2}$$

(b) *If $\Phi \notin L^q(B_1)$, then $\lim_{|x|\to 0} u(x)/\Phi(x) = 0$ for any positive solution of (1.2).*

Proof (a) We set $\theta := \limsup_{|x|\to 0} u(x)$. From Lemma 4 and Proposition 1(a), we conclude that

$$0 < \liminf_{|x|\to 0} u(x) \le \theta < \infty. \tag{4.3}$$

We fix $r_0 > 0$ small such that $B_{2r_0} \subset\subset B_1$. We define $F(r) := \sup_{|x|=r} u(x)$ for $r \in (0, 2r_0)$. Clearly, we have $\limsup_{r\to 0} F(r) = \theta$. We now show that $\liminf_{r\to 0} F(r) = \theta$. If we assume the contrary, that is $\liminf_{r\to 0} F(r) < \theta$, then there exist $\epsilon > 0$ small and a sequence of positive numbers $(t_n)_{n\ge 1}$ decreasing to zero as $n \to \infty$ such that $F(t_n) \le \theta - \epsilon$ for every $n \ge 1$. Since $\limsup_{r\to 0} F(r) = \theta$, then we have

$$F(t_*) > \theta - \epsilon \quad \text{for some small } t_* > 0.$$

Without loss of generality, we can assume that $t_* < t_1 < 1$. Let $n_1 > 1$ be large enough such that $t_{n_1} < t_*$. We fix $n \ge n_1$ and set $\Omega := \{x \in \mathbb{R}^N : \ t_n < |x| < t_1\}$. Since $\max\{F(t_n), F(t_1)\} < F(t_*)$, we have that $\sup_\Omega u = \sup_B u$ for some ball $B \subset\subset \Omega$ (that is $\overline{B} \subset \Omega$). Recall that u is a solution of (3.9). Then by Theorem 8.19 in [11, p. 198] with L defined here as in (3.10) , we conclude that u is constant in Ω. This is a contradiction, which proves that

$$\lim_{r\to 0} F(r) = \theta, \quad \text{where } F(r) := \sup_{|x|=r} u(x). \tag{4.4}$$

From (4.3), there exists a positive constant C_1 such that

$$0 < u(x) \le C_1 \quad \text{for every } 0 < |x| \le 2r_0. \tag{4.5}$$

By applying Lemma 3 with $g \equiv 1$, we find positive constants C and $\alpha \in (0,1)$ such that

$$|\nabla u(x)| \le \frac{C}{|x|} \quad \text{and} \quad |\nabla u(x) - \nabla u(x')| \le C \frac{|x - x'|^\alpha}{|x|^{1+\alpha}} \tag{4.6}$$

for any x, x' in $\mathbb{R}^N$ with $0 < |x| \le |x'| < r_0$. For each $r \in (0, r_0)$, we introduce the function $U_{(r)}$ by

$$U_{(r)}(\xi) := u(r\xi) \quad \text{for every } \xi \in \mathbb{R}^N \text{ with } 0 < |\xi| < r_0/r. \tag{4.7}$$

This, together with (4.5) and (4.6), implies that

$$\begin{cases} |U_{(r)}(\xi)| \le C_1, \quad |\nabla U_{(r)}(\xi)| \le \dfrac{C}{|\xi|}, \\ |\nabla U_{(r)}(\xi) - \nabla U_{(r)}(\xi')| \le C \dfrac{|\xi - \xi'|^\alpha}{|\xi|^{1+\alpha}} \end{cases} \tag{4.8}$$

for every ξ, ξ' in $\mathbb{R}^N$ with $0 < |\xi| \le |\xi'| < r_0/r$. Since u is a solution of (3.9), we obtain that $U_{(r)}$ is a positive weak solution of the equation

$$\Delta U_{(r)}(\xi) + \frac{r|\xi|\mathcal{A}'(r|\xi|)}{\mathcal{A}(r|\xi|)} \frac{\xi}{|\xi|^2} \cdot \nabla U_{(r)}(\xi) = r^2 \frac{U_{(r)}^q(\xi)}{\mathcal{A}(r|\xi|)} \quad \text{for } 0 < |\xi| < r_0/r. \tag{4.9}$$

For every fixed $\xi \in \mathbb{R}^N \setminus \{0\}$, we have $0 < |\xi| < r_0/r$ provided that $r > 0$ is sufficiently small. Recall that $\mathcal{A} \in RV_\vartheta(0+)$ and (1.3) holds. Hence, using also Proposition 9, we deduce that

$$\lim_{r\to 0} \frac{r|\xi|\mathcal{A}'(r|\xi|)}{\mathcal{A}(r|\xi|)} = \vartheta \quad \text{and} \quad \lim_{r\to 0} \frac{r^2}{\mathcal{A}(r|\xi|)} = 0 \quad \text{for every fixed } \xi \in \mathbb{R}^N \setminus \{0\}.$$

Using (4.8), (4.9) and the Arzela–Ascoli Theorem, we have that any sequence $(\overline{r}_n)$ decreasing to zero as $n \to \infty$ contains a subsequence r_n such that

$$U_{(r_n)} \to U \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}), \tag{4.10}$$

where U satisfies the equation

$$\Delta U(x) + \vartheta \frac{x}{|x|^2} \cdot \nabla U(x) = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{4.11}$$

We next show that $U \equiv \theta$ in $\mathbb{R}^N \setminus \{0\}$. Let ξ_{r_n} be on the $(N-1)$-dimensional unit sphere S^{N-1} in $\mathbb{R}^N$ such that $F(r_n) = u(r_n \xi_{r_n})$. We may assume that $\xi_{r_n} \to \xi_0$ as $n \to \infty$. Using (4.7), we have

$$U_{(r_n)}(\xi) \le F(r_n|\xi|) \quad \text{for } 0 < |\xi| < \frac{r_0}{r_n} \quad \text{and} \quad U_{(r_n)}(\xi_{r_n}) = F(r_n). \tag{4.12}$$

Letting $n \to \infty$ in (4.12), then using (4.4) and (4.10), we obtain that $U(\xi) \le \theta$ for every $\xi \in \mathbb{R}^N \setminus \{0\}$ and $U(\xi_0) = \theta$. Then by the strong maximum principle of Theorem 8.19 in [11] applied to $U - \theta$ satisfying (4.11), we conclude that $U(x) = \theta$ for every $x \in \mathbb{R}^N \setminus \{0\}$. From (4.10), we find that $\lim_{n\to\infty} U_{(r_n)}(x) = \theta$ and $\lim_{n\to\infty} \nabla U_{(r_n)}(x) = 0$ for every $x \in \mathbb{R}^N \setminus \{0\}$. Since $(\overline{r}_n)$ is an arbitrary sequence decreasing to 0, we derive that $\lim_{r\to 0} U_{(r)}(x) = \theta$ and $\lim_{r\to 0} \nabla U_{(r)}(x) = 0$ for every $x \in \mathbb{R}^N \setminus \{0\}$. When $x = r\xi$ with $|\xi| = 1$, we conclude the assertion of (4.1).

To finish the proof of (a), it remains to show (4.2). Let $\varphi \in C^1_c(B_1)$ be fixed arbitrarily. In light of (4.1) and Assumption A, we see that all the integrals in (4.2) are well-defined. For every $\epsilon > 0$ small, we let w_ϵ be a non-decreasing and smooth function on $(0, \infty)$ such that

$$\begin{cases} 0 < w_\epsilon(r) < 1 & \text{for every } r \in (\epsilon, 2\epsilon), \\ w_\epsilon(r) = 1 & \text{for } r \ge 2\epsilon, \\ w_\epsilon(r) = 0 & \text{for } r \in (0, \epsilon]. \end{cases}$$

Using $\varphi w_\epsilon \in C^1_c(B_1^*)$ as a test function for (3.9), we find that

$$\int_{B_1} w_\epsilon \mathcal{A}(|x|) \nabla u \cdot \nabla \varphi \, dx + \int_{B_1} u^q \varphi w_\epsilon \, dx = -J_\epsilon, \tag{4.13}$$

where we denote

$$J_\epsilon := \int_{\epsilon < |x| < 2\epsilon} \mathcal{A}(|x|) \varphi(x) w_\epsilon'(|x|) \frac{x}{|x|} \cdot \nabla u \, dx.$$

We now claim that $\lim_{\epsilon\to 0} J_\epsilon = 0$. By (4.1), it is enough to show that as $\epsilon \to 0$

$$\int_{\epsilon<|x|<2\epsilon} \frac{\mathcal{A}(|x|)}{|x|} w_\epsilon'(|x|)\, dx \to 0, \text{ that is } \int_\epsilon^{2\epsilon} r^{N-2}\mathcal{A}(r) w_\epsilon'(r)\, dr \to 0 \quad (4.14)$$

as $\epsilon \to 0$. This is true because $r \longmapsto r^{N-2}\mathcal{A}(r)$ is regularly varying at 0 with positive index $\vartheta + N - 2$ so that $r^{N-2}\mathcal{A}(r) \sim \zeta(r)$ as $r \to 0$ for some non-decreasing function ζ with $\lim_{r\to 0} \zeta(r) = 0$ (see Propositions 9 and 10 in Appendix A). Then passing to the limit $\epsilon \to 0$ in (4.13), we conclude (4.2).

(b) The assertion follows from Lemma 4 and Proposition 2(b2).

Proposition 4 *Let $\eta \in C^1(\partial B_1)$ be a non-negative and non-trivial function. Then there exists a unique solution of the problem*

$$\begin{cases} -\operatorname{div}(\mathcal{A}(|x|)\nabla u) + |u|^{q-1}u = 0 & \text{in } \mathcal{D}'(B_1^*), \\ \lim\limits_{|x|\to 0} \dfrac{u(x)}{\Phi(x)} = 0, \quad u = \eta & \text{on } \partial B_1, \\ u > 0 \quad \text{in } B_1^*. \end{cases} \quad (4.15)$$

Moreover, $u \in C^{1,\alpha}_{\mathrm{loc}}(B_1^*) \cap H^1(B_1)$ for some $0 < \alpha < 1$.

Proof By Lemma 3, we have that any solution u of (4.15) is in $C^{1,\alpha}_{\mathrm{loc}}(B_1^*)$ for some $0 < \alpha < 1$. Moreover, we have $u \in H^1(B_1)$. Indeed, from Proposition 3, we infer that $u \in W^{1,p}_{\mathrm{loc}}(B_1)$ for every $1 < p < N$. Since $\eta \in C^1(\partial B_1)$ and $u \in C^1(B_1^*)$, by the classical trace theory, there exists a function $\tilde\eta \in H^1(B_1 \setminus \overline{B_{1/2}})$ such that $\tilde\eta = \eta$ on ∂B_1 and $\tilde\eta = u$ on $\partial B_{1/2}$. By the classical regularity theory, we conclude that $u \in H^1(B_1 \setminus \overline{B_{1/2}})$. This proves that $u \in H^1(B_1)$.

Existence. We show that (4.15) has at least a solution. Let $C_0 > 0$ be a large constant such that $C_0 \geq \max_{\partial B_1} \eta$. For any integer $n \geq 2$, let u_n be the unique solution of the problem

$$\begin{cases} -\operatorname{div}(\mathcal{A}(|x|)\nabla u) + |u|^{q-1}v = 0 \ \text{ in } B_1 \setminus \overline{B_{1/n}}, \\ u(x) = C_0 \ \text{ for } |x| = 1/n \ \text{ and } \ u = \eta \ \text{ on } \partial B_1, \\ u > 0 \ \text{ in } B_1 \setminus \overline{B_{1/n}}. \end{cases}$$

By the comparison principle, we obtain that

$$u_{n+1} \leq u_n \leq C_0 \ \text{ in } B_1 \setminus \overline{B_{1/n}}. \quad (4.16)$$

By Lemma 3, we conclude that, up to a subsequence, $u_n \to u_\infty$ in $C^1_{\mathrm{loc}}(B_1^*)$ and u_∞ is a non-negative solution of (1.2) such that $u_\infty = \eta$ on ∂B_1. Since $\eta \not\equiv 0$ on ∂B_1, by the strong maximum principle (see, for example, [18]), we find that $u_\infty > 0$ in B_1^*.

Uniqueness. Let u_1 and u_2 be two solutions of (4.15). Then we have $u_1 - u_2 \in H^1_0(B_1)$. For any small $\epsilon > 0$, we define w_ϵ as in the proof of Proposition 3.

We set $\Omega_\epsilon := B_1 \setminus \overline{B_\epsilon}$. Since $(u_1-u_2)w_\epsilon \in H_0^1(\Omega_\epsilon)$, it follows that there exists a sequence $(\varphi_n)_{n\geq 1}$ in $C_c^1(\Omega_\epsilon)$ such that

$$\varphi_n \to (u_1-u_2)w_\epsilon \quad \text{in } H^1(\Omega_\epsilon). \tag{4.17}$$

Using φ_n as a test function in (1.4) for the solutions u_1 and u_2 on (1.2), then subtracting these equations, we obtain that

$$\int_{\Omega_\epsilon} \mathcal{A}(|x|)\nabla(u_1-u_2)\cdot\nabla\varphi_n\,dx + \int_{\Omega_\epsilon}(u_1^q-u_2^q)\varphi_n\,dx = 0 \tag{4.18}$$

for every $n \geq 1$. Since $\mathcal{A} \in C^1[\epsilon,1]$ and (4.17) holds, by passing to the limit $n\to\infty$ in (4.18), we arrive at

$$\int_{\Omega_\epsilon} w_\epsilon(|x|)\mathcal{A}(|x|)|\nabla(u_1-u_2)|^2\,dx + \int_{\Omega_\epsilon}(u_1^q-u_2^q)(u_1-u_2)w_\epsilon\,dx = -I_\epsilon, \tag{4.19}$$

where by I_ϵ we denote

$$I_\epsilon := \int_{\epsilon<|x|<2\epsilon} \mathcal{A}(|x|)(u_1-u_2)w_\epsilon'(|x|)\frac{x}{|x|}\cdot\nabla(u_1-u_2)\,dx.$$

By Proposition 3 and (4.14), we see that $\lim_{\epsilon\to 0} I_\epsilon = 0$. Hence, letting $\epsilon \to 0$ in (4.19), we find that

$$\int_{B_1} \mathcal{A}(|x|)|\nabla(u_1-u_2)|^2\,dx + \int_{B_1}(u_1^q-u_2^q)(u_1-u_2)\,dx = 0. \tag{4.20}$$

Since both integrals in (4.20) are non-negative, we find that $u_1 \equiv u_2$ in B_1^*.

Proposition 5 *Let $\eta \in C^1(\partial B_1)$ be a non-negative function. If $\Phi \in L^q(B_1)$, then for any $\lambda \in (0,\infty]$, there exists a unique solution of the problem*

$$\begin{cases} -\operatorname{div}(\mathcal{A}(|x|)\nabla u) + |u|^{q-1}u = 0 \quad \textit{in } \mathcal{D}'(B_1^*), \\ \lim\limits_{|x|\to 0}\dfrac{u(x)}{\Phi(x)} = \lambda, \quad \textit{and} \quad u = \eta \quad \textit{on } \partial B_1, \\ u > 0 \quad \textit{in } B_1^*. \end{cases} \tag{4.21}$$

Conversely, if (1.2) *admits a positive solution such that $u(x)/\Phi(x)$ converges to a positive number as $|x|\to 0$, then $\Phi \in L^q(B_1)$.*

Proof We first prove the uniqueness of the solution of (4.21) when $\lambda \in (0,\infty]$.

Uniqueness. Let u_1 and u_2 be two solutions of (4.21). If $\lambda \in (0,\infty)$, then clearly $\lim_{|x|\to 0} u_1(x)/u_2(x) = 1$. This property also holds if $\lambda = \infty$ by Theorem 3(III). Hence, for any $\epsilon > 0$ fixed, there exists $r_\epsilon \in (0,1)$ such that

$$u_1(x) \leq (1+\epsilon)u_2(x) \quad \text{for } 0 < |x| \leq r_\epsilon.$$

By the comparison principle, we find that

$$u_1(x) \leq (1+\epsilon)u_2(x) \quad \text{for } r_\epsilon \leq |x| \leq 1.$$

Hence, $u_1 \leq (1+\epsilon)u_2$ in B_1^*. Letting $\epsilon \to 0$ and interchanging u_1 and u_2, we have $u_1 \equiv u_2$ in B_1^*.

Existence. We show that if $\lambda \in (0,\infty)$, then (4.21) has a solution. By Proposition 1(b1), there exists a unique positive solution $u_* \in C^2(0,1]$ of

$$\begin{cases} u_*''(r) + \left(N-1+\dfrac{r\mathcal{A}'(r)}{\mathcal{A}(r)}\right)\dfrac{u_*'(r)}{r} = \dfrac{u_*^q(r)}{\mathcal{A}(r)} & \text{for } 0<r<1, \\ \lim\limits_{r\to 0}\dfrac{u_*(r)}{\Phi(r)} = \lambda, \quad u_*(1)=1. \end{cases} \tag{4.22}$$

Let $C_0 \geq 1$ be chosen such that $C_0 \geq \max_{\partial B_1}\eta$. By the comparison principle, we find that

$$u_* \leq \lambda\Phi + C_0 \quad \text{for } 0<r\leq 1. \tag{4.23}$$

For any integer $n\geq 2$, we denote by v_n the unique solution of the problem

$$\begin{cases} -\operatorname{div}(\mathcal{A}(|x|)\nabla v) + |v|^{q-1}v = 0 \ \text{ in } B_1\setminus\overline{B_{1/n}}, \\ v(x) = \lambda\Phi(x)+C_0 \ \text{ for } |x|=1/n \ \text{ and } \ v=\eta \ \text{ on } \partial B_1, \\ v>0 \ \text{ in } B_1\setminus\overline{B_{1/n}}. \end{cases}$$

By the comparison principle and the method of sub-super-solutions, we have

$$\begin{cases} v_{n+1} \leq v_n \leq \lambda\Phi + C_0 & \text{in } B_1\setminus\overline{B_{1/n}}, \\ u_* \leq v_n + C_0 & \text{in } B_1\setminus\overline{B_{1/n}}. \end{cases} \tag{4.24}$$

Using again Lemma 3, we obtain that, up to a subsequence, $v_n \to v_\infty$ in $C^1_{\text{loc}}(B_1^*)$ and v_∞ is a positive solution of (1.2) for some $0<\alpha<1$ such that $v_\infty = \eta$ on ∂B_1. From (4.24), we have

$$u_*(|x|) - C_0 \leq v_\infty(x) \leq \lambda\Phi(x) + C_0 \ \text{ for } 0<|x|<1.$$

Using (4.22), we conclude that $\lim_{|x|\to 0} v_\infty(x)/\Phi(x) = \lambda$.

We prove that (4.21) has also a solution when $\lambda = \infty$. To this aim, for any integer $n\geq 1$, we denote by u_n the unique solution of (4.21) with $\lambda = n$. By the comparison principle, we infer that $u_n \leq u_{n+1}$ in B_1^*. Then, by using Lemmas 1 and 3, we obtain that, up to a subsequence, $u_n \to u_\infty$ in $C^1_{\text{loc}}(B_1^*)$, where u_∞ is a solution of (4.21) with $\lambda=\infty$. Moreover, u_∞ is in $C^{1,\alpha}_{\text{loc}}(B_1^*)$ for some $\alpha \in (0,1)$.

Finally, let u be a positive solution of (1.2) such that $u(x)/\Phi(x)$ converges to a positive number as $|x|\to 0$. We show that $\Phi \in L^q(B_1)$. By Lemma 4, there exists a positive radial solution u^* of (1.2) in $B_{1/2}^*$ such that $u \leq u^* \leq Ku$ in $B_{1/2}^*$ for some constant $K>1$. It follows that $\limsup_{r\to 0} u^*(r)/\Phi(r)$ is a positive number. We prove that $\lim_{r\to 0} u^*(r)/\Phi(r)$ exists in $(0,\infty)$, then by applying Proposition 1(b), we conclude that $\Phi \in L^q(B_1)$. Assume by contradiction that there exists a constant C such that

$$\liminf_{r\to 0}\frac{u^*(r)}{\Phi(r)} < C < \limsup_{r\to 0}\frac{u^*(r)}{\Phi(r)}. \tag{4.25}$$

Then for some sequence $(r_n)_{n\geq 1}$ decreasing to 0, we have $u^*(r_n) \leq C\Phi(r_n)$ for every $n \geq 1$. By the comparison principle, we obtain that $u^*(r) \leq C\Phi(r)$ for every $r \in (0, r_1)$. This is a contradiction with (4.25). This completes the proof of Proposition 5.

5 Proof of Theorem 3

Case (I) occurs when $\lim_{|x|\to 0} u(x)/\Phi(x) = 0$ (see Proposition 3). In Proposition 6, we prove that Case (II) is valid when $\limsup_{|x|\to 0} u(x)/\Phi(x) \in (0, \infty)$. Finally, in Proposition 7 we conclude that Case (III) occurs for any positive solution u of (1.2) with $\limsup_{|x|\to 0} u(x)/\Phi(x) = \infty$.

We first show that

$$\lim_{r\to 0} \frac{\Phi(r)}{\Upsilon(r)} = 0, \quad \text{where } \Upsilon \text{ is defined by (3.1).} \tag{5.1}$$

By Theorem 5(b) in Appendix A, we have

$$\lim_{r\to 0} \frac{r^N \Phi^q(r)}{\int_0^r t^{N-1}\Phi^q(t)\,dt} = N - q(N-2+\vartheta) \geq 0. \tag{5.2}$$

From (3.1) and (5.2), we infer that

$$\frac{\Phi^{q-1}(r)}{\Upsilon^{q-1}(r)} \sim [N - q(N-2+\vartheta)] \frac{\int_0^r \frac{t}{\mathcal{A}(t)}\,dt}{r^N \Phi(r)} \int_0^r t^{N-1}\Phi^q(t)\,dt \quad \text{as } r \to 0. \tag{5.3}$$

By using (3.7) and $\lim_{r\to 0} r\Phi'(r)/\Phi(r) = 2 - N - \vartheta$, we obtain that

$$\lim_{r\to 0} \frac{\int_0^r \frac{t}{\mathcal{A}(t)}\,dt}{r^N \Phi(r)} = \frac{1}{2-\vartheta} \lim_{r\to 0} \frac{r^{2-N}}{\mathcal{A}(r)\Phi(r)} = \frac{N\omega_N(\vartheta + N - 2)}{2-\vartheta}. \tag{5.4}$$

By combining (5.3) and (5.4), we arrive at

$$\frac{\Phi^{q-1}(r)}{\Upsilon^{q-1}(r)} \sim [N - q(N-2+\vartheta)] \frac{N\omega_N(\vartheta+N-2)}{2-\vartheta} \int_0^r t^{N-1}\Phi^q(t)\,dt \quad \text{as } r \to 0,$$

which proves (5.1) since $\int_0^r t^{N-1}\Phi^q(t)\,dt \to 0$ as $r \to 0$.

Proposition 6 *Let $q > 1$ and $\Phi \in L^q(B_1)$. If u is a positive solution of (1.2) such that*

$$\lambda := \limsup_{|x|\to 0} \frac{u(x)}{\Phi(x)} \in (0, \infty), \tag{5.5}$$

then we have

$$\begin{cases} \lim_{|x|\to 0} \frac{u(x)}{\Phi(x)} = \lambda, \quad \lim_{|x|\to 0} \frac{x \cdot \nabla u(x)}{\Phi(x)} = -(\vartheta - 2 + N)\lambda, \\ \lim_{|x|\to 0} \frac{|x|\,|\nabla u(x)|}{\Phi(x)} = (\vartheta - 2 + N)\lambda. \end{cases} \tag{5.6}$$

Moreover, $u \in C^{1,\alpha}_{\text{loc}}(B_1^*)$ for some $0 < \alpha < 1$ and u satisfies (1.8).

Proof Let $r_0 > 0$ be small such that $B_{2r_0}(0) \subset\subset B_1$. We define

$$v(x) := \frac{u(x)}{\Phi(x)} \quad \text{for } x \in B_1^* \text{ and } G(r) := \sup_{|x|=r} v(x) \quad \text{for } r \in (0, 2r_0).$$

Claim A. *We have* $\lim_{r\to 0} G(r) = \lambda$.

By using that Φ is the fundamental solution of (1.7), we obtain that v is a weak solution of

$$\Delta v + \left(2\frac{\Phi'(|x|)}{\Phi(x)} + \frac{\mathcal{A}'(|x|)}{\mathcal{A}(|x|)}\right)\frac{x}{|x|}\cdot\nabla v = \frac{v^q \Phi^{q-1}(|x|)}{\mathcal{A}(|x|)} \quad \text{in } B_1^*. \tag{5.7}$$

We rewrite (5.7) as

$$Lv := \Delta v + \sum_{i=1}^{N} c^i(x)\frac{\partial v}{\partial x_i} + d(x)v = 0 \quad \text{in } B_1^*,$$

where $c^i(x)$ for $i = 1, \ldots, N$ and $d(x)$ are given in B_1^* by

$$c^i(x) := \frac{x_i}{|x|}\left(2\frac{\Phi'(|x|)}{\Phi(x)} + \frac{\mathcal{A}'(|x|)}{\mathcal{A}(|x|)}\right) \quad \text{and} \quad d(x) := -v^{q-1}\frac{\Phi^{q-1}(|x|)}{\mathcal{A}(|x|)}.$$

By applying Theorem 8.19 in [11] to v and an argument similar to (4.4), we conclude the claim.

For each $r \in (0, r_0)$, we define $V_{(r)}$ as follows

$$V_{(r)}(\xi) := v(r\xi) \quad \text{for every } \xi \in \mathbb{R}^N \text{ with } 0 < |\xi| < \frac{r_0}{r}. \tag{5.8}$$

Claim B. *There exist positive constants C_1, C_2 and C_3 such that*

$$\begin{cases} |V_{(r)}(\xi)| \le C_1, \quad |\nabla V_{(r)}(\xi)| \le \dfrac{C_2}{|\xi|}, \\ |\nabla V_{(r)}(\xi) - \nabla V_{(r)}(\xi')| \le \dfrac{|\xi-\xi'|^\alpha}{|\xi|^{1+\alpha}}\left[C + C_3\left(\dfrac{|\xi-\xi'|}{|\xi|}\right)^{1-\alpha}\right] \end{cases} \tag{5.9}$$

for every ξ, ξ' in $\mathbb{R}^N$ with $0 < |\xi| \le |\xi'| < r_0/r$.

From (5.5), we infer that there exists a positive constant C_1 such that

$$0 < u(x) \le C_1\Phi(x) \quad \text{for every } 0 < |x| \le 2r_0. \tag{5.10}$$

Since Φ is regularly varying at zero with index $-(N-2+\vartheta)$ such that $0 \le N-2+\vartheta \le (2-\vartheta)/(q-1)$ and (5.1) holds, we can apply Lemma 3 with $g \equiv \Phi$. Hence, there exist positive constants C and $\alpha \in (0,1)$ such that

$$|\nabla u(x)| \le C\frac{\Phi(x)}{|x|} \quad \text{and} \quad |\nabla u(x) - \nabla u(x')| \le C\frac{\Phi(x)}{|x|^{1+\alpha}}|x-x'|^\alpha \tag{5.11}$$

for any x, x' in $\mathbb{R}^N$ satisfying $0 < |x| \le |x'| < r_0$. We see that $V_{(r)}(\xi)$ given by (5.8) satisfies

$$\nabla V_{(r)}(\xi) = r\left(\frac{(\nabla u)(r\xi)}{\Phi(r|\xi|)} - \frac{u(r\xi)}{\Phi^2(r|\xi|)}\Phi'(r|\xi|)\frac{\xi}{|\xi|}\right) \tag{5.12}$$

for every $0 < |\xi| < r_0/r$. The first two inequalities in (5.9) are immediate since

$$\lim_{r\to 0}\frac{r\Phi'(r)}{\Phi(r)} = 2 - N - \vartheta \quad \text{and } C_0 := \sup_{0<t<r_0}\frac{t|\Phi'(t)|}{\Phi(t)} \in (0,\infty). \tag{5.13}$$

We now show the last inequality in (5.9). In view of (5.12), we can write

$$\nabla V_{(r)}(\xi) - \nabla V_{(r)}(\xi') = r[T_1(r,\xi,\xi') - T_2(r,\xi,\xi')], \tag{5.14}$$

where we define T_1 and T_2 as follows

$$\begin{cases} T_1(r,\xi,\xi') := \dfrac{(\nabla u)(r\xi)}{\Phi(r|\xi|)} - \dfrac{(\nabla u)(r\xi')}{\Phi(r|\xi'|)}, \\ T_2(r,\xi,\xi') := V_{(r)}(\xi)\dfrac{\Phi'(r|\xi|)}{\Phi(r|\xi|)}\dfrac{\xi}{|\xi|} - V_{(r)}(\xi')\dfrac{\Phi'(r|\xi'|)}{\Phi(r|\xi'|)}\dfrac{\xi'}{|\xi'|}. \end{cases}$$

Using the triangle inequality, then (5.11) and the mean value theorem, we have

$$\begin{aligned} |T_1(r,\xi,\xi')| &\le \frac{|(\nabla u)(r\xi) - (\nabla u)(r\xi')|}{\Phi(r|\xi|)} + \frac{|(\nabla u)(r\xi')|}{\Phi(r|\xi|)\Phi(r|\xi'|)}|\Phi(r|\xi|) - \Phi(r|\xi'|)| \\ &\le C\frac{|\xi-\xi'|^\alpha}{r|\xi|^{1+\alpha}} + CC_0\frac{||\xi| - |\xi'||}{r|\xi'||\xi|} \le C\left(\frac{|\xi-\xi'|^\alpha}{r|\xi|^{1+\alpha}} + C_0\frac{|\xi-\xi'|}{r|\xi|^2}\right) \end{aligned}$$

for every $0 < |\xi| \le |\xi'| < r_0/r$, where C_0 is given by (5.13). We define

$$T_3(r,\xi,\xi') := \frac{\Phi'(r|\xi|)}{\Phi(r|\xi|)\,|\xi|}\,\xi - \frac{\Phi'(r|\xi'|)}{\Phi(r|\xi'|)\,|\xi'|}\,\xi'.$$

By the mean value theorem and the asymptotic properties of Φ near zero, we infer that there exists a positive constant C' such that

$$|T_3(r,\xi,\xi')| \le C'\frac{|\xi-\xi'|}{r|\xi|^2} \quad \text{for every } 0 < |\xi| \le |\xi'| < \frac{r_0}{r}.$$

By the triangle inequality and the first two inequalities in (5.9), we find that

$$\begin{aligned} |T_2(r,\xi,\xi')| &\le \frac{|\Phi'(r|\xi|)|}{\Phi(r|\xi|)}|V_{(r)}(\xi) - V_{(r)}(\xi')| + V_{(r)}(\xi')|T_3(r,\xi,\xi')| \\ &\le C_0C_2\frac{|\xi-\xi'|}{r|\xi|^2} + C_1|T_3(r,\xi,\xi')| \le (C_0C_2 + C_1C')\frac{|\xi-\xi'|}{r|\xi|^2} \end{aligned}$$

for every $0 < |\xi| \le |\xi'| < r_0/r$. By (5.14) and the above estimates on T_1 and T_2, we reach the last inequality in (5.9).

Claim C. *The assertion of* (5.6) *holds.*

The function $V_{(r)}$ given by (5.8) is a positive weak solution of the equation

$$\Delta V_{(r)}(\xi)+r\left(\frac{2\Phi'(r|\xi|)}{\Phi(r|\xi|)}+\frac{\mathcal{A}'(r|\xi|)}{\mathcal{A}(r|\xi|)}\right)\frac{\xi\cdot\nabla V_{(r)}(\xi)}{|\xi|}=r^2\frac{\Phi^{q-1}(r|\xi|)}{\mathcal{A}(r|\xi|)}V^q_{(r)}(\xi) \tag{5.15}$$

for $0<|\xi|<r_0/r$. Let $\xi\in\mathbb{R}^N\setminus\{0\}$ be fixed. Then $0<|\xi|<r_0/r$ for every $r>0$ small enough. Using (5.1) and the limit in (3.7), we deduce that

$$\lim_{r\to 0}(r|\xi|)^2\frac{\Phi^{q-1}(r|\xi|)}{\mathcal{A}(r|\xi|)}=\lim_{t\to 0}\frac{t^2\Phi^{q-1}(t)}{\mathcal{A}(t)}=(2-\vartheta)\lim_{t\to 0}\frac{\Phi^{q-1}(t)}{\Upsilon^{q-1}(t)}=0. \tag{5.16}$$

Claim B and the Arzela–Ascoli Theorem imply that any sequence $(\bar r_n)$ decreasing to zero as $n\to\infty$ contains a subsequence (r_n) such that

$$V_{(r_n)}\to V \quad\text{in } C^1_{\text{loc}}(\mathbb{R}^N\setminus\{0\}).$$

Moreover, V satisfies the equation

$$\Delta V(x)+(4-2N-\vartheta)\frac{x}{|x|^2}\cdot\nabla V(x)=0 \quad\text{in } \mathbb{R}^N\setminus\{0\}. \tag{5.17}$$

This follows from (5.15) by using (1.3), (5.13) and (5.16). From Claim A and Theorem 8.19 in [11] applied to $V-\gamma$ satisfying (5.17), we conclude that $V\equiv\gamma$ in $\mathbb{R}^N\setminus\{0\}$. The argument is the same as in Proposition 3 regarding $U\equiv\theta$ in $\mathbb{R}^N\setminus\{0\}$. Thus $\lim_{r\to 0}V_{(r)}(x)=\gamma$ and $\lim_{r\to 0}\nabla V_{(r)}(x)=0$ for every $x\in\mathbb{R}^N\setminus\{0\}$. By letting $x=r\xi$ with $|\xi|=1$ and using (5.12), we obtain (5.6).

Claim D. *We have $u\in C^{1,\alpha}_{\text{loc}}(B_1^*)$ for some $0<\alpha<1$ and u satisfies (1.8), that is*

$$\int_{B_1}\mathcal{A}(|x|)\nabla u\cdot\nabla\varphi\,dx+\int_{B_1}u^q\varphi\,dx=\lambda\varphi(0)\quad\text{for every } \varphi\in C^1_c(B_1). \tag{5.18}$$

Let $\varphi\in C^1_c(B_1)$ be arbitrarily fixed. By (1.6) and Theorem 5 in Appendix A, we find that

$$\lim_{r\to 0}r^{N-1}\Phi(r)\mathcal{A}(r)=\frac{1}{N\omega_N(\vartheta+N-2)}. \tag{5.19}$$

Hence, using (5.6), we obtain that the integrals in (5.18) are well-defined. Let $\epsilon>0$ be small. We define w_ϵ as in the proof of Proposition 3 and with a similar argument, we recover (4.13). Using (5.19), together with (5.6), we obtain that the right-hand side of (4.13), converges to $\lambda\varphi(0)$ as $\epsilon\to 0$. Hence, letting $\epsilon\to 0$ in (4.13), we conclude (5.18). This finishes the proof of Proposition 6.

Proposition 7 *Let $q>1$. Assume that $\Phi\in L^q(B_1)$. If u is a positive solution of (1.2) such that $\limsup_{|x|\to 0}u(x)/\Phi(x)=\infty$, then we have*

$$u(x)\sim\tilde u(|x|)\ \text{ as } |x|\to 0, \tag{5.20}$$

where $\tilde u$ is given by (1.9).

Proof Let u be a positive solution of (1.2) with $\limsup_{|x|\to 0} u(x)/\Phi(x) = \infty$. Then Corollary 4 gives that $\lim_{|x|\to 0} u(x)/\Phi(x) = \infty$. By Lemma 4, there exist two radial solutions u_* and u^* of (1.2) in $B^*_{1/2}$ such that (3.26) holds. If $q = N/(N-2+\vartheta)$, then by Corollary 2, we know that $u_*(r) \sim u^*(r) \sim \tilde{u}(r)$ as $r \to 0$ so that (5.20) holds.

To conclude the proof, it remains to show (5.20) when $1 < q < N/(N-2+\vartheta)$. We distinguish two cases:

Case 1: If $q(N-2+\vartheta) \neq 4-N-2\vartheta$, then by Corollary 1, we obtain (5.20).

Case 2: If $q(N-2+\vartheta) = 4-N-2\vartheta$, then we cannot use Corollary 1. We thus introduce in the spirit of [8] a suitable pair of sub-super-solutions with known asymptotic behaviour at 0. We divide the proof of (5.20) into three steps.

Step 1: *We have* $\lim_{|x|\to 0} u(x)/f(|x|) = \infty$ *for every* $f \in RV_p(0+)$ *with* $0 > p > (\vartheta-2)/(q-1)$.

Let $f \in RV_p(0+)$ with $0 > p > (\vartheta-2)/(q-1)$. We fix $q_1 \in \mathbb{R}$ such that

$$q < q_1 < \min\left\{\frac{2-\vartheta-p}{-p}, \frac{N}{N-2+\vartheta}\right\}. \tag{5.21}$$

Since $\lim_{|x|\to 0} u(x) = \infty$ and u is a solution of (1.2), we obtain that

$$-\Delta u - \frac{\mathcal{A}'(|x|)}{\mathcal{A}(|x|)}\frac{x}{|x|}\cdot\nabla u(x) + \frac{1}{\mathcal{A}(|x|)}u^{q_1} \geq 0 \quad \text{for } 0 < |x| < 1/2. \tag{5.22}$$

Similar to u_* in Lemma 4, we construct a positive solution v_∞ of

$$-v''(r) - \left(N-1+\frac{r\mathcal{A}'(r)}{\mathcal{A}(r)}\right)\frac{v'(r)}{r} + \frac{1}{\mathcal{A}(r)}v^{q_1} = 0 \quad \text{for } 0 < r < 1/2$$

such that $\lim_{r\to 0} v_\infty(r)/\Phi(r) = \infty$ and $v_\infty(|x|) \leq u(x)$ for every $0 < |x| \leq 1/2$. In view of (5.21) we can use Corollary 1 with $q = q_1$ to obtain that

$$v_\infty(r) \sim \left[\frac{(q_1-1)^2}{N-(N-2+\vartheta)q_1}\int_0^r \frac{t}{\mathcal{A}(t)}\,dt\right]^{-\frac{1}{q_1-1}} \quad \text{as } r \to 0.$$

Hence, v_∞ behaves near zero as a regularly varying function at zero with index $(\vartheta-2)/(q_1-1)$. This index is less than p by virtue of (5.21). Thus by Proposition 9 in Appendix A, we obtain that $\lim_{r\to 0} v_\infty(r)/f(r) = \infty$. Since $v_\infty(|x|) \leq u(x)$ for every $0 < |x| \leq 1/2$, we conclude Step 1.

Step 2: *Construction of sub- and super-solutions.*

Let $\lambda \in (0,1)$ be any suitably small number. Fix $\epsilon \in (0, 2-\vartheta)$ small. We define

$$\ell_\pm(\epsilon,\lambda) := \left\{\frac{1\pm\lambda}{q-1}\left[\left(\frac{q\pm\lambda}{q-1}\right)(2-\vartheta\pm\epsilon) - N\right]\right\}^{\frac{1}{q-1}}. \tag{5.23}$$

Recall that Υ is defined by (3.1), namely

$$\Upsilon(r) := \left(\int_0^r \frac{t}{\mathcal{A}(t)}\, dt\right)^{-\frac{1}{q-1}} \quad \text{for every } r \in (0,1). \tag{5.24}$$

Claim: *For any small $\epsilon > 0$, there exists $r_\epsilon > 0$ such that for every $\lambda \in [0, \lambda_0]$, the function $\ell_+(\epsilon,\lambda)\Upsilon^{1+\lambda}(r)$ (respectively, $\ell_-(\epsilon,\lambda)\Upsilon^{1-\lambda}$) is a radial super-solution (respectively, sub-solution) of (1.2) in $B^*_{r_\epsilon}$.*

Fix $\epsilon > 0$ sufficiently small. We need to show that there exists $r_\epsilon > 0$ such that for every $\lambda \in [0, \lambda_0]$, the function $v = \ell_+(\epsilon,\lambda)\Upsilon^{1+\lambda}(r)$ satisfies

$$-\left[v''(r) + \left(N - 1 + \frac{r\mathcal{A}'(r)}{\mathcal{A}(r)}\right)\frac{v'(r)}{r}\right] + \frac{[v(r)]^q}{\mathcal{A}(r)} \geq 0 \quad \text{for } 0 < r < r_\epsilon, \tag{5.25}$$

respectively, $v = \ell_-(\epsilon,\lambda)\Upsilon^{1-\lambda}(r)$ satisfies the reverse inequality in (5.25). By (3.7), we have that for every $\epsilon > 0$, there exists $r_\epsilon > 0$ such that

$$|\mathcal{B}(r) - 2 + \vartheta| < \epsilon \text{ for all } r \in (0, r_\epsilon), \text{ where } \mathcal{B}(r) := \frac{r^2}{\mathcal{A}(r)\int_0^r \frac{t}{\mathcal{A}(t)}\, dt}. \tag{5.26}$$

Since $\lim_{r\to 0}\Upsilon(r) = \infty$, we can diminish $r_\epsilon > 0$ so that $\Upsilon(r) > 1$ for every $r \in (0, r_\epsilon)$. Using a simple calculation, we find that $v_\pm(r) = \ell_\pm(\epsilon,\lambda)\Upsilon^{1\pm\lambda}(r)$ satisfies

$$\begin{cases} \dfrac{v_\pm'(r)}{r} = \ell_\pm(\epsilon,\lambda)\left(\dfrac{1\pm\lambda}{1-q}\right)\dfrac{[\Upsilon(r)]^{q\pm\lambda}}{\mathcal{A}(r)}, \\ v_\pm''(r) = \dfrac{v_\pm'(r)}{r}\left[1 - \dfrac{r\mathcal{A}'(r)}{\mathcal{A}(r)} - \left(\dfrac{q\pm\lambda}{q-1}\right)\mathcal{B}(r)\right]. \end{cases}$$

Thus the left-hand side of (5.25), in short LHS of (5.25), with $v = \ell_\pm(\epsilon,\lambda)\Upsilon^{1\pm\lambda}(r)$ is given by

$$\text{LHS of (5.25)} = \ell_\pm(\epsilon,\lambda)[\Upsilon(r)]^{q\pm\lambda}\frac{T^\pm_{\epsilon,\lambda}(r)}{\mathcal{A}(r)}, \tag{5.27}$$

where we define

$$T^\pm_{\epsilon,\lambda}(r) := \left(\frac{1\pm\lambda}{q-1}\right)\left[N - \left(\frac{q\pm\lambda}{q-1}\right)\mathcal{B}(r)\right] + \ell_\pm^{q-1}(\epsilon,\lambda)\,[\Upsilon(r)]^{\pm\lambda(q-1)}. \tag{5.28}$$

From (5.27), we see that the claim is proved if $\pm T^\pm_{\epsilon,\lambda}(r) \geq 0$ for every $r \in (0, r_\epsilon)$ and for each $\lambda \in [0, \lambda_0]$. Using (5.28), jointly with (5.26), we obtain that

$$\pm T^\pm_{\epsilon,\lambda}(r) \geq \left(\frac{1\pm\lambda}{q-1}\right)\left[N - \left(\frac{q\pm\lambda}{q-1}\right)(2 - \vartheta \pm \epsilon)\right] + \ell_\pm^{q-1}(\epsilon,\lambda). \tag{5.29}$$

Our definition of $\ell_\pm(\epsilon,\lambda)$ in (5.23) gives that the right-hand side of the inequality in (5.29) is equal to zero. Hence, we conclude the claim.

Step 3: *Proof of* (5.20) *completed.*

Fix $\epsilon \in (0, 2-\vartheta)$ sufficiently small and $\lambda \in (0, \lambda_0]$. Let $r_\epsilon > 0$ and λ_0 be as in Step 2. Observe that $\Upsilon^{1\pm\lambda}$ is regularly varying at 0 with index $\frac{(1\pm\lambda)(\vartheta-2)}{q-1}$. Thus Corollary 3 and Step 1 give that

$$\lim_{|x|\to 0} \frac{u(x)}{\Upsilon^{1+\lambda}(|x|)} = 0, \qquad \lim_{|x|\to 0} \frac{u(x)}{\Upsilon^{1-\lambda}(|x|)} = \infty. \tag{5.30}$$

Set $M_\epsilon := \max_{|x|=r_\epsilon} u(x)$ and $\ell = \lim_{\epsilon\to 0}(\lim_{\lambda\to 0} \ell_\pm(\epsilon,\lambda))$. Then from (5.23), we obtain that $\ell_-(\epsilon,\lambda) < \ell < \ell_+(\epsilon,\lambda)$ for $\lambda \in (0,\lambda_0]$. Clearly, $u(x) + \ell\Upsilon(r_\epsilon)$ is a super-solution of (1.2) in $B^*_{r_\epsilon}$, while Step 2 gives that $\ell_+(\epsilon,\lambda)\Upsilon^{1+\lambda} + M_\epsilon$ is a super-solution of (1.2) in $B^*_{r_\epsilon}$. From (5.30) and the comparison principle, we find that for every $\lambda \in (0,\lambda_0]$

$$\ell_+(\epsilon,\lambda)\Upsilon^{1+\lambda}(|x|) + M_\epsilon \geq u(x), \qquad u(x) + \ell\Upsilon(r_\epsilon) \geq \ell_-(\epsilon,\lambda)\Upsilon^{1-\lambda}(|x|)$$

for every $0 < |x| \leq r_\epsilon$. The above inequalities also hold when λ is replaced by 0 (by taking $\lambda \to 0$). Hence, we have

$$\limsup_{|x|\to 0} \frac{u(x)}{\Upsilon(|x|)} \leq \lim_{\lambda\to 0} \ell_+(\epsilon,\lambda), \qquad \liminf_{|x|\to 0} \frac{u(x)}{\Upsilon(|x|)} \geq \lim_{\lambda\to 0} \ell_-(\epsilon,\lambda). \tag{5.31}$$

By letting ϵ go to zero in (5.31), we obtain that

$$\lim_{|x|\to 0} \frac{u(x)}{\Upsilon(|x|)} = \left[\frac{N-(N-2+\vartheta)q}{(q-1)^2}\right]^{\frac{1}{q-1}}. \tag{5.32}$$

From (5.24) and (5.32), we conclude the proof of (5.20).

A Regular variation theory

We recall the notion of regular variation at zero and some properties of regularly varying functions. For more details, we refer to [1] and [21].

Definition 3 A positive measurable function L defined on an interval $(0, b]$ for some $b > 0$ is called *slowly varying at (the right of) zero* if

$$\lim_{t\to 0} \frac{L(\lambda t)}{L(t)} = 1 \quad \text{for every } \lambda > 0.$$

A function f is called *regularly varying at 0 with real index* ρ, or $f \in RV_\rho(0+)$ in short, if $f(t) = t^\rho L(t)$ for some function L which is slowly varying at 0.

Thus for almost all purposes, it is enough to study the properties of slowly varying functions.

Proposition 8 (Uniform Convergence Theorem) *If L is a slowly varying function at zero, then $L(\xi t)/L(t) \to 1$ as $t \to 0$, uniformly on each compact ξ-set in $(0,\infty)$.*

It may be easily proved that a slowly varying function is locally bounded.

Theorem 4 (Representation Theorem) *The function L is slowly varying at 0 if and only if it can be written in the form*

$$L(t) = \eta(t) \exp\left(\int_b^t \frac{\varepsilon(r)}{r} dr \right), \qquad 0 < t \leq b$$

for some $b > 0$, where η is a measurable function on $(0, b]$ satisfying $\lim_{t \to 0} \eta(t) = \eta \in (0, +\infty)$ and ε is a continuous function such that $\lim_{t \to 0} \varepsilon(t) = 0$.

If $\eta(t)$ is replaced by a constant η, then the new function is referred to as a *normalized slowly varying function*. In this case, $\varepsilon(t) = tL'(t)/L(t)$ for $0 < t \leq b$. Conversely, any function $\tilde{L} \in C^1(0, b]$ which is positive and satisfies $\lim_{t \to 0} t\tilde{L}'(t)/\tilde{L}(t) = 0$ is a normalized slowly varying function.

Remark 5 Any slowly varying function at zero is asymptotically equivalent to a normalized slowly varying one.

Proposition 9 (Elementary properties of slowly varying functions) *Let L be a slowly varying function at zero. The following properties hold:*

(a) $t^m L(t) \to 0$ and $t^{-m} L(t) \to \infty$ as $t \to 0$ for every $m > 0$;
(b) $L(t)^m$ is slowly varying for every $m \in \mathbb{R}$;
(c) If L_1 is also slowly varying, then so are $L(t)L_1(t)$ and $L(t) + L_1(t)$.

Proposition 10 (Monotone equivalents) *A positive, measurable function L is slowly varying at zero if and only if, for every $m > 0$, there exist a non-increasing function f and a non-decreasing function g such that*

$$t^{-m} L(t) \sim f(t), \qquad t^m L(t) \sim g(t) \quad (as\, t \to 0).$$

Theorem 5 (Karamata's Theorem; direct half) *Let f vary regularly at zero with index ρ and be locally bounded on $(0, b]$. Then*

(a) for any $\sigma \geq \rho - 1$, we have

$$\lim_{t \to 0} \frac{t^{-\sigma-1} f(t)}{\int_t^b r^{-\sigma-2} f(r)\, dr} = \sigma - \rho + 1;$$

(b) for any $\sigma < \rho - 1$ (and for $\sigma = \rho - 1$ if $\int_0 r^{-\rho-1} f(r)\, dr < +\infty$), we have

$$\lim_{t \to 0} \frac{t^{-\sigma-1} f(t)}{\int_0^t r^{-\sigma-2} f(r)\, dr} = -\sigma + \rho - 1.$$

Karamata's Theorem tells us how regularly varying functions behave when multiplied by powers and integrated. It is a remarkable fact that such a behaviour can only arise in the case of regular variation, as we can deduce from the following result.

Theorem 6 (Karamata's theorem: converse half) *Let f be a positive and locally integrable function on $(0, b]$.*

(a) If for some $\sigma > \rho - 1$, we have

$$\lim_{t \to 0} \frac{t^{-\sigma-1} f(t)}{\int_t^b r^{-\sigma-2} f(r)\, dr} = \sigma - \rho + 1,$$

then f varies regularly at zero with index ρ;

(b) If for some $\sigma < \rho - 1$, we have

$$\lim_{t \to 0} \frac{t^{-\sigma-1} f(t)}{\int_0^t r^{-\sigma-2} f(r)\, dr} = -\sigma + \rho - 1,$$

then again f varies regularly at zero with index ρ.

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References

1. N.H. Bingham, C.M. Goldie, J.L. Teugels, *Regular Variation*, Vol. 27 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1987.
2. H. Brezis, E.H. Lieb, *Long range atomic potentials in Thomas-Fermi theory*, Comm. Math. Phys. 65 (1979), no. 3, 231–246.
3. H. Brezis, L. Oswald, *Singular solutions for some semilinear elliptic equations*, Arch. Rational Mech. Anal. 99 (1987) 249–259.
4. H. Brezis, L. Véron, *Removable singularities for some nonlinear elliptic equations*, Arch. Rational Mech. Anal. 75 (1980/81) 1–6.
5. N. Chaudhuri, F.C. Cîrstea, *On trichotomy of positive singular solutions associated with the Hardy–Sobolev operator*, C. R. Acad. Sci. Paris, Ser. I **347** (2009), 153–158.
6. F.C. Cîrstea, *A complete classification of the isolated singularities for nonlinear elliptic equations with inverse square potentials*, Memoirs AMS (in press).
7. F.C. Cîrstea, Y. Du, *Asymptotic behavior of solutions of semilinear elliptic equations near an isolated singularity*, J. Funct. Anal. 250 (2007) 317–346.
8. F.C. Cîrstea, Y. Du, *Isolated singularities for weighted quasilinear elliptic equations* J. Funct. Anal. 259 (2010), 174–202.
9. P.L. Felmer, A. Quaas, *Fundamental solutions and two properties of elliptic maximal and minimal operators*, Trans. Amer. Math. Soc. 361 (2009), 5721–5736.
10. A. Friedman, L. Véron, *Singular solutions of some quasilinear elliptic equations*, Arch. Rational Mech. Anal. 96 (1986) 359–387.
11. D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*. Reprint of the 1998 edition. Classics in Mathematics, Springer-Verlag, Berlin, 2001.
12. B. Guerch, L. Véron, *Local properties of stationary solutions of some nonlinear singular Schrödinger equations*, Rev. Mat. Iberoamericana 7 (1991), no. 1, 65–114.
13. V. Julin, *Solutions of Aronsson equation near isolated points*, Calc. Var. Partial Differential Equations 37 (2010), no. 3-4, 303–328.
14. J. Karamata, *Sur un mode de croissance régulière. Théorèmes fondamentaux'*, Bull. Soc. Math. France 61 (1933) 55–62.
15. D.A. Labutin, *Removable singularities for fully nonlinear elliptic equations*, Arch. Ration. Mech. Anal. **155** (2000), 201–214.
16. V. Liskevich, I.I. Skrypnik, *Isolated singularities of solutions to quasi-linear elliptic equations with absorption*, J. Math. Anal. Appl. 338 (2008), 536–544.
17. F.I. Mamedov, A. Harman, *On the removability of isolated singular points for degenerating nonlinear elliptic equations*, Nonlinear Anal. 71 (2009), 6290–6298.
18. P. Pucci, J. Serrin, *The Maximum Principle*, no. 73 in Progress in Nonlinear Differential Equations and their Applications, Birkhäuser Verlag, Basel, 2007.
19. S.I. Resnick, *Extreme Values, Regular Variation, and Point Processes*, Springer-Verlag, New York, Berlin, 1987.
20. O. Savin, C. Wang, Y. Yu, *Asymptotic behavior of infinity harmonic functions near an isolated singularity*, Int. Math. Res. Not. IMRN, **2008** (2008). Article ID rnm 163, 23 pp.
21. E. Seneta, *Regularly Varying Functions*, Vol. 508 of Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, 1976.
22. S.D. Taliaferro, *Asymptotic behavior of solutions of* $y'' = \varphi(t)y^\lambda$, J. Math. Anal. Appl. (1978), 95–134.
23. N.S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. 20 (1967) 721–747.
24. J.L. Vázquez, L. Véron, *Removable singularities of some strongly nonlinear elliptic equations*, Manuscripta Math. 33 (1980/81) 129–144.

25. J.L. Vázquez, L. Véron, *Isolated singularities of some semilinear elliptic equations*, J. Differential Equations **60** (1985), 301–321.
26. L. Véron, *Singular solutions of some nonlinear elliptic equations*, Nonlinear Anal. 5 (1981) 225–242.
27. L. Véron, *Weak and strong singularities of nonlinear elliptic equations*, in: Nonlinear functional analysis and its applications, Part 2 (Berkeley, Calif., 1983), Vol. 45, Part 2 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 1986, pp. 477–495.
28. L. Véron, *Singularities of solutions of second order quasilinear equations*, Longman Scientific & Technical, Pitman Research Notes in Mathematics Series, 353, Harlow UK, 1996.