W-graph ideals and biideals

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Abstract We further develop the theory of W-graph ideals, first introduced in [6]. We discuss W-graph subideals, and induction and restriction of W-graph ideals for parabolic subgroups. We introduce W-graph biideals: those W-graph ideals that yield $(W \times W^{\circ})$-graphs, where W° is the group opposite to W. We determine all W-graph ideals and biideals in finite Coxeter groups of rank 2.

Keywords Coxeter groups · Hecke algebras · W-graphs · Kazhdan–Lusztig polynomials · cells

1 Introduction

Let (W,S) be a Coxeter system and $\mathcal{H}(W)$ its Hecke algebra over $\mathbb{Z}[q,q^{-1}]$, the ring of Laurent polynomials in the indeterminate q. The Coxeter system (W,S) is naturally equipped with the left weak order and the Bruhat order, denoted by $\leqslant_{\mathsf{L}}$ and $\leqslant$, respectively. In [6], an algorithm was given to produce from an ideal (down set) $\mathscr{I}$ of $(W,\leqslant_{\mathsf{L}})$ and a subset J of $S \setminus \mathscr{I}$ a weighted digraph $\Gamma(\mathscr{I},J)$ with vertices indexed by the elements of $\mathscr{I}$ and coloured with subsets of S. If, in the terminology of [6], $\mathscr{I}$ is a W-graph ideal with respect to J, then $\Gamma(\mathscr{I},J)$ is a W-graph. In the present paper we use the terminology "$(\mathscr{I},J)$ is a W-graph ideal" to mean the same thing as "$\mathscr{I}$ is a W-graph ideal with respect to J".

The algorithm referred to above proceeds chiefly by recursively computing polynomials $q_{y,w}$ for all $y,w \in \mathscr{I}$ such that $y < w$. These polynomials are anologous to Kazhdan–Lusztig polynomials, and the Kazhdan–Lusztig W-graph ([8]) and Deodhar's parabolic analogues ([2]) are obtained as special cases. Moreover, it was shown in [10] that W-graphs for the Kazhdan–Lusztig left cells that contain longest elements of standard parabolic subgroups can be constructed this way. In type A, this provides a practical procedure for calculating a W-graph for a cell module (which is known to be isomorphic to the corresponding Specht module) from standard tableaux of a given shape.

In general, it is still unknown which subsets of W generate W-graph ideals, and the problem of describing them combinatorially is still open, even in type A. Preliminary results

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concerning these matters in type A are established in [11], using the results of the present paper combined with those of [6, 10].

In this paper, we define a W-graph subideal of a W-graph ideal $(\mathscr{I}, J)$ to be a W-graph ideal $(\mathscr{L}, K)$ such that $\mathscr{L} \subseteq \mathscr{I}$ and $K = J$. It was shown in [10] that if $(\mathscr{I}, J)$ is a W-graph ideal and $\mathscr{L} \subseteq \mathscr{I}$ then $(\mathscr{L}, J)$ is a W-graph subideal of $(\mathscr{I}, J)$ if the complement $\mathscr{I} \setminus \mathscr{L}$ is closed when regarded as a subset of the vertex set of $\Gamma = \Gamma(\mathscr{I}, J)$ (in the sense that it is an ideal with respect to the Kazhdan–Lusztig preorder $\leqslant_\Gamma$ on the vertex set). We call W-graph subideals of this form *strong* W-graph subideals. We show that this strong W-graph subideal relation is preserved by induction of W-graph ideals, as defined in [6, Section 9]. More precisely, if W_K is a standard parabolic subgroup of W (where $K \subseteq S$), and D_K denotes the set of minimal length representatives of left cosets of W_K in W, then $(D_K\mathscr{L}, J)$ is a strong W-graph subideal of $(D_K\mathscr{I}, J)$ if $(\mathscr{L}, J)$ is a strong W_K-graph subideal of $(\mathscr{I}, J)$.

Recall that the original construction given by Kazhdan and Lusztig in [8] produces a $(W \times W^{\circ})$-graph, where W° is the Coxeter group opposite to W. Thus it is natural to seek a generalization the results of [6] that produces $(W \times W^{\circ})$-graphs. This is the motivation for the W-graph biideal concept.

As mentioned earlier, for an arbitrary Coxeter system (W, S), the algorithm in [6] takes as input an ideal $\mathscr{I}$ of $(W, \leqslant_{\mathsf{L}})$ and a subset J of $S \setminus \mathscr{I}$, and produces a (decorated) graph $\Gamma(\mathscr{I}, J)$ as output. If $\mathscr{I}$ is a W-graph ideal with respect to J, then $\Gamma(\mathscr{I}, J)$ is W-graph. It is natural to ask whether this condition characterizes W-graph ideals. The answer to this question is affirmative: W-graph ideals are precisely the ideals for which the above construction produces W-graphs. This is useful in practice as a computational means of determining whether or not a given ideal is a W-graph ideal.

In [6, Section 9] it was shown that if $J \subseteq K \subseteq S$ and $(\mathscr{I}_0, J)$ is a W_K-graph ideal then $(D_K\mathscr{I}_0, J)$ is a W-graph ideal. This construction corresponds to inducing modules. In the present paper we prove a dual result relating to restriction of modules: if $(\mathscr{I}, J)$ is a W-graph ideal and $K \subseteq S$ then for each right coset $W_K d \subseteq W$ the intersection $\mathscr{I} \cap W_K d$ is a translate of a W_K-graph ideal. Indeed, for each $d \in D_K^{-1}$, the set of minimal right coset representatives for W_K, the set $\mathscr{I}_d = W_K \cap \mathscr{I} d^{-1}$ is a W_K-graph ideal with respect to $K \cap dJd^{-1}$. Thus

$$\mathscr{I} = \bigsqcup_{d \in D_K^{-1} \cap \mathscr{I}} \mathscr{I}_d d,$$

where $(\mathscr{I}_d, K \cap dJd^{-1})$ is a W_K-graph ideal in each case.

Finally, as an example, we provide a complete list of W-graph ideals and biideals for Coxeter groups of type $I_2(m)$, where $m \geqslant 2$.

The present paper is organized as follows. In Section 2, we provide basic definitions and facts concerning Coxeter groups and Hecke algebras. In Section 3 we review the definition of a W-graph and related concepts, and in Section 4 we recall the notion of a W-graph ideal and the procedure for constructing a W-graph from a W-graph ideal. In Section 5 we define W-graph subideals and show that parabolic induction preserves the strong W-graph subideal relation, as described above. In Section 6 we define W-graph biideals and show that they do indeed produce $(W \times W^{\circ})$-graphs. Section 7 deals mainly with the computational characterization of W-graph ideals. In Section 8 we prove the decomposition formula mentioned above: if $\mathscr{I}$ is a W-graph ideal then the intersection of $\mathscr{I}$ with any right coset of any standard parabolic subgroup W_K is a translate of a W_K-graph ideal. The paper ends with Section 9, in which W-graph ideals and biideals are investigated for Coxeter groups of rank 2.

2 Coxeter groups and Hecke algebras

Let (W,S) be a Coxeter system and l the length function on W determined by S. The Bruhat order, denoted by $\leqslant$, is the partial order on W such that 1 (the identity element) is the unique minimal element and the following property holds.

Lemma 2.1 [1, Theorem 1.1] *Let $s \in S$ and $u, w \in W$ satisfy $u \leqslant su$ and $w \leqslant sw$. Then $u \leqslant w$ if and only if $u \leqslant sw$, and $u \leqslant sw$ if and only if $su \leqslant sw$.*

The following result follows easily from Lemma 2.1

Lemma 2.2 *Let $u, v, w \in W$ with $l(uv) = l(u) + l(v)$ and $l(uw) = l(u) + l(w)$. Then $uv \leqslant uw$ if and only if $v \leqslant w$.*

As well as the Bruhat order, we shall make extensive use of the left weak order, defined by the condition that if $v, w \in W$ then $v \leqslant_{\mathsf{L}} w$ if and only if $l(w) = l(wv^{-1}) + l(v)$. The right weak order is defined similarly, and satisfies $v \leqslant_{\mathsf{R}} w$ if and only if $v^{-1} \leqslant_{\mathsf{L}} w^{-1}$.

For each $J \subseteq S$ let W_J be the (standard parabolic) subgroup of W generated by J, and let D_J the set of distinguished (or minimal) representatives of the left cosets of W_J in W. Thus each $w \in W$ has a unique factorization $w = du$ with $d \in D_J$ and $u \in W_J$, and $l(du) = l(d) + l(u)$ holds for all $d \in D_J$ and $u \in W_J$. It is easily seen that D_J is an ideal of $(W, \leqslant_{\mathsf{L}})$: if $w \in D_J$ and $v \in W$ with $v \leqslant_{\mathsf{L}} w$ then $v \in D_J$.

If $L \subseteq J \subseteq S$ then we define $D_L^J = W_J \cap D_L$, the set of minimal representatives of the left cosets of W_L in W_J.

If W_J is finite then we denote the longest element of W_J by w_J. If W is finite then $D_J = \{\, w \in W \mid w \leqslant_{\mathsf{L}} d_J \,\}$ ([5, Lemma 2.2.1]), where d_J is the unique element in $D_J \cap w_S W_J$.

The map $W \to D_J$ given by $w = du \mapsto d$ preserves the Bruhat order, as the following proposition shows.

Proposition 2.3 [1, Lemma 3.5] *Let $w_1 = d_1u_1$ and $w_2 = d_2u_2$, where $d_1, d_2 \in D_J$ and $w_1, w_2 \in W_J$. If $w_1 \leqslant w_2$ then $d_1 \leqslant d_2$.*

The following result will be used frequently later.

Lemma 2.4 [2, Lemma 2.1 (iii)] *Let $J \subseteq S$. For each $s \in S$ and each $w \in D_J$, exactly one of the following occurs:*
(i) $l(sw) < l(w)$ *and* $sw \in D_J$;
(ii) $l(sw) > l(w)$ *and* $sw \in D_J$;
(iii) $l(sw) > l(w)$ *and* $sw \notin D_J$, *and* $w^{-1}sw \in J$.

Let $K \subseteq S$. Applying the anti-automorphism of W given by $w \mapsto w^{-1}$ shows that D_K^{-1} is the set of minimal representatives of the right cosets of W_K in W. It is well known that each double coset $W_K w W_J$ contains a unique element $d \in D_{K,J} = D_K^{-1} \cap D_J$, and that $W_K \cap dW_Jd^{-1} = W_{K\cap dJd^{-1}}$ whenever $d \in D_{K,J}$. It follows that each element of $W_K dW_J$ has a factorization vdu with $v \in D^K_{K\cap dJd^{-1}}$ and $u \in W_J$, and satisfying $l(vdu) = l(v) + l(d) + l(u)$. Applying this to elements of D_J gives the following result.

Lemma 2.5 *Let $J, K \subseteq S$. Then* $D_J = \displaystyle\bigsqcup_{d\in D_{K,J}} D^K_{K\cap dJd^{-1}}d.$

Remark 2.6 Each element w of D_J has a unique factorization vd with $d \in D_{K,J}$ and $v \in D_L^K$, where $L = K \cap dJd^{-1}$, satisfying $l(w) = l(v) + l(d)$.

As in [6], if $X \subseteq W$ we define $\operatorname{Pos}(X) = \{\, s \in S \mid l(xs) > l(x) \text{ for all } x \in X \,\}$. Thus $\operatorname{Pos}(X)$ is the largest subset J of S such that $X \subseteq D_J$.

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$, the ring of Laurent polynomials with integer coefficients in the indeterminate q, and let $\mathcal{A}^+ = \mathbb{Z}[q]$. The Hecke algebra corresponding to the Coxeter system (W,S) is the associative $\mathcal{A}$-algebra $\mathcal{H} = \mathcal{H}(W)$ generated by elements $\{T_s \mid s \in S\}$, subject to the defining relations

$$T_s^2 = 1 + (q - q^{-1})T_s \quad \text{for all } s \in S,$$
$$T_s T_t T_s \cdots = T_t T_s T_t \cdots \quad \text{for all } s,t \in S,$$

where in the second of these there are $m(s,t)$ factors on each side, $m(s,t)$ being the order of st in W.

It is well known that $\mathcal{H}$ is $\mathcal{A}$-free with an $\mathcal{A}$-basis $\{\, T_w \mid w \in W \,\}$ and multiplication satisfying

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ T_{sw} + (q - q^{-1})T_w & \text{if } l(sw) < l(w). \end{cases}$$

for all $s \in S$ and $w \in W$.

Let $a \mapsto \overline{a}$ be the involutory automorphism of $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ defined by $\overline{q} = q^{-1}$. This extends to an involutory automorphism of $\mathcal{H}$ satisfying

$$\overline{T_s} = T_s^{-1} = T_s - (q - q^{-1}) \quad \text{for all } s \in S.$$

If $J \subseteq S$ then $\mathcal{H}(W_J)$, the Hecke algebra associated with the Coxeter system (W_J, J), is isomorphic to the subalgebra of $\mathcal{H}(W)$ generated by $\{\, T_s \mid s \in J \,\}$. We shall identify $\mathcal{H}(W_J)$ with this subalgebra.

3 *W*-graphs

A W-graph is a triple (V, μ, τ) consisting of a set V, a function $\mu \colon V \times V \to \mathbb{Z}$ and a function τ from V to the power set of S, subject to the requirement that the free $\mathcal{A}$-module with basis V admits an $\mathcal{H}$-module structure satisfying

$$T_s v = \begin{cases} -q^{-1} v & \text{if } s \in \tau(v) \\ qv + \sum_{\{u \in V \mid s \in \tau(u)\}} \mu(u,v)u & \text{if } s \notin \tau(v), \end{cases} \tag{3.1}$$

for all $s \in S$ and $v \in V$. The elements of V are the vertices of the graph, and if $v \in V$ then $\tau(v)$ is the colour of the vertex. By definition there is a directed edge from a vertex v to a vertex u if and only if $\mu(u,v) \neq 0$, in which case $\mu(u,v)$ is the weight of the edge. We say that the edge is *superfluous* if $\tau(u) \subseteq \tau(v)$ (since the formulas in Eq. (3.1) would be unchanged by the deletion of any such edge).

Notation. If $\Gamma = (V, \mu, \tau)$ is a W-graph, we denote the $\mathcal{H}$-module $\mathcal{A}V$ by M_Γ. When there is no ambiguity we write $\Gamma(V)$ for the W-graph whose vertex set is V.

Since M_Γ is $\mathcal{A}$-free on V it admits a unique $\mathcal{A}$-semilinear involution $\alpha \mapsto \overline{\alpha}$ such that $\overline{v} = v$ for all $v \in V$. We call this involution the bar involution on M_Γ. It is an easy consequence of Eq. (3.1) that $\overline{h\alpha} = \overline{h}\overline{\alpha}$ for all $h \in \mathcal{H}$ and $\alpha \in \mathcal{A}V$.

Following [8], define a preorder $\leqslant_\Gamma$ on V as follows: $u \leqslant_\Gamma v$ if there exists a sequence of vertices $u = x_0, x_1, \ldots, x_m = v$ such that $\mu(x_{i-1}, x_i) \neq 0$ and $\tau(x_{i-1}) \nsubseteq \tau(x_i)$ for all $i \in [1, m]$.

That is, $u \leqslant_\Gamma v$ if there is a directed path from v to u along non-superfluous edges. Let $\sim_\Gamma$ be the equivalence relation on V corresponding to $\leqslant_\Gamma$. The $\sim_\Gamma$ equivalence classes in V are called the *cells* of Γ. For each cell $\mathcal{C}$ the corresponding full subgraph of Γ is itself a W-graph, the μ and τ functions being the restrictions of those for Γ. The preorder $\leqslant_\Gamma$ on V induces a partial order on the cells, as follows: $\mathcal{C} \leqslant_\Gamma \mathcal{C}'$ if $u \leqslant_\Gamma v$ for some $u \in \mathcal{C}$ and $v \in \mathcal{C}'$.

It follows readily from Eq. (3.1) that a subset of V spans a $\mathcal{H}(W)$ -submodule of M_Γ if and only if it is closed, in the sense that for every vertex v in the subset, each $u \in V$ satisfying $\mu(u,v) \neq 0$ and $\tau(u) \nsubseteq \tau(v)$ is also in the subset. Thus $U \subseteq V$ is a closed subset of V if and only if $U = \bigcup_{v \in U} \{\, u \in V \mid u \leqslant_{\Gamma(V)} v \,\}$. Clearly, a subset of V is closed if and only if it is the union of cells that form an ideal with respect to the partial ordering of cells. If U is a closed subset of V then the subgraphs $\Gamma(U)$ and $\Gamma(V \setminus U)$ induced by U and $V \setminus U$ are themselves W-graphs, with edge weights $\mu(v,w)$and vertex colours $\tau(v)$ inherited from $\Gamma(V)$, and we have $M_{\Gamma(V\setminus U)} \cong M_{\Gamma(V)}/M_{\Gamma(U)}$ as $\mathcal{H}(W)$-modules.

It is trivial to check that if $\Gamma = (V, \mu, \tau)$ is a W-graph and $J \subseteq S$ then the $\mathcal{H}(W_J)$-module obtained from M_Γ by restriction is afforded by a W_J-graph, namely $\Gamma_J = (V, \mu, \tau_J)$, where τ_J is defined by $\tau_J(v) = \tau(v) \cap J$ for all $v \in V$. We remark that, by the main theorem of [7], if N is an $\mathcal{H}(W_J)$-module afforded by a W_J-graph with vertex set U, then the induced module $\mathcal{H} \otimes_{\mathcal{H}(W_J)} N$ is afforded by a W-graph with vertex set $D_J \times U$.

We end this section by recalling the original Kazhdan–Lusztig W-graph for the regular representation of $\mathcal{H}(W)$. For each $w \in W$, define

$$\mathcal{L}(w) = \{s \in S \mid l(sw) < l(w)\},$$
$$\mathcal{R}(w) = \{s \in S \mid l(ws) < l(w)\},$$

the elements of which are called the left descents of w and the right descents of w, respectively. Kazhdan and Lusztig give a recursive procedure that defines polynomials $P_{y,w}$ whenever $y, w \in W$ and $y < w$. These polynomials satisfy $\deg P_{y,w} \leqslant \frac{1}{2}(l(w) - l(y) - 1)$, and $\mu_{y,w}$ is defined to be the leading coefficient of $P_{y,w}$ if the degree is $\frac{1}{2}(l(w) - l(y) - 1)$, or 0 otherwise. Now define W° to be the group opposite to W, writing $w \mapsto w^{\circ}$ for the natural antiisomorphism from W to W°. Observe that $(W \times W^{\circ}, S \sqcup S^{\circ})$ is a Coxeter system. Kazhdan and Lusztig show that defining μ and τ by the formulas

$$\mu(y,w) = \begin{cases} \mu_{y,w} & \text{if } y < w \\ \mu_{w,y} & \text{if } w < y \end{cases}$$
$$\tau(w) = \mathcal{L}(w) \sqcup \mathcal{R}(w)^{\circ}$$

makes $\Gamma(W) = (W, \mu, \tau)$ into a $(W \times W^{\circ})$-graph. Thus the module $M_{\Gamma(W)}$ may be regarded as an $(\mathcal{H}, \mathcal{H})$-bimodule.

4 *W*-graph ideals

Let (W,S) be a Coxeter sytem and $\mathcal{H} = \mathcal{H}(W)$. Let $\mathscr{I}$ be a nonempty ideal in the poset $(W, \leqslant_{\mathsf{L}})$, and note that this implies that $\mathrm{Pos}(\mathscr{I}) = S \setminus \mathscr{I} = \{\, s \in S \mid s \notin \mathscr{I} \,\}$. Let J be a subset of $\mathrm{Pos}(\mathscr{I})$, so that $\mathscr{I} \subseteq D_J$. For each $w \in \mathscr{I}$ the following subsets of S give a partition of S:

$$\mathrm{SD}(\mathscr{I}, w) = \{\, s \in S \mid sw < w \,\},$$
$$\mathrm{SA}(\mathscr{I}, w) = \{\, s \in S \mid sw > w \text{ and } sw \in \mathscr{I} \,\},$$
$$\mathrm{WD}_J(\mathscr{I}, w) = \{\, s \in S \mid sw > w \text{ and } sw \notin D_J \,\},$$
$$\mathrm{WA}_J(\mathscr{I}, w) = \{\, s \in S \mid sw > w \text{ and } sw \in D_J \setminus \mathscr{I} \,\}.$$

We call the elements of these sets the strong ascents, strong descents, weak ascents and weak descents of w relative to $\mathscr{I}$ and J. If $\mathscr{I}$ and J are clear from the context then we may omit reference to them, and write, for example, $\mathrm{WA}(w)$ rather than $\mathrm{WA}_J(\mathscr{I}, w)$. We also define $\mathrm{D}_J(\mathscr{I}, w) = \mathrm{SD}(\mathscr{I}, w) \cup \mathrm{WD}_J(\mathscr{I}, w)$ and $\mathrm{A}_J(\mathscr{I}, w) = \mathrm{SA}(\mathscr{I}, w) \cup \mathrm{WA}_J(\mathscr{I}, w)$, the descents and ascents of w relative to $\mathscr{I}$ and J.

Remark 4.1 It follows from Lemma 2.4 that

$$\mathrm{WA}(w) = \{\, s \in S \mid sw \notin \mathscr{I} \text{ and } w^{-1}sw \notin J \,\},$$
$$\mathrm{WD}(w) = \{\, s \in S \mid sw \notin \mathscr{I} \text{ and } w^{-1}sw \in J \,\},$$

since $sw \notin \mathscr{I}$ implies that $sw > w$, given that $\mathscr{I}$ is an ideal in $(W, \leqslant_{\mathsf{L}})$. Clearly all descents of the identity element are weak descents, and in fact $\mathrm{D}(1) = \mathrm{WD}(1) = J$.

Definition 4.2 With the above notation, we say that $\mathscr{I}$ is a *W-graph ideal with respect to J*, or that $(\mathscr{I}, J)$ is a W-graph ideal, if the following hypotheses are satisfied.

(i) There is an $\mathcal{A}$-free $\mathcal{H}$-module $\mathscr{S} = \mathscr{S}(\mathscr{I}, J)$ with an $\mathcal{A}$-basis $B = \{\, b_w \mid w \in \mathscr{I} \,\}$ on which the generators T_s act by

$$T_s b_w = \begin{cases} b_{sw} & \text{if } s \in \mathrm{SA}(w), \\ b_{sw} + (q - q^{-1}) b_w & \text{if } s \in \mathrm{SD}(w), \\ -q^{-1} b_w & \text{if } s \in \mathrm{WD}(w), \\ q b_w - \sum\limits_{\substack{y \in \mathscr{I} \\ y < sw}} r^s_{y,w} b_y & \text{if } s \in \mathrm{WA}(w), \end{cases} \tag{4.1}$$

for some polynomials $r^s_{y,w} \in q\mathcal{A}^+$.

(ii) The module $\mathscr{S}$ admits an $\mathcal{A}$-semilinear involution $\alpha \mapsto \overline{\alpha}$ satisfying $\overline{b_1} = b_1$ and $\overline{h\alpha} = \overline{h}\overline{\alpha}$ for all $h \in \mathcal{H}$ and $\alpha \in \mathscr{S}$.

The basis B in (i) is called the *standard basis* of $\mathscr{S}$, and the involution $\alpha \mapsto \overline{\alpha}$ in (ii) is called the *bar involution* on $\mathscr{S}$.

Remark 4.3 An obvious induction on $l(w)$ shows that $b_w = T_w b_1$ for all $w \in \mathscr{I}$.

Remark 4.4 In view of the relation $T_s(T_s - q) = -q^{-1}(T_s - q)$, it follows from Eq. (4.1) that $\{\, b_w \mid s \in \mathrm{WD}(w) \,\} \cup \{\, b_{sw} - q b_w \mid s \in \mathrm{SA}(w) \,\}$ spans the $(-q^{-1})$-eigenspace of T_s in $\mathscr{S}$. In the case $s \in \mathrm{WA}(w)$ we deduce that $r^s_{y,w} = q r^s_{sy,w}$ whenever $s \in \mathrm{SA}(y)$, and that $r^s_{y,w} = 0$ whenever $s \in \mathrm{WA}(y)$. In particular, $r^s_{w,w} = 0$.

Definition 4.5 If $w \in W$ and $\mathscr{I} = \{\, u \in W \mid u \leqslant_{\mathsf{L}} w \,\}$ is a W-graph ideal with respect to some $J \subseteq S$ then we say that w is a *W-graph determining element* associated with J.

Remark 4.6 If $\mathscr{I}$ is a W-graph ideal generated by a W-graph determining element then it follows from [6, Proposition 7.9] that, in the case $s \in \mathrm{WA}_J(\mathscr{I})$ in Part (i) of Definition 5.2, the sum $\sum_{y \in \mathscr{I}, y < sw} r^s_{y,w} b_y$ can be replaced by the simpler $\sum_{y \in \mathscr{I}, y < w} r^s_{y,w} b_y$.

Let $(\mathscr{I}, J)$ be a W-graph ideal and let $\mathscr{S}(\mathscr{I}, J)$ be the corresponding $\mathcal{H}$-module, as given in Definition 4.2. From these data one can construct a W-graph $\Gamma = \Gamma(\mathscr{I}, J)$ with $M_\Gamma = \mathscr{S}(\mathscr{I}, J)$. Specifically, the following results are proved in [6].

Lemma 4.7 [6, Lemma 7.2.] *The module $\mathscr{S}(\mathscr{I},J)$ in Definition 4.2 has a unique $\mathcal{A}$-basis $C=\{\,c_w \mid w\in\mathscr{I}\,\}$ such that for all $w\in\mathscr{I}$ we have $\overline{c_w}=c_w$ and*

$$b_w = c_w + q\sum_{y<w} q_{y,w}c_y \tag{4.2}$$

for certain polynomials $q_{y,w}\in\mathcal{A}^+$.

Define $\mu_{y,w}$ to be the constant term of $q_{y,w}$. The polynomials $q_{y,w}$, where $y<w$, can be computed recursively by the following formulas.

Corollary 4.8 [6, Corollary 7.4] *Suppose that $w<sw\in\mathscr{I}$ and $y<sw$. If $y=w$ then $q_{y,sw}=1$, and if $y\neq w$ we have the following formulas:*

(i) $q_{y,sw}=qq_{y,w}$ *if $s\in \mathrm{A}(y)$,*
(ii) $q_{y,sw}=-q^{-1}(q_{y,w}-\mu_{y,w})+q_{sy,w}+\sum_x \mu_{y,x}q_{x,w}$ *if $s\in \mathrm{SD}(y)$,*
(iii) $q_{y,sw}=-q^{-1}(q_{y,w}-\mu_{y,w})+\sum_x \mu_{y,x}q_{x,w}$ *if $s\in \mathrm{WD}(y)$,*

where $q_{y,w}$ and $\mu_{y,w}$ are regarded as 0 if $y\not< w$, and in (ii) *and* (iii) *the sums extend over all $x\in\mathscr{I}$ such that $y<x<w$ and $s\notin \mathrm{D}(x)$.*

Corollary 4.9 *Suppose that $y,w\in\mathscr{I}$ with $y<w$. If $l(w)-l(y)$ is odd then $q_{y,w}$ is a polynomial in q^2, while if $l(w)-l(y)$ is even then $\mu_{y,w}=0$ and $q^{-1}q_{y,w}$ is a polynomial in q^2.*

Proof This follows from Corollary 4.8 by a straightforward induction on $l(w)-l(y)$. □

Let $\mu\colon C\times C\to\mathbb{Z}$ be given by

$$\mu(c_y,c_w)=\begin{cases}\mu_{y,w} & \text{if } y<w\\ \mu_{w,y} & \text{if } w<y\\ 0 & \text{otherwise,}\end{cases} \tag{4.3}$$

and let τ from C to the power set of S be given by $\tau(c_w)=\mathrm{D}(w)$ for all $y\in\mathscr{I}$.

Theorem 4.10 [6, Theorem 7.5.] *The triple (C,μ,τ) is a W-graph.*

Definition 4.11 We call $C=\{\,c_w \mid w\in\mathscr{I}\,\}$ the *W-graph basis* of $\mathscr{S}(\mathscr{I},J)$.

The generators T_s act on the basis elements c_w as described in the following theorem.

Theorem 4.12 [6, Theorem 7.3.] *Let $s\in S$ and $w\in\mathscr{I}$. Then*

$$T_sc_w=\begin{cases}-q^{-1}c_w & \text{if } s\in \mathrm{D}(w),\\ qc_w+\sum_{y\in\mathcal{R}(s,w)}\mu_{y,w}c_y & \text{if } s\in \mathrm{WA}(w),\\ qc_w+c_{sw}+\sum_{y\in\mathcal{R}(s,w)}\mu_{y,w}c_y & \text{if } s\in \mathrm{SA}(w),\end{cases}$$

where the set $\mathcal{R}(s,w)$ consists of all $y\in\mathscr{I}$ such that $y<w$ and $s\in \mathrm{D}(y)$.

Corollary 4.13 [10, Corollary 3.6.(i)] *Let $x,y\in\mathscr{I}$. If $x\leqslant_{\mathsf{L}} y$ then $c_y\leqslant_{\Gamma(C)} c_x$.*

Remark 4.14 It is an easy consequence of Theorem 4.12 that $\{\,c_w \mid s\in \mathrm{D}(w)\,\}$ is a basis for the $(-q^{-1})$-eigenspace of T_s in M_Γ. In particular, since Eq. (4.1) shows that b_w is in this eigenspace when $s\in \mathrm{WD}(w)$, it follows from Lemma 4.7 that $q_{y,w}=0$ whenever there is an $s\in \mathrm{WD}(w)$ such that $s\notin \mathrm{D}(y)$.

Corollary 4.15 *Let $y,w\in\mathscr{I}$ with $y<w$ and $l(y)<l(w)-1$. If $\mu_{y,w}\neq 0$ then* $\mathrm{D}(w)\subseteq \mathrm{D}(y)$.

Proof Suppose, for a contradiction, that $\mathrm{D}(w)\cap \mathrm{A}(y)\neq\emptyset$, and choose $s\in \mathrm{D}(w)\cap A(y)$. If $s\in \mathrm{SD}(w)$ then the first formula in Corollary 4.8 gives $q_{y,w}=qq_{y,sw}$, whence $\mu_{y,w}=0$, since $\mu_{y,w}$ is the constant term of $q_{y,w}$. But if $s\in \mathrm{WD}(w)$ then $q_{y,w}=0$ by Remark 4.14, so that $\mu_{y,w}=0$ in this case also. In either case, the assumption that $\mu_{y,w}\neq 0$ is contradicted. □

5 Strong subideals of a W-graph ideal

As above, let (W,S) be a Coxeter system, and $\mathcal{H} = \mathcal{H}(W)$.

Definition 5.1 Suppose that $(\mathscr{I},J)$ and $(\mathscr{I}_0,J_0)$ are W-graph ideals. We say that $\mathscr{I}$ is a *W-graph subideal of $\mathscr{I}_0$* if $\mathscr{I} \subseteq \mathscr{I}_0$ and $J = J_0$.

The following result is Theorem 4.4 of [10]. See Remark 5.4 below for some comments relating to its proof.

Theorem 5.2 *Let $(\mathscr{I}_0,J)$ be a W-graph ideal, and let $C_0 = \{\, c^0_w \mid w \in \mathscr{I}_0 \,\}$ be the W-graph basis of the module $\mathscr{S}_0 = \mathscr{S}(\mathscr{I}_0,J)$. Suppose that $\mathscr{I} \subseteq \mathscr{I}_0$ and $\{\, c^0_w \mid w \in \mathscr{I}_0 \setminus \mathscr{I} \,\}$ is a closed subset of C_0. Then $\mathscr{I}$ is a W-graph subideal of $\mathscr{I}_0$. Moreover, the corresponding W-graph $\Gamma(\mathscr{I})$ is isomorphic to the full subgraph of $\Gamma(\mathscr{I}_0)$ on the vertex set $\{\, c^0_w \mid w \in \mathscr{I} \,\} \subseteq C_0$, with τ and μ functions inherited from $\Gamma(\mathscr{I}_0)$.*

In view of Theorem 5.2 we make the following definition.

Definition 5.3 Let $(\mathscr{I}_0,J)$ be a W-graph ideal and let $C_0 = \{\, c^0_w \mid w \in \mathscr{I}_0\}$ be the W-graph basis of the module $\mathscr{S}(\mathscr{I}_0,J)$. A *strong W-graph subideal of $\mathscr{I}_0$* is a W-graph subideal $\mathscr{I}$ such that $\{\, c^0_w \mid w \in \mathscr{I}_0 \setminus \mathscr{I}\}$ is a closed subset of C_0.

Remark 5.4 Given the hypotheses of Theorem 5.2, let $\Gamma(\mathscr{I}_0) = (C_0,\mu,\tau)$ be the W-graph obtained from $(\mathscr{I}_0,J)$, and let $\mathscr{S}'$ be the $\mathcal{A}$-submodule of $\mathscr{S}_0 = M_\Gamma$ spanned by the set $C' = \{\, c^0_w \mid w \in \mathscr{I}_0 \setminus \mathscr{I} \,\}$. The assumption that C' is closed ensures, by Corollary 4.13, that $\mathscr{I}$ is an ideal of $(W,\leqslant_{\mathsf{L}})$. Moreover, $\mathscr{S}'$ is an $\mathcal{H}(W)$-submodule of $\mathscr{S}_0$. Now, defining f to be the natural map $\mathscr{S}_0 \to \mathscr{S}_0/\mathscr{S}'$, it is readily checked that for all $s \in S$ and $w \in \mathscr{I}$,

$$T_s f(c^0_w) = \begin{cases} -q^{-1} f(c^0_w) & \text{if } s \in \tau(w) \\ q f(c^0_w) + \sum_{\{x \in \mathscr{I} \mid s \in \tau(x)\}} \mu(x,w) f(c^0_x) & \text{if } s \notin \tau(w), \end{cases}$$

since $f(c^0_y) = 0$ whenever $y \in \mathscr{I}_0 \setminus \mathscr{I}$. The proof of Theorem 5.2 proceeds by showing that if $\{\, b^0_w \mid w \in \mathscr{I}_0 \,\}$ is the standard basis of $\mathscr{S}_0$ then for all $w \in \mathscr{I}_0 \setminus \mathscr{I}$,

$$f(b^0_w) = \sum_{y \in \mathscr{I},\, y<w} r_{y,w} f(b^0_y)$$

for some polynomials $r_{y,w} \in q\mathcal{A}^+$, with $r_{y,w} = q$ if $y = sw$ for some $s \in S$. Then Lemma 5.5 below, which extends part of the proof of Theorem 5.2 given in [10], shows that $\mathscr{I}$ satisfies Definition 4.2, with $\mathscr{S}(\mathscr{I},J) = \mathscr{S}_0/\mathscr{S}'$ and with $\{\, f(b^0_w) \mid w \in \mathscr{I} \,\}$ as its standard basis. The proof of Lemma 5.5 also shows that $\Gamma(\mathscr{I})$ inherits its μ and τ functions from $\Gamma(\mathscr{I}_0)$.

Lemma 5.5 is needed in the proof of Theorem 5.9 below.

Lemma 5.5 *Assume that $(\mathscr{I}_0,J)$ is a W-graph ideal and that $\mathscr{I} \subseteq \mathscr{I}_0$ is an ideal of $(W,\leqslant_{\mathsf{L}})$. Let $B_0 = \{\, b^0_w \mid w \in \mathscr{I}_0 \,\}$ be the standard basis of $\mathscr{S}_0 = \mathscr{S}(\mathscr{I}_0,J)$, and suppose that there exists an $\mathcal{A}$-free $\mathcal{H}$-module $\mathscr{S}$ and an $\mathcal{H}$-module homomorphism $f\colon \mathscr{S}_0 \to \mathscr{S}$ such that*

(i) *$\{\, f(b^0_w) \mid w \in \mathscr{I} \,\}$ is an $\mathcal{A}$-basis of $\mathscr{S}$,*

(ii) *the kernel of f is invariant under the bar involution on $\mathscr{S}_0$, and*

(iii) *for each $w \in \mathscr{I}_0 \setminus \mathscr{I}$ and $y \in \mathscr{I}$ there is a polynomial $r_{y,w} \in q\mathcal{A}^+$ such that $r_{y,w} = q$ if $y = sw$ for some $s \in S$, and $f(b^0_w) = \sum_{\{y \in \mathscr{I} \mid y<w\}} r_{y,w} f(b^0_y)$.*

Then $\mathscr{I}$ is a strong W-graph subideal of $\mathscr{I}_0$.

Proof The first step is to show that $(\mathscr{I}, J)$ is a W-graph ideal. We define $b_w = f(b^0_w)$ for all $w \in \mathscr{I}$, so that by hypothesis $B = \{\, b_w \mid w \in \mathscr{I}\,\}$ is an $\mathcal{A}$-basis of $\mathscr{S}$, and proceed to show that the requirements of Definition 4.2 are satisfied. Hypothesis (ii) above ensures that $\mathscr{S}$ admits a bar involution such that $\overline{f(\alpha)} = f(\overline{\alpha})$ for all $\alpha \in \mathscr{S}_0$, and the requirements that $\overline{b_1} = b_1$ and that $\overline{h\alpha} = \overline{h}\overline{\alpha}$ for all $h \in \mathcal{H}$ and $\alpha \in \mathscr{S}$ follow immediately by applying f to the corresponding formulas in $\mathscr{S}_0$.

Since $(\mathscr{I}_0, J)$ is a W-graph ideal and f is an $\mathcal{H}$-module homomorphism, it follows from Definition 4.2 that for all $s \in S$ and $w \in \mathscr{I}_0$,

$$T_s f(b^0_w) = \begin{cases} f(b^0_{sw}) & \text{if } s \in \mathrm{SA}(\mathscr{I}_0, w),\\ f(b^0_{sw}) + (q-q^{-1})f(b^0_w) & \text{if } s \in \mathrm{SD}(\mathscr{I}_0, w),\\ -q^{-1}f(b^0_w) & \text{if } s \in \mathrm{WD}_J(\mathscr{I}_0, w),\\ qf(b^0_w) - \sum_{\{y \in \mathscr{I} \mid y<sw\}} r^s_{y,w} f(b^0_y) & \text{if } s \in \mathrm{WA}_J(\mathscr{I}_0, w), \end{cases}$$

for some polynomials $r^s_{y,w} \in q\mathcal{A}^+$. Note that since $\mathscr{I} \subseteq \mathscr{I}_0$ it follows immediately from the definitions that if $w \in \mathscr{I}$ then $\mathrm{SD}(\mathscr{I}, w) = \mathrm{SD}(\mathscr{I}_0, w)$ and $\mathrm{WD}_J(\mathscr{I}, w) = \mathrm{WD}_J(\mathscr{I}_0, w)$, and $\mathrm{SA}(\mathscr{I}) \subseteq \mathrm{SA}(\mathscr{I}_0)$. Thus if $s \in S$ and $w \in \mathscr{I}$ then

$$T_s b_w = \begin{cases} b_{sw} & \text{if } s \in \mathrm{SA}(\mathscr{I}, w),\\ b_{sw} + (q-q^{-1})b_w & \text{if } s \in \mathrm{SD}(\mathscr{I}, w),\\ -q^{-1}b_w & \text{if } s \in \mathrm{WD}_J(\mathscr{I}, w),\\ qb_w - \sum_{\{y \in \mathscr{I} \mid y<sw\}} r^s_{y,w} b_y & \text{if } s \in \mathrm{WA}_J(\mathscr{I}_0, w), \end{cases}$$

and to complete the proof that Eq. (4.1) holds in all cases it remains to show that it holds whenever s is in $\mathrm{WA}_J(\mathscr{I}, w)$ and in $\mathrm{SA}(\mathscr{I}_0, w)$. In this case we have $sw \in \mathscr{I}_0$ and $sw \notin \mathscr{I}$, and in view of hypothesis (iii) it follows that

$$T_s b_w = f(b^0_{sw}) = \sum_{\substack{y \in \mathscr{I}\\ y<sw}} r_{y,sw} b_y = qb_w + \sum_{\substack{y \in \mathscr{I}\\ y<w}} r_{y,sw} b_y$$

by Lemma 2.1 and the fact that $r_{w,sw} = q$ (by hypothesis). So Eq. (4.1) does indeed hold, with $r^s_{y,w} = -r_{y,sw}$ when $s \in \mathrm{WA}_J(\mathscr{I}, w) \cap \mathrm{SA}(\mathscr{I}_0, w)$, and hence $(\mathscr{I}, J)$ is a W-graph ideal.

Now let $C_0 = \{\, c^0_w \mid w \in \mathscr{I}_0 \,\}$ be the W-graph basis of $\mathscr{S}_0$ and let $C = \{\, c_w \mid w \in \mathscr{I}\,\}$ be the W-graph basis of $\mathscr{S}$. Thus, by Theorem 4.7, for all $w \in \mathscr{I}_0$ there exist polynomials $q^0_{y,w} \in \mathcal{A}^+$ such that

$$c^0_w = b^0_w - q \sum_{\substack{y<w\\ y \in \mathscr{I}_0}} q^0_{y,w} c^0_y \tag{5.1}$$

and for all $w \in \mathscr{I}$ there exist polynomials $q_{y,w} \in \mathcal{A}^+$ such that

$$c_w = b_w - q \sum_{\substack{y<w\\ y \in \mathscr{I}}} q_{y,w} c_y. \tag{5.2}$$

We use induction on $l(w)$ to show that for all $w \in \mathscr{I}_0$,

$$f(c^0_w) = \begin{cases} c_w & \text{if } w \in \mathscr{I},\\ 0 & \text{if } w \notin \mathscr{I}. \end{cases}$$

In the course of this we shall also show that $q_{y,w} = q^0_{y,w}$ whenever $y, w \in \mathscr{I}$ with $y < w$.

In the case $l(w) = 0$ we have $w = 1$ and $f(c_w^0) = f(b_w^0) = b_w = c_w$, as required. Now assume that $w \in \mathscr{I}_0$ and $l(w) > 1$. Applying f to both sides of Eq. (5.1) gives

$$\begin{aligned} f(c_w^0) &= f(b_w^0) - q \sum_{\substack{y<w \\ y \in \mathscr{I}_0}} q_{y,w}^0 f(c_y^0) \\ &= f(b_w^0) - q \sum_{\substack{y<w \\ y \in \mathscr{I}}} q_{y,w}^0 c_y \end{aligned}$$

by the inductive hypothesis. If $w \in \mathscr{I}$ then $f(b_w^0) = b_w$, and using Eq. (5.2) we find that

$$f(c_w^0) - c_w = \sum_{\substack{y<w \\ y \in \mathscr{I}}} q(q_{y,w} - q_{y,w}^0) c_y.$$

But the left hand side is fixed by the bar involution, as are the basis elements c_y on the right hand side. So the coefficients $q(q_{y,w} - q_{y,w}^0)$ must also be fixed. But since $q(q_{y,w} - q_{y,w}^0)$ is a polynomial in q with zero constant term, and since $\overline{q} = q^{-1}$, this forces $q(q_{y,w} - q_{y,w}^0) = 0$. Hence $f(c_w^0) = c_w$ and $q_{y,w} = q_{y,w}^0$, as required. On the other hand, if $w \notin \mathscr{I}$ then by our hypothesis (iii),

$$f(b_w^0) = \sum_{\substack{y<w \\ y \in \mathscr{I}}} r_{y,w} b_y$$

where the $r_{y,w}$ are polynomials in q with zero constant term, and so (using Eq. 5.2)

$$f(c_w^0) = \sum_{\substack{y<w \\ y \in \mathscr{I}}} r_{y,w} \Big(c_y + q \sum_{\substack{z<y \\ z \in \mathscr{I}}} q_{z,y} c_z \Big) - q \sum_{\substack{y<w \\ y \in \mathscr{I}}} q_{y,w}^0 c_y.$$

Since $f(c_w^0)$ is fixed by the bar involution, while the right hand side is a linear combination of the basis elements c_y in which all the coefficients are polynomials with zero constant term, it follows that $f(c_w^0) = 0$, as required.

It is now clear that $C' = \{ c_w^0 \mid w \in \mathscr{I}_0 \setminus \mathscr{I} \}$ spans an $\mathcal{H}$-submodule of $\mathscr{S}_0$, namely the kernel of f. Hence C' is a closed subset of C_0, and so $\mathscr{I}$ is a strong W-graph subideal of $\mathscr{I}_0$. □

Remark 5.6 In the situation of Lemma 5.5, let $\Gamma_0 = (C_0, \mu_0, \tau_0)$ be the W-graph obtained from $\mathscr{I}_0$ and $\Gamma = (C, \mu, \tau)$ the W-graph obtained from $\mathscr{I}$. Recall that if $\mu_{y,w}$ denotes the constant term of the polynomial $q_{y,w}$, then for all $y, w \in \mathscr{I}$,

$$\mu(c_y, c_w) = \begin{cases} \mu_{y,w} & \text{if } y < w, \\ \mu_{w,y} & \text{if } w < y, \\ 0 & \text{otherwise.} \end{cases}$$

The parameters $\mu_0(c_y^0, c_w^0)$, for $y, w \in \mathscr{I}_0$, are similarly obtained from the polynomials $q_{y,w}^0$. Since we showed in the proof that $q_{y,w}^0 = q_{y,w}$ whenever $y, w \in \mathscr{I}$ with $y < w$, it follows that $\mu(c_y, c_w) = \mu_0(c_y^0, c_w^0)$ whenever $y, w \in \mathscr{I}$. Furthermore, $\tau(c_w) = \tau_0(c_w^0)$ whenever $w \in \mathscr{I}$, since by definition $\tau(c_w) = D_J(\mathscr{I}, w)$ and $\tau(c_w^0) = D_J(\mathscr{I}_0, w)$, and, as we noted in the proof, these are equal if $w \in \mathscr{I}$, since $\mathrm{SD}(\mathscr{I}, w) = \mathrm{SD}(\mathscr{I}_0, w)$ and $\mathrm{WD}_J(\mathscr{I}, w) = \mathrm{WD}_J(\mathscr{I}_0, w)$. Thus Γ is isomorphic to the full (decorated) subgraph of Γ_0 on the vertices $\{ c_w^0 \mid w \in \mathscr{I} \}$.

Remark 5.7 The converse of Lemma 5.5 is also true: if $(\mathscr{I}_0, J)$ is a *W*-graph ideal and $\mathscr{I}$ is a strong *W*-graph subideal of $\mathscr{I}_0$, then $\mathscr{S} = \mathscr{S}(\mathscr{I}, J)$ is an $\mathcal{A}$-free $\mathcal{H}$-module, and there is an $\mathcal{H}$-module homomorphism $f\colon \mathscr{S}(\mathscr{I}_0, J) \to \mathscr{S}$ satisfying conditions (i), (ii) and (iii) of Lemma 5.5. Indeed, the proof of Theorem 5.2 proceeded by constructing the required f, and in the course of this the following properties of f were established:

(i) $f(c^0_w) = c_w$ for all $w \in \mathscr{I}$ and $f(c^0_w) = 0$ for all $w \in \mathscr{I}_0 \setminus \mathscr{I}$,

(ii) $f(b^0_w) = b_w$ for all $w \in \mathscr{I}$, while for all $w \in \mathscr{I}_0 \setminus \mathscr{I}$ there exist polynomials $r_{y,w} \in q\mathcal{A}^+$ with $r_{y,w} = q$ if $wy^{-1} \in S$ and $f(b^0_w) = \sum_{\{y\in\mathscr{I} \mid y<w\}} r_{y,w} f(b^0_y)$,

(iii) $f(\overline{\alpha}) = \overline{f(\alpha)}$ for all $\alpha \in \mathscr{S}_0$.

Proposition 5.8 *If $\mathscr{I}_0$ is a W-graph ideal and $\mathscr{I}_1$ and $\mathscr{I}_2$ are strong W-graph subideals of $\mathscr{I}_0$, then $\mathscr{I}_1 \cup \mathscr{I}_2$ and $\mathscr{I}_1 \cap \mathscr{I}_2$ are strong W-graph subideals of $\mathscr{I}_0$.*

Proof This is clear, since intersections and unions of ideals of $(W, \leqslant_{\mathsf{L}})$ are ideals, and, for any *W*-graph, intersections and unions of closed sets are closed. □

We now come to the main result of this section: induction of *W*-graph ideals preserves the strong subideal relationship.

Theorem 5.9 *Suppose that $J \subseteq K \subseteq S$ and that $(\mathscr{I}_0, J)$ is a W_K-graph ideal. If $\mathscr{I}$ is a strong W_K-graph subideal of $\mathscr{I}_0$ then $D_K\mathscr{I}$ is a strong W-graph subideal of $D_K\mathscr{I}_0$.*

Proof Write $\mathcal{H}_K$ for the Hecke algebra associated with the Coxeter system (W_K, K), regarded as a subalgebra of $\mathcal{H}$. Let $\mathscr{S}_0$ and $\mathscr{S}$ be the $\mathcal{H}_K$-modules derived from the W_K-graphs $(\mathscr{I}_0, J)$ and $(\mathscr{I}, J)$, and let $B_0 = \{\, b^0_w \mid w \in \mathscr{I}_0 \,\}$ and $B = \{\, b_w \mid w \in \mathscr{I} \,\}$ be their standard bases. By Remark 5.7 there is an $\mathcal{H}_K$-module homomorphism $f\colon \mathscr{S}_0 \to \mathscr{S}$ satisfying

(i) $f(\overline{\alpha}) = \overline{f(\alpha)}$ for all $\alpha \in \mathscr{S}_0$,

(ii) $f(b^0_w) = b_w$ for all $w \in \mathscr{I}$, and for all $w \in \mathscr{I}_0 \setminus \mathscr{I}$ there exist $r_{y,w} \in q\mathcal{A}^+$ with $r_{y,w} = q$ if $wy^{-1} \in S$ and $f(b^0_w) = \sum_{\{y\in\mathscr{I} \mid y<w\}} r_{y,w} b_y$.

We know from Theorem 9.2 of [6] that $D_K\mathscr{I}_0$ and $D_K\mathscr{I}$ are *W*-graph ideals, and the associated $\mathcal{H}$-modules are the induced modules $\mathscr{S}^*_0 = \mathcal{H} \otimes_{\mathcal{H}_K} \mathscr{S}_0$ and $\mathscr{S}^* = \mathcal{H} \otimes_{\mathcal{H}_K} \mathscr{S}$. Moreover, $B^*_0 = \{\, T_d \otimes b^0_w \mid d \in D_K, w \in \mathscr{I}_0 \,\}$ and $B^* = \{\, T_d \otimes b_w \mid d \in D_K, w \in \mathscr{I} \,\}$ are the standard bases of $\mathscr{S}^*_0$ and $\mathscr{S}^*$, and the bar involutions satisfy $\overline{h \otimes \alpha} = \overline{h} \otimes \overline{\alpha}$ for all $h \in \mathcal{H}$ and α in $\mathscr{S}_0$ or $\mathscr{S}$. Let $f^*\colon \mathscr{S}^*_0 \to \mathscr{S}^*$ be the $\mathcal{H}$-module homomorphism induced from the $\mathcal{H}_K$-module homomorphism f, so that $f^*(h \otimes \alpha) = h \otimes f(\alpha)$ for all $h \in \mathcal{H}$ and $\alpha \in \mathscr{S}_0$. The conclusion that $D_K\mathscr{I}$ is a strong *W*-graph subideal of $D_K\mathscr{I}_0$ will follow by an application of Lemma 5.5, if it can be shown that f^* satisfies conditions (i), (ii) and (iii) of Lemma 5.5.

For all $d \in D_K$ and $w \in \mathscr{I}$ we have $f^*(T_d \otimes b^0_w) = T_d \otimes f(b^0_w) = T_d \otimes b_w$, and since $\{\, T_d \otimes b_w \mid d \in D_K, w \in \mathscr{I} \,\}$ is an $\mathcal{A}$-basis of $\mathscr{S}^*$, condition (i) of Lemma 5.5 is satisfied.

For all $h \in \mathcal{H}$ and $\alpha \in \mathscr{S}$ we have

$$f^*(\overline{h \otimes \alpha}) = f^*(\overline{h} \otimes \overline{\alpha}) = \overline{h} \otimes f(\overline{\alpha}) = \overline{h} \otimes \overline{f(\alpha)} = \overline{h \otimes f(\alpha)} = \overline{f^*(h \otimes \alpha)},$$

whence $f^*(\overline{\beta}) = \overline{f^*(\beta)}$ for all $\beta \in \mathscr{S}^*_0$, and condition (ii) of Lemma 5.5 is satisfied.

For all $d \in D_K$ and $w \in \mathscr{I}_0 \setminus \mathscr{I}$ we have

$$f^*(T_d \otimes b^0_w) = T_d \otimes f(b^0_w) = T_d \otimes \Big(\sum\nolimits_y r_{y,w} b_y\Big) = \sum\nolimits_y r_{y,w}(T_d \otimes b_y) = \sum\nolimits_y r_{y,w} f^*(T_d \otimes b^0_y),$$

where the sums extend over all $y \in \mathscr{I}$ such that $y < w$. Since $r_{y,w} \in q\mathcal{A}^+$ and $r_{y,w} = q$ if $wy^{-1} \in S$, condition (iii) of Lemma 5.5 is satisfied. □

Let $(\mathscr{I}, J)$ be a W-graph ideal and $C = \{ c_w \mid w \in \mathscr{I} \}$ the W-graph basis of $\Gamma = \Gamma(\mathscr{I}, J)$. To simplify our terminology, we shall use the preorder $\leqslant_\Gamma$ on C to define a preorder on $\mathscr{I}$, writing $x \leqslant_{\mathscr{I}} y$ if and only if $c_x \leqslant_\Gamma c_y$, whenever $x, y \in \mathscr{I}$. In the same spirit, if $X \subseteq \mathscr{I}$ then we shall say that X is $(\mathscr{I}, J)$-closed if $\{ c_x \mid x \in X \}$ is a closed subset of C, and we shall call X a cell of $(\mathscr{I}, J)$ if $\{ c_x \mid x \in X \}$ is a cell of Γ.

Proposition 5.10 *Suppose that $(\mathscr{I}, J)$ is a W-graph ideal and that X is a cell of $(\mathscr{I}, J)$. Let $o(X) = \{ y \in \mathscr{I} \mid x \leqslant_{\mathscr{I}} y$ for some $x \in X \}$, the union of the cells Y of $(\mathscr{I}, J)$ with $X \leqslant_{\mathscr{I}} Y$. Then $o(X)$ is a strong W-graph subideal of $(\mathscr{I}, J)$. Moreover, if $\mathscr{Z} \subseteq \mathscr{I}$ then $\mathscr{Z}$ is a strong W-graph subideal of $(\mathscr{I}, J)$ if and only if it is a union of subideals of the above form.*

Proof Let Γ be the W-graph $\Gamma(\mathscr{I}, J)$. If $w \in \mathscr{I}$ and $s \in \mathrm{SA}(w)$ then $sw \leqslant_{\mathscr{I}} w$, since $\mu(sw, w) = 1$ (by Theorem 4.12) and $D(sw) \nsubseteq D(w)$. It follows by an induction on $l(v) - l(w)$ that if $w, v \in \mathscr{I}$ with $w \leqslant_{\mathsf{L}} v$ then $v \leqslant_{\mathscr{I}} w$. Hence $o(X)$ is an ideal of $(W, \leqslant_{\mathsf{L}})$. Now suppose that $z \in \mathscr{I} \setminus o(X)$ and $y \leqslant_{\mathscr{I}} z$. Since $z \in \mathscr{I} \setminus o(X)$ there is no $x \in X$ with $x \leqslant_{\mathscr{I}} z$, and by transitivity of $\leqslant_{\mathscr{I}}$ there is no $x \in X$ with $x \leqslant_{\mathscr{I}} y$. So $y \in \mathscr{I} \setminus o(X)$. Hence $\mathscr{I} \setminus o(X)$ is $(\mathscr{I}, J)$-closed, and, by Theorem 5.2, $o(X)$ is a strong W-graph subideal of $(\mathscr{I}, J)$.

As noted in Proposition 5.8, any union of strong W-graph subideals is a strong W-graph subideal. Now let $\mathscr{Z}$ be an arbitrary strong W-graph subideal of $\mathscr{I}$, and suppose that X and Y are cells of $(\mathscr{I}, J)$ with $X \leqslant_{\mathscr{I}} Y$. Since $\mathscr{I} \setminus \mathscr{Z}$ is a closed set, if $Y \subseteq (\mathscr{I} \setminus \mathscr{Z})$ then $X \subseteq (\mathscr{I} \setminus \mathscr{Z})$. Equivalently, if $X \subseteq \mathscr{Z}$ then $Y \subseteq \mathscr{Z}$. So if $X \subseteq \mathscr{Z}$ is a cell then $o(X) \subseteq \mathscr{Z}$, and it follows that $\mathscr{Z}$ is the union of those strong subideals $o(X)$ that it contains. □

Combining Theorem 5.9 and Proposition 5.10 yields the following corollary.

Corollary 5.11 *Suppose that $J \subseteq K \subseteq S$ and that $(\mathscr{I}, J)$ is a W_K-graph ideal. If $X \subseteq \mathscr{I}$ is a cell of $(\mathscr{I}, J)$ then $D_K X$ is a union of cells of the induced W-graph ideal $(D_K \mathscr{I}, J)$.*

Proof By Proposition 5.10, the sets $o(X) = \{ y \in \mathscr{I} \mid x \leqslant_{\mathscr{I}} y \text{ for some } x \in X \}$ and $o(X) \setminus X$ are both strong W_K-graph subideals of $\mathscr{I}$. So by Theorem 5.9 it follows that $D_K o(X)$ and $D_K(o(X) \setminus X)$ are strong W-graph subideals of $(D_K \mathscr{I}, J)$, and hence their complements in $D_K \mathscr{I}$ are unions of cells. Since $D_K X = D_K o(X) \setminus D_K(o(X) \setminus X)$ we deduce that $D_K X$ is a union of cells. □

Remark 5.12 Applying Corollary 5.11 in the case $(\mathscr{I}, J) = (W_K, \emptyset)$ recovers the equal parameters case of [4, Theorem 1].

Let $\Gamma = (C, \mu, \tau)$ be the W-graph obtained from W-graph ideal $(\mathscr{I}, J) = (W, \emptyset)$, so that $\mathscr{S}(\mathscr{I}, J)$ can be identified with the left regular $\mathcal{H}$-module, the basis $C = \{ c_w \mid w \in W \}$ is the Kazhdan–Lusztig basis of $\mathcal{H}$, and $\tau(c_w) = \mathcal{L}(w) = \{ s \in S \mid sw < w \}$, for all $w \in W$. Observe that every edge of Γ with tail c_1 is superfluous, since $\mathcal{L}(1) = \emptyset \subseteq \mathcal{L}(w)$ for all $w \in W$. Hence $W \setminus \{1\}$ is a closed set of $(W, \emptyset)$, and, since $\{1\}$ is an ideal of $(W, \leqslant_{\mathsf{L}})$, it follows that $\{1\}$ is a strong W-graph subideal of W. Similarly, if W is finite and w_S is the longest element of W, then every edge of Γ with head c_{w_S} is superfluous, since $\mathcal{L}(w) \subseteq S = \mathcal{L}(w_S)$ for all $w \in W$. So $\{w_S\}$ is $(W, \emptyset)$-closed. Since $W \setminus \{w_S\}$ is an ideal of $(W, \leqslant_{\mathsf{L}})$, it follows that $W \setminus \{w_S\}$ is strong W-graph subideal of W.

Since $\{w_S\}$ is $(W, \emptyset)$-closed, $\mathcal{A} c_{w_S}$ is an $\mathcal{H}$-submodule of $\mathcal{H}$, as was already obvious from the fact that $T_s c_{w_S} = -q^{-1} c_{w_S}$ for all $s \in S$ (by Theorem 4.12). Using this it is also easy to show that $c_{w_S} = \sum_{w \in W} (-q)^{l(w_S) - l(w)} T_w$.

Now let $K \subseteq S$. By the above discussion, $\{1\}$ is a strong W_K-graph subideal of $(W_K, \emptyset)$, and so by Theorem 5.9 it follows that D_K is a strong W-graph subideal of $(D_K W_K, \emptyset) = (W, \emptyset)$.

Thus $W \setminus D_K$ is a closed subset of $(W,\emptyset)$, whence $W \setminus D_K$ and D_K are both unions of left cells. Furthermore, if W_K is finite and w_K is its longest element, then $W_K \setminus \{w_K\}$ is a strong W_K-graph ideal of $(W_K,\emptyset)$, and by Theorem 5.9 it follows that $W \setminus D_K w_K$ is a strong W-graph subideal of $(W,\emptyset)$. Hence $D_K w_K$ is $(W,\emptyset)$-closed, and, in particular, $D_K w_K$ is a union of left cells. (This result was proved by Geck in [3, Lemma 2.8].)

It is easily checked, using Definition 4.2, that if K is any subset of S then $(1,K)$ is a W_K-graph ideal, associated with the one-dimensional representation ε of $\mathcal{H}_K$ given by $\varepsilon(T_s) = -q^{-1}$ for all $s \in K$. By Theorem 5.2 it follows that (D_K,K) is a W-graph ideal, associated with the representation of $\mathcal{H}$ induced from ε. (This corresponds to the case $u = -1$ in the construction given by Deodhar in [2].) In the case that W_K is finite with w_K its longest element, the $(W_K,\emptyset)$-closed set $\{w_K\}$ also affords the representation ε, and the $(W,\emptyset)$-closed set $D_K w_K$ also affords the representation of $\mathcal{H}$ induced from ε. The following proposition confirms that the W-graph $\Gamma(D_K,K)$ is isomorphic to the full subgraph of $\Gamma(W,\emptyset)$ spanned by the vertices corresponding to $D_K w_K$.

Proposition 5.13 *Let $K \subseteq S$ with W_K finite. Let $C = \{\, c_w \mid w \in W \,\}$ be the W-graph basis of $\mathscr{S}(W,\emptyset)$ and $\Gamma = (C,\mu,\tau)$ the corresponding W-graph, and let $C^K = \{\, c_d^K \mid d \in D_K \,\}$ be the W-graph basis of $\mathscr{S}(D_K,K)$ and $\Gamma^K = (C^K,\mu^K,\tau^K)$ the corresponding W-graph. Define $\varphi\colon C^K \to C$ by $\varphi(c_d^K) = c_{dw_K}$ for all $d \in D_K$, where w_K is the longest element of W_K. Then $\tau^K(v) = \tau(\varphi(v))$ for all $v \in C^K$, and $\mu^K(u,v) = \mu(\varphi(u),\varphi(v))$ for all $u, v \in C^K$.*

Proof As above, we identify $\mathscr{S}(W,\emptyset)$ with $\mathcal{H}$. Since the set $D_K w_K$ is $(W,\emptyset)$-closed, the $\mathcal{A}$-submodule of $\mathcal{H}$ spanned by $\{\, c_w \mid w \in D_K w_K \,\}$ is an $\mathcal{H}$-submodule. It clearly coincides with the left ideal $\mathcal{H}c_{w_K} = \bigoplus_{d \in D_K} T_d \mathcal{H}_K c_{w_K}$. Here each summand has dimension 1.

The module $\mathscr{S}(D_K,K)$ can be identified with $\mathcal{H}c_{w_K}$, with $\{\, T_d c_{w_K} \mid d \in D_K \,\}$ as the standard basis, since the bar involution on $\mathcal{H}$ fixes $T_1 c_{w_K}$, and for all $s \in S$ and $d \in D_K$,

$$T_s T_d c_{w_K} = \begin{cases} T_{sd} c_{w_K} & \text{if } sd \in D_K \text{ and } sd > d, \\ T_{sd} c_{w_K} + (q-q^{-1}) T_d c_{w_K} & \text{if } sd < d, \\ -q^{-1} T_d c_{w_K} & \text{if } sd = dt \text{ for some } t \in K, \end{cases}$$

in accordance with the requirements of Definition 4.2. The first of the three cases corresponds to $s \in \mathrm{SA}(D_K,d)$, the second to $s \in \mathrm{SD}(D_K,d)$, the third to $s \in \mathrm{WD}_K(D_K,d)$. It is immediate from the definition that $\mathrm{WA}_K(D_K,d)$ is always empty.

Note that if $d \in D_K$ and $s \in S$ then $sdw_K < dw_K$ if and only if either $sd < d$ or $sd = dt$ for some $t \in K$. Since $\tau^K(c_d) = \mathrm{SD}(D_K,d) \cup \mathrm{WD}_K(D_K,d)$ and $\tau(\varphi(c_d)) = \tau(c_{dw_K}) = \mathcal{L}(dw_K)$, this establishes the first assertion of the proposition.

It follows from Lemma 4.7 that the W-graph basis and standard basis of $\mathscr{S}(D_K,K)$ are related by

$$c_d^K = T_d c_{w_K} - q \sum_{e<d} p_{e,d}^K T_e c_{w_K} \qquad \text{for all } d \in D_K, \tag{5.3}$$

for some $p_{e,d}^K \in \mathcal{A}^+$. Moreover, the W-graph basis is the only basis of bar-invariant elements satisfying such a system of equations. Similarly, in $\mathscr{S}(W,\emptyset)$ we have

$$c_w = T_w - q \sum_{y<w} p_{y,w} T_y \qquad \text{for all } w \in D_K,$$

for some $p_{y,w} \in \mathcal{A}^+$. We apply this with $w = dw_K$, where $d \in D_K$, and group the terms on the right hand side according to cosets of W_K, thus obtaining the components of c_{dw_K} in the direct sum decomposition $\mathcal{H} = \bigoplus_{e\in D_K} T_e\mathcal{H}_K$. We find that

$$c_{dw_K} = T_d\big(T_{w_K} - q\sum_{v<w_K} p_{dv,dw_K}T_v\big) - q\sum_{e\in D_K,\,e<d} T_e\big(\sum_{v\in W_K} p_{ev,dw_K}T_v\big). \tag{5.4}$$

Since $c_{dw_K} \in \mathcal{H}c_{w_K}$ its component in each summand $T_e\mathcal{H}_K$ must lie in the one-dimensional subspace $T_e\mathcal{H}_K c_{w_S}$. So it follows that $T_{w_K} - q\sum_{v<w_K} p_{dv,dw_K}T_v$ and each $\sum_{v\in W_K} p_{ev,dw_K}T_v$ in Eq. (5.4) must be scalar multiples of $c_{w_K} = T_{w_K} - q\sum_{v<w_K}(-q)^{l(w_s)-l(v)-1}T_v$. So

$$c_{dw_K} = T_d c_{w_K} - q\sum_{e<d} p_{ew_K,dw_K}T_e c_{w_K}.$$

Comparing this with Eq. (5.3), uniqueness tells us that $c^K_d = c_{dw_K}$ for all $d\in D_K$, and that $p^K_{e,d} = p_{ew_K,dw_K}$ for all $e,d\in D_K$. Since $\mu^K(c_e,c_d)$ is the constant term of $p^K_{e,d}$ and $\mu(c_{ew_K},c_{dw_K})$ is the constant term of p_{ew_K,dw_K}, this establishes the other assertion of the proposition. □

Remark 5.14 The equation $p^K_{e,d} = p_{ew_K,dw_K}$, which is the key part of the above proof, is due to Deodhar [2, Proposition 3.4]. The proof also shows that $p_{ev,dw_K} = q^{l(w_K)-l(v)}p_{ew_K,dw_K}$ whenever $e,d\in D_K$ and $v\in W_K$, a fact that was already known.

6 *W*-graph biideals

It is clear from the defining presentation that the Hecke algebra $\mathcal{H}$ possesses an involutive antiautomorphism $h\mapsto h^\flat$ that fixes each element of the generating set $\{\,T_s \mid s\in S\,\}$. This can be used to convert left $\mathcal{H}$-modules into right $\mathcal{H}$-modules, and vice versa. The corresponding antiautomorphism of W, given by $w\mapsto w^{-1}$, maps ideals of $(W,\leqslant_{\mathsf{L}})$ to ideals of $(W,\leqslant_{\mathsf{R}})$, and vice versa. Since, moreover, $(\overline{T_s})^\flat = (T_s^{-1})^\flat = (T_s^\flat)^{-1} = T_s^{-1} = \overline{T_s} = (\overline{T_s^\flat})$ for all $s\in S$, it follows that $\overline{h^\flat} = (\overline{h})^\flat$ for all $h\in H$. So there is a theory of W-graph right ideals that is completely parallel to the theory of W-graph (left) ideals as presented above, with $(W,\leqslant_{\mathsf{R}})$ replacing $(W,\leqslant_{\mathsf{L}})$ and right $\mathcal{H}$-modules replacing left $\mathcal{H}$-modules. Just as W-graph ideals give rise to W-graphs, so W-graph right ideals give rise to W^{o}-graphs. If $\mathscr{I}\subseteq W$ and $K\subseteq S$ then $(\mathscr{I},K)$ is a W-graph right ideal if and only if $(\mathscr{I}^{-1},K)$ is a W-graph ideal.

If $(\mathscr{I},K)$ is a W-graph right ideal we write $\mathscr{S}^{\mathrm{o}}(\mathscr{I},K)$ for the associated right $\mathcal{H}$-module, $B^{\mathrm{o}} = \{\,b^{\mathrm{o}}_w \mid w\in\mathscr{I}\,\}$ for its standard basis and $C^{\mathrm{o}} = \{\,c^{\mathrm{o}}_w \mid w\in\mathscr{I}\,\}$ for its W^{o}-graph basis. The module $\mathscr{S}^{\mathrm{o}}(\mathscr{I},K)$ admits an $\mathcal{A}$-semilinear involution $\alpha\mapsto\underline{\alpha}$ such that $\underline{\alpha h} = \underline{\alpha}\overline{h}$ for all $h\in\mathcal{H}$ and $\alpha\in\mathscr{S}^{\mathrm{o}}(\mathscr{I},K)$ and $\underline{c^{\mathrm{o}}_w} = c^{\mathrm{o}}_w$ for all $w\in\mathscr{I}$. Moreover, as in Lemma 4.7, the c^{o}_w are uniquely determined by the requirements that $\underline{c^{\mathrm{o}}_w} = c^{\mathrm{o}}_w$ and $b^{\mathrm{o}}_w = c^{\mathrm{o}}_w + q\sum_{y<w} q^{\mathrm{o}}_{y,w}c^{\mathrm{o}}_y$ for some $q^{\mathrm{o}}_{y,w}\in\mathcal{A}^+$. We write $\mu^{\mathrm{o}}_{y,w}$ for the constant term of the polynomial $q^{\mathrm{o}}_{y,w}$.

Remark 6.1 If $(\mathscr{I},K)$ is a W-graph right ideal then the module $\mathscr{S}^{\mathrm{o}}(\mathscr{I},K)$ can be identified with $\mathscr{S}(\mathscr{I}^{-1},K)$, made into a right module by defining $\alpha h = h^\flat\alpha$ for all $\alpha\in\mathscr{S}(\mathscr{I}^{-1},K)$ and $h\in\mathcal{H}$. With this convention, $b^{\mathrm{o}}_w = b_{w^{-1}}$, and Eq. (4.1) says that for all $w\in\mathscr{I}$ and $s\in S$,

$$b^{\mathrm{o}}_w T_s = \begin{cases} b^{\mathrm{o}}_{ws} & \text{if } s\in \mathrm{SA}(w^{-1},\mathscr{I}^{-1}),\\ b^{\mathrm{o}}_{ws} + (q-q^{-1})b^{\mathrm{o}}_w & \text{if } s\in \mathrm{SD}(w^{-1},\mathscr{I}^{-1}),\\ -q^{-1}b^{\mathrm{o}}_w & \text{if } s\in \mathrm{WD}_K(w^{-1},\mathscr{I}^{-1}),\\ qb^{\mathrm{o}}_w - \sum_{y\in\mathscr{I},\,y<ws} r^s_{y^{-1},w^{-1}}b^{\mathrm{o}}_y & \text{if } s\in \mathrm{WA}_K(w^{-1},\mathscr{I}^{-1}), \end{cases} \tag{6.1}$$

where the coefficients $r^s_{y^{-1},w^{-1}}$ lie in $q\mathcal{A}^+$. Note that the first of these four cases corresponds to $w<ws\in\mathscr{I}$, the second to $w>ws$, the third to $ws\notin D_K^{-1}$, and the last to $ws\in D_K^{-1}\setminus\mathscr{I}$.

Remark 6.2 It is conceivably possible for some $\mathscr{I} \subseteq W$ to be simultaneously a W-graph ideal with respect to J and a W-graph right ideal with respect to K, where $J, K \subseteq S$. However, if this happens then $\mathscr{I}$ must be contained in the standard parabolic subgroup generated by the complement of $J \cup K$ in S. To see this, observe that since $\mathscr{I}$ is both an ideal of $(W, \leqslant_{\mathsf{L}})$ and an ideal of $(W, \leqslant_{\mathsf{R}})$, if $w \in \mathscr{I}$ and $u \in W$ has the property that there exist $x, y \in W$ with $w = xuy$ and $l(w) = l(x) + l(u) + l(y)$, then $u \in \mathscr{I}$. In particular, if $s \in S$ occurs in any reduced expression for any $w \in \mathscr{I}$ then $s \in \mathscr{I}$, whence $s \notin J \cup K$ (since $\mathscr{I} \subseteq D_J \cap D_K^{-1}$). Of course this will automatically hold if $J = K = \emptyset$.

If it is the case that $(\mathscr{I}, J)$ is a W-graph ideal and $(\mathscr{I}, K)$ is a W-graph right ideal then there is an $\mathcal{A}$-isomorphism from the left $\mathcal{H}$-module $\mathscr{S}(\mathscr{I}, J)$ to the right $\mathcal{H}$-module $\mathscr{S}^{\mathrm{o}}(\mathscr{I}, K)$ mapping the standard basis of $\mathscr{S}(\mathscr{I}, J)$ to the standard basis of $\mathscr{S}^{\mathrm{o}}(\mathscr{I}, K)$. It is therefore natural to ask whether it is possible to obtain an $(\mathcal{H}, \mathcal{H})$-bimodule by identifying b_w^{o} with b_w for all $w \in \mathscr{I}$. Accordingly, we make the following definition.

Definition 6.3 Let $\mathscr{I} \subseteq W$ and $J, K \subseteq S$, and suppose that $(\mathscr{I}, J)$ is a W-graph ideal and $(\mathscr{I}, K)$ is a W-graph right ideal. Identify $\mathscr{S}^{\mathrm{o}}(\mathscr{I}, K)$ with $\mathscr{S}(\mathscr{I}, J)$ by putting $b_w^{\mathrm{o}} = b_w$ for all $w \in \mathscr{I}$. We say that $\mathscr{I}$ is a *W-graph biideal with respect to J and K* (or that $(\mathscr{I}, J, K)$ is a W-graph biideal) if $\mathscr{S} = \mathscr{S}(\mathscr{I}, J) = \mathscr{S}^{\mathrm{o}}(\mathscr{I}, K)$ is an $(\mathcal{H}, \mathcal{H})$-bimodule with the left and right $\mathcal{H}$-actions defined in Eq. (4.1) and Eq. (6.1).

Notation. When $(\mathscr{I}, J, K)$ is a W-graph biideal the $(\mathcal{H}, \mathcal{H})$-bimodule $\mathscr{S}(\mathscr{I}, J) = \mathscr{S}^{\mathrm{o}}(\mathscr{I}, K)$ will be denoted by $\mathscr{S}(\mathscr{I}, J, K)$.

Suppose now that $(\mathscr{I}, J)$ is simultaneously a W-graph ideal and a W-graph right ideal, and that $\mathscr{S} = \mathscr{S}(\mathscr{I}, J) = \mathscr{S}^{\mathrm{o}}(\mathscr{I}, J)$ with $b_w^{\mathrm{o}} = b_w$ for all $w \in \mathscr{I}$. By Remark 4.3 and its analogue for the right action, we see that $T_w b_1 = b_w = b_1 T_w$ for all $w \in \mathscr{I}$. The following result shows that $(\mathscr{I}, J, J)$ is a W-graph biideal if and only if $T_w b_1 = b_1 T_w$ for all $w \in W$.

Lemma 6.4 *With the assumptions of the above preamble, $\mathscr{S}$ is an $(\mathcal{H}, \mathcal{H})$-bimodule if and only if $hb_1 = b_1 h$ for all $h \in \mathcal{H}$.*

Proof Suppose first that $hb_1 = b_1 h$ for all $h \in \mathcal{H}$. Then for all $h, g \in \mathcal{H}$, we have

$$(hb_1)g = (b_1 h)g = b_1(hg) = (hg)b_1 = h(gb_1) = h(b_1 g). \tag{6.2}$$

Now let w be an arbitrary element of $\mathscr{I}$. By Remark 4.3 we have $b_w = T_w b_1$, and so it follows from Eq. (6.2) that for all $h, g \in \mathcal{H}$,

$$h(b_w g) = h((T_w b_1)g) = h(T_w(b_1 g)) = (hT_w)(b_1 g) = ((hT_w)b_1)g = (h(T_w b_1))g = (hb_w)g.$$

Since $\{ b_w \mid w \in \mathscr{I} \}$ spans $\mathscr{S}$ it follows from this that $h(\alpha g) = (h\alpha)g$ for all $h, g \in \mathcal{H}$ and $\alpha \in \mathscr{S}$, whence $\mathscr{S}$ is an $(\mathcal{H}, \mathcal{H})$-bimodule, as required.

Conversely, suppose that $\mathscr{S}$ is a $(\mathcal{H}, \mathcal{H})$-bimodule. We must show that $hb_1 = b_1 h$ for all $h \in \mathcal{H}$, and since $\{ T_w \mid w \in W \}$ spans $\mathcal{H}$ it suffices to show that $T_w b_1 = b_1 T_w$ for all $w \in W$. We use induction on $l(w)$ to do this. The case $l(w) = 0$ is trivial. For the inductive step, suppose that $l(w) > 0$ and write $w = sv$ with $s \in S$ and $l(v) = l(w) - 1$. By Eq. (4.1) we find that

$$T_s b_1 = \begin{cases} b_s & \text{if } s \in \mathscr{I}, \\ -q^{-1} b_1 & \text{if } s \notin D_J, \\ q b_1 & \text{if } s \in D_J \setminus \mathscr{I}, \end{cases} \tag{6.3}$$

and by Eq. (6.1) it follows that $b_1 T_s = T_s b_1$ (since $s \notin D_J^{-1}$ if and only if $s \notin D_J$, as $s = s^{-1}$). Hence, by the inductive hypothesis and the assumption that $\mathscr{S}$ is a bimodule, it follows that

$$T_w b_1 = (T_s T_v) b_1 = T_s(T_v b_1) = T_s(b_1 T_v) = (T_s b_1) T_v = (b_1 T_s) T_v = b_1 (T_s T_v) = b_1 T_w$$

as required. □

If $(\mathscr{I}, J, K)$ is a W-graph biideal then the bimodule $\mathscr{S}(\mathscr{I}, J, K) = \mathscr{S}(\mathscr{I}, J) = \mathscr{S}^{\circ}(\mathscr{I}, K)$ possesses a W-graph basis $C = \{\, c_w \mid w \in \mathscr{I} \,\}$ and a W°-graph basis $C^{\circ} = \{\, c_w^{\circ} \mid w \in \mathscr{I} \,\}$. By Lemma 4.7 the c_w are characterized by the properties that $\overline{c_w} = c_w$ and $b_w = c_w + q \sum_{y<w} q_{y,w} c_y$ for some $q_{y,w} \in \mathcal{A}^+$, and similarly the c_w° are characterized by the properties that $\underline{c_w^{\circ}} = c_w^{\circ}$ and $b_w = c_w^{\circ} + q \sum_{y<w} q_{y,w}^{\circ} c_y^{\circ}$ for some $q_{y,w}^{\circ} \in \mathcal{A}^+$. It follows that if $\underline{\alpha} = \overline{\alpha}$ for all $\alpha \in \mathscr{S}(\mathscr{I}, J, K)$ then the W-graph basis C and the W°-graph basis C° coincide.

Proposition 6.5 *If $(\mathscr{I}, J, K)$ is a W-graph biideal then $\underline{\alpha} = \overline{\alpha}$ for all $\alpha \in \mathscr{S}(\mathscr{I}, J, K)$.*

Proof We use induction on $l(w)$ to show that $\underline{b_w} = \overline{b_w}$ for all $w \in \mathscr{I}$. Since the case $l(w) = 0$ is trivial, assume that $l(w) > 0$ and let $w = sv$ with $s \in S$ and $l(v) = l(w) - 1$. Note that since $\mathscr{I}$ is an ideal of $(W, \leqslant_{\mathsf{L}})$ and of $(W, \leqslant_{\mathsf{R}})$, both v and s are elements of $\mathscr{I}$. Observe that

$$\overline{T_s} b_1 = (T_s - (q - q^{-1})) b_1 = b_s - (q - q^{-1}) b_1 = b_1 (T_s - (q - q^{-1})) = b_1 \overline{T_s}.$$

Hence, by the inductive hypothesis and the fact that $\mathscr{S}(\mathscr{I}, J, K)$ is a bimodule, we find that

$$\begin{aligned} \underline{b_w} &= \underline{b_1 T_w} = b_1 \overline{T_w} = b_1 \overline{T_s T_v} = b_1 (\overline{T_s}\, \overline{T_v}) = (b_1 \overline{T_s}) \overline{T_v} \\ &= (\overline{T_s} b_1) \overline{T_v} = \overline{T_s} (b_1 \overline{T_v}) = \overline{T_s} (\underline{b_1 T_v}) = \overline{T_s} \underline{b_v} = \overline{T_s}\, \overline{b_v} = \overline{T_s} (\overline{T_v b_1}) \\ &= \overline{T_s} (\overline{T_v} b_1) = (\overline{T_s}\, \overline{T_v}) b_1 = \overline{T_s T_v} b_1 = \overline{T_w} b_1 = \overline{T_w b_1} = \overline{b_w} \end{aligned}$$

as required. □

So if $(\mathscr{I}, J, K)$ is a W-graph biideal then it is indeed true that $C = C^{\circ}$. Moreover, we also see that $q_{y,w}^{\circ} = q_{y,w}$ for all $y, w \in \mathscr{I}$ with $y < w$, and hence $\mu_{y,w}^{\circ} = \mu_{y,w}$ for all $y, w \in \mathscr{I}$ with $y < w$. It follows from this that $\Gamma = (C, \mu, \tau)$ is a $(W \times W^{\circ})$-graph, where μ is defined by

$$\mu(c_y, c_w) = \begin{cases} \mu_{y,w} & \text{if } y < w \\ \mu_{w,y} & \text{if } w < y \\ 0 & \text{otherwise,} \end{cases}$$

and τ is defined by $\tau(c_w) = \mathrm{D}_J(w, \mathscr{I}) \sqcup \mathrm{D}_K(w^{-1}, \mathscr{I}^{-1})^{\circ}$ for all $w \in \mathscr{I}$.

Theorem 6.6 *If $(\mathscr{I}, J, K)$ is a W-graph biideal, then the triple $\Gamma = (C, \mu, \tau)$ defined in the above preamble is a $(W \times W^{\circ})$-graph.*

Remark 6.7 The work of Kazhdan and Lusztig [8] shows that $(W, \emptyset, \emptyset)$ is a W-graph biideal.

Remark 6.8 With the notation as in Theorem 6.6, let $\tau_{\mathsf{L}} \colon C \to \mathcal{P}(S)$ and $\tau_{\mathsf{R}} \colon C \to \mathcal{P}(S^{\circ})$ be defined by $\tau_{\mathsf{L}}(c) = \tau(c) \cap S$ and $\tau_{\mathsf{R}}(c) = \tau(c) \cap S^{\circ}$ for all c in C, so that $\Gamma_{\mathsf{L}} = (C, \mu, \tau_{\mathsf{L}})$ is the W-graph $\Gamma(\mathscr{I}, J)$ and $\Gamma_{\mathsf{R}} = (C, \mu, \tau_{\mathsf{R}})$ is the W°-graph $\Gamma(\mathscr{I}, K)$. As in Section 3 above, the functions μ and τ determine a preorder $\leqslant_{\Gamma}$ on C; we call the corresponding equivalence classes the two-sided cells of C. Similarly Γ_{L} and Γ_{R} yield preorders $\leqslant_{\Gamma_{\mathsf{L}}}$ and $\leqslant_{\Gamma_{\mathsf{R}}}$ on C; the corresponding equivalence classes are called the left cells and right cells of C.

Remark 6.9 It is obvious from the definitions that if $(\mathscr{I}, J, K)$ is a *W*-graph biideal then so is $(\mathscr{I}^{-1}, K, J)$. If $f\colon \mathscr{S}(\mathscr{I}, J, K) \to \mathscr{S}(\mathscr{I}^{-1}, K, J)$ is the $\mathcal{A}$-isomorphism defined by $f(b_w) = b_{w^{-1}}$ then $hf(b) = f(bh^\flat)$ and $f(b)h = f(h^\flat b)$ for all $b \in \mathscr{S}(\mathscr{I}, J, K)$ and $h \in \mathcal{H}$. Furthermore, for all $y, w \in \mathscr{I}$, the polynomial $q_{y,w}$ for $(\mathscr{I}, J, K)$ equals the polynomial $q^{\mathrm{o}}_{y^{-1},w^{-1}} = q_{y^{-1},w^{-1}}$ for $(\mathscr{I}^{-1}, K, J)$. So, in the important special case that $\mathscr{I} = \mathscr{I}^{-1}$ and $J = K$, we have $q_{y^{-1},w^{-1}} = q_{y,w}$ for all $y, w \in \mathscr{I}$. This corresponds to the well known identity $P_{y^{-1},w^{-1}} = P_{y,w}$ for Kazhdan–Lusztig polynomials, established in [9, 5.6].

In Definition 6.3, the requirement that $\mathscr{S}$ is a $(\mathcal{H},\mathcal{H})$-bimodule is not implied by other requirements, as the following example shows.

Example 6.10 Let *W* be the Weyl group of type A_2, with $S = \{s,t\}$. We shall show that $(\mathscr{I}, J) = (\{1,t\}, \{s\})$ is both a *W*-graph ideal and a *W*-graph right ideal, but $(\mathscr{I}, J, J)$ is not a *W*-graph biideal.

Recall first that $D_J = \{1,t,st\}$, and that (D_J, J) is a *W*-graph ideal (by [6, Theorem 9.2]). Let $C = \{c_1, c_t, c_{st}\}$ be the *W*-graph basis of the corresponding $\mathcal{H}$-module. Since *s* is a strong descent of *st* and *t* is a weak descent of *st*, it follows that $T_s c_{st} = T_t c_{st} = -q^{-1} c_{st}$. So the set $\{st\}$ is a (D_J, J)-closed subset of D_J, and it follows by Theorem 5.2 that $\mathscr{I}$ is a (strong) *W*-graph subideal of (D_J, J) (since $\mathscr{I} = D_J \setminus \{st\}$). In particular, $(\mathscr{I}, J)$ is a *W*-graph ideal. Since $\mathscr{I} = \mathscr{I}^{-1}$ we conclude that $(\mathscr{I}, J)$ is also a *W*-graph right ideal.

Suppose, for a contradiction, that $(\mathscr{I}, J, J)$ is a *W*-graph biideal, and let $\Gamma = (C, \mu, \tau)$ be the corresponding $(W \times W^{\mathrm{o}})$-graph, defined as in the preamble to Theorem 6.6. Thus $C = \{c_1, c_t\}$ is an $\mathcal{A}$-basis for M_Γ, which is an $(\mathcal{H},\mathcal{H})$-bimodule. Since $\mathrm{D}_J(\mathscr{I}, 1) = J = \{s\}$ and $\mathrm{D}_J(\mathscr{I}, t) = \{t\}$ it follows that $\tau(c_1) = \{s, s^{\mathrm{o}}\}$ and $\tau(c_t) = \{t, t^{\mathrm{o}}\}$, and since it is immediate from Corollary 4.8 that $\mu_{1,t} = q_{1,t} = 1$ we conclude that

$$\begin{aligned} T_s c_1 &= c_1 T_s = -q^{-1} c_1, \\ T_s c_t &= c_t T_s = c_1 + q c_t, \end{aligned} \qquad \text{and} \qquad \begin{aligned} T_t c_1 &= c_1 T_t = q c_1 + c_t, \\ T_t c_t &= c_t T_t = -q^{-1} c_t. \end{aligned}$$

The observation that $(T_s c_1) T_t = -q^{-1} c_1 T_t \neq T_s (c_1 T_t)$ gives the desired contradiction.

Definition 6.11 Suppose that $(\mathscr{I}, J, K)$ and $(\mathscr{I}_0, J_0, K_0)$ are *W*-graph biideals. We say that $\mathscr{I}$ is a *W-graph subbiideal of* $\mathscr{I}_0$ if $\mathscr{I} \subseteq \mathscr{I}_0$ and $(J, K) = (J_0, K_0)$.

The following result is the biideal analogue of Theorem 5.2.

Theorem 6.12 *Let $(\mathscr{I}_0, J, K)$ be a W-graph biideal with corresponding $(W \times W^{\mathrm{o}})$-graph $\Gamma = (C_0, \mu, \tau)$, so that $C_0 = \{\, c^0_w \mid w \in \mathscr{I}_0 \,\}$ is an $\mathcal{A}$-basis of the bimodule $\mathscr{S}_0 = \mathscr{S}(\mathscr{I}_0, J, K)$. Let $\mathscr{I} \subseteq \mathscr{I}_0$ be such that $\{\, c^0_w \mid w \in \mathscr{I}_0 \setminus \mathscr{I} \,\} \subseteq C_0$ is closed with respect to the (two-sided) preorder $\leqslant_\Gamma$ on C_0. Then $(\mathscr{I}, J, K)$ is a W-graph biideal, and the $(W \times W^{\mathrm{o}})$-graph $\Gamma(\mathscr{I}, J, K)$ is isomorphic to the full subgraph of Γ on the vertex set $\{\, c^0_w \mid w \in \mathscr{I} \,\} \subseteq C_0$, with μ and τ functions inherited from Γ.*

Proof Since the set $C' = \{\, c^0_w \mid w \in \mathscr{I}_0 \setminus \mathscr{I} \,\}$ is closed with respect to $\leqslant_\Gamma$, it follows from the theory described in Section 3 that $\mathcal{A}C'$ is an $(\mathcal{H},\mathcal{H})$-bimodule, and also that C' is closed with respect to the left and right preorders $\leqslant_{\Gamma_{\mathsf{L}}}$ and $\leqslant_{\Gamma_{\mathsf{R}}}$ defined as in Remark 6.8 above. Hence it follows from Theorem 5.2 that $(\mathscr{I}, J)$ is a *W*-graph ideal and also that $(\mathscr{I}, K)$ is a *W*-graph right ideal. Moreover, by Remark 5.4 the left $\mathcal{H}$-module $\mathscr{S}(\mathscr{I}, J)$ and the right $\mathcal{H}$-module $\mathscr{S}^{\mathrm{o}}(\mathscr{I}, K)$ can both be identified with $\mathscr{S}_0/\mathcal{A}C'$ (which is an $(\mathcal{H},\mathcal{H})$-bimodule), with the standard basis of $\mathscr{S}(\mathscr{I}, J)$ and that of $\mathscr{S}^{\mathrm{o}}(\mathscr{I}, K)$ both equal to $\{\, f(b^0_w) \mid w \in \mathscr{I} \,\}$, where $\{\, b_w \mid w \in \mathscr{I}_0 \,\}$ is the standard basis of $\mathscr{S}_0$ and f is the natural map $\mathscr{S}_0 \to \mathscr{S}_0/\mathcal{A}C'$. Hence $(\mathscr{I}, J, K)$ is a *W*-graph biideal, by Definition 6.3. The remaining assertions follow from Theorem 5.2 and its right ideal analogue applied to $(\mathscr{I}_0, J)$ and $(\mathscr{I}_0, K)$. □

Remark 6.13 Let $(\mathscr{I}, J, K)$ be a W-graph biideal and $C = \{\, c_w \mid w \in \mathscr{I} \,\}$ the $(W \times W^{\circ})$-graph basis of $\Gamma = \Gamma(\mathscr{I}, J, K)$. In keeping with the conventions we adopted in the preamble to Proposition 5.10 above, we say that a subset X of $\mathscr{I}$ is $(\mathscr{I}, J, K)$-closed if $\{\, c_x \mid x \in X \,\}$ is closed with respect to the preorder $\leqslant_\Gamma$, and call X a two-sided cell of $(\mathscr{I}, J, K)$ if $\{\, c_x \mid x \in X \,\}$ is a cell of Γ. Clearly $\leqslant_\Gamma$ induces a partial ordering on the set of two-sided cells, and $X \subseteq \mathscr{I}$ is $(\mathscr{I}, J, K)$-closed if and only if it is a union of two-sided that form an ideal with respect to this order. Theorem 6.12 shows that the complement in $\mathscr{I}$ of any such union is a W-graph biideal with respect to J and K.

7 Computational characterization of *W*-graph ideals

Let (W,S) be a Coxeter system, $\mathscr{I}$ an ideal of $(W, \leqslant_{\mathsf{L}})$ and J a subset of $\mathrm{Pos}(\mathscr{I})$. We know that if $(\mathscr{I},J)$ is a W-graph ideal then we can construct an $\mathcal{H}$-module that has an $\mathcal{A}$-basis $\{\, c_w \mid w \in \mathscr{I} \,\}$ on which the generators of $\mathcal{H}$ via the formulas given in Theorem 4.12, where the parameters $\mu_{y,w}$ are the constant terms of a family of polynomials $q_{y,w}$ that can be computed recursively using the formulas in Corollary 4.8. In this section we prove the converse: if $(\mathscr{I},J)$ gives rise to an $\mathcal{H}$-module via this construction then $(\mathscr{I},J)$ must be a W-graph ideal.

Note that if $(\mathscr{I},J)$ is not a W-graph ideal then the polynomials $q_{y,w}$ are not necessarily uniquely determined by the formulas in Corollary 4.8. If $z \in \mathscr{I}$ and the $q_{y,w}$ have been found for all $y, w \in \mathscr{I}$ with $y < w < z$, then computing the polynomials $q_{y,z}$ involves first choosing some $s \in \mathrm{SD}(z)$, so that $z = sw$ with $w < z$, after which the formulas for $q_{y,sw}$ can be applied. A different sequence of choices of the elements $s \in \mathrm{SD}(z)$ could conceivably produce a different family of polynomials. We show that if some sequence of choices produces polynomials that give rise to an $\mathcal{H}$-module then $(\mathscr{I},J)$ must be a W-graph ideal. So, to be precise, our assumptions are as follows:

(A1) $\mathscr{I}$ is an ideal of $(W, \leqslant_{\mathsf{L}})$ and $J \subseteq \mathrm{Pos}(\mathscr{I})$, and $\mathscr{S}$ is an $\mathcal{A}$-free $\mathcal{H}$-module;

(A2) $\mathscr{S}$ has an $\mathcal{A}$-basis $C = \{\, c_w \mid w \in \mathscr{I} \,\}$ in bijective correspondence with $\mathscr{I}$, such that for certain integers $\mu_{y,w}$

$$T_s c_w = \begin{cases} -q^{-1} c_w & \text{if } s \in \mathrm{D}(w), \\ q c_w + \sum_{y \in \mathcal{R}(s,w)} \mu_{y,w} c_y & \text{if } s \in \mathrm{WA}(w), \\ q c_w + c_{sw} + \sum_{y \in \mathcal{R}(s,w)} \mu_{y,w} c_y & \text{if } s \in \mathrm{SA}(w), \end{cases}$$

where the set $\mathcal{R}(s,w)$ consists of all $y \in \mathscr{I}$ such that $y < w$ and $s \in \mathrm{D}(y)$;

(A3) there exist polynomials $q_{y,w} \in \mathcal{A}^+$, defined whenever $y, w \in \mathscr{I}$, such that $\mu_{y,w}$ is the constant term of $q_{y,w}$, and $q_{y,w} = 0$ whenever $y \not< w$;

(A4) for each $z \in \mathscr{I}$ with $z \neq 1$ there exists $s \in S$ with $l(sz) < l(z)$ such that $q_{sz,z} = 1$, and for all $y \in \mathscr{I}$ with $y < z$ we have

(1) $q_{y,z} = q q_{y,sz}$ if $s \in \mathrm{A}(y)$,

(2) $q_{y,z} = -q^{-1}(q_{y,sz} - \mu_{y,sz}) + q_{sy,sz} + \sum_x \mu_{y,x} q_{x,sz}$ if $s \in \mathrm{SD}(y)$,

(3) $q_{y,z} = -q^{-1}(q_{y,sz} - \mu_{y,sz}) + \sum_x \mu_{y,x} q_{x,sz}$ if $s \in \mathrm{WD}(y)$,

where the sums in (2) and (3) extend over all $x \in \mathscr{I}$ such that $y < x < sz$ and $s \notin \mathrm{D}(x)$.

The conclusion is that $(\mathscr{I},J)$ is a W-graph ideal. The proof consists of showing that the module $\mathscr{S}$ satisfies the conditions of Definition 4.2.

Since C is an $\mathcal{A}$-basis of $\mathscr{S}$ there is an $\mathcal{A}$-semilinear involution $\alpha \mapsto \overline{\alpha}$ on $\mathscr{S}$ such that $\overline{c_w} = c_w$ for all $w \in \mathscr{I}$. Since $\overline{T_s - q} = T_s - q$ and $\overline{T_s + q^{-1}} = T_s + q^{-1}$it follows from assumption (A2) that $\overline{T_s\overline{c_w}} = \overline{T_s c_w}$ in each of the three cases, and hence $\overline{h\alpha} = \overline{h}\overline{\alpha}$ for all $h \in \mathcal{H}$ and $\alpha \in \mathscr{S}$. The remaining task is to show that $\mathscr{S}$ has an $\mathcal{A}$-basis $\{\, b_w \mid w \in \mathscr{I}\,\}$ such that the formulas in Eq. (4.1) hold. We define $b_w = T_w c_1$ for all $w \in \mathscr{I}$, and observe first that Eq. (4.1) is satisfied in three of the four cases.

Proposition 7.1 *Let $w \in \mathscr{I}$ and $s \in S$, and suppose that $s \notin \mathrm{WA}(w)$. Then*

$$T_s b_w = \begin{cases} b_{sw} & \text{if } s \in \mathrm{SA}(w),\\ b_{sw} + (q-q^{-1})b_w & \text{if } s \in \mathrm{SD}(w),\\ -q^{-1}b_w & \text{if } s \in \mathrm{WD}(w).\end{cases}$$

Proof If $s \in \mathrm{SA}(w)$ then $w < sw \in \mathscr{I}$, by the definition of $\mathrm{SA}(w)$, and by the definition of b_w and b_{sw} it follows that $T_s b_w = T_s(T_w c_1) = (T_s T_w)c_1 = T_{sw}c_1 = b_{sw}$, as required.

If $s \in \mathrm{SD}(w)$ then $s \in \mathrm{SA}(sw)$, and so from the case we have just done it follows that $T_s b_w = T_s(T_s b_{sw}) = T_s^2 b_{sw} = (1+(q-q^{-1})T_s)b_{sw} = b_{sw} + (q-q^{-1})b_w$, as required.

Now suppose that $s \in \mathrm{WD}(w)$. Since this gives $w \in D_J$ and $sw \notin D_J$, it follows from Lemma 2.4 that $l(sw) = l(w)+1$ and $sw = wt$ for some $t \in J$. So $T_sT_w = T_{sw} = T_{wt} = T_wT_t$. Furthermore, $T_t c_1 = -q^{-1}c_1$, since $t \in J = \mathrm{WD}(1)$. Hence

$$T_s b_w = T_s(T_w c_1) = (T_s T_w)c_1 = (T_w T_t)c_1 = T_w(T_t c_1) = -q^{-1}T_w c_1 = -q^{-1}b_w,$$

as required. □

Lemma 7.2 *We have $b_z = c_z + q\sum_{\{y\in\mathscr{I}\mid y<z\}} q_{y,z}c_y$ for all $z \in \mathscr{I}$.*

Proof The proof is by induction on $l(z)$, the case $l(z) = 0$ being trivial. So we assume that $l(z) > 1$, and choose s as in assumption (A4) above. We write

$$\begin{aligned} \mathcal{R} &= \{\, x \in \mathscr{I} \mid x < sz \text{ and } s \in \mathrm{D}(x)\,\},\\ \mathcal{T}_1 &= \{\, x \in \mathscr{I} \mid x < sz \text{ and } s \in \mathrm{SA}(x)\,\},\\ \mathcal{T}_2 &= \{\, x \in \mathscr{I} \mid x < sz \text{ and } s \in \mathrm{WA}(x)\,\},\end{aligned}$$

so that $\mathcal{R}$ is the set $\mathcal{R}(s,sz)$ of assumption (A2) above, and we also write $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$. The inductive hypothesis gives $b_{sz} = c_{sz} + q\sum_{x<sz} q_{x,sz}c_x$, and Proposition 7.1 gives $b_z = T_s b_{sz}$, since $s \in \mathrm{SA}(sz)$. So, using (A2) to evaluate $T_s c_{sz}$ and $T_s c_x$ for $x \in \mathcal{R}$,

$$\begin{aligned} b_z &= T_s c_{sz} + q\sum_{x\in\mathcal{R}} q_{x,sz}T_s c_x + q\sum_{x\in\mathcal{T}_1} q_{y,sz}T_s c_x + q\sum_{x\in\mathcal{T}_2} q_{y,sz}T_s c_x\\ &= (c_z + qc_{sz} + \sum_{x\in\mathcal{R}} \mu_{x,sz}c_x) - \sum_{y\in\mathcal{R}} q_{x,sz}c_x + q\sum_{x\in\mathcal{T}_1} q_{x,sz}T_s c_x + q\sum_{x\in\mathcal{T}_2} q_{y,sz}T_s c_x\\ &= c_z + qc_{sz} - \sum_{x\in\mathcal{R}} (q_{x,sz} - \mu_{x,sz})c_x + q\sum_{x\in\mathcal{T}_1} q_{x,sz}T_s c_x + q\sum_{x\in\mathcal{T}_2} q_{x,sz}T_s c_x.\end{aligned}$$

Now using (A2) to evaluate $T_s c_x$ for $x \in \mathcal{T}_1$ and $x \in \mathcal{T}_2$, and making use of the similarity between the two formulas, we find that

$$b_z - c_z = qc_{sz} - \sum_{x\in\mathcal{R}} (q_{x,sz} - \mu_{x,sz})c_x + q\sum_{x\in\mathcal{T}_1} q_{x,sz}c_{sx} + q\sum_{x\in\mathcal{T}} q_{x,sz}\Big(qc_x + \sum_{y\in\mathcal{R}(s,x)} \mu_{y,x}c_y\Big).$$

We proceed to collect the coefficients of the various elements of C in the right hand side. Note first that if $x \in \mathcal{T}_1$ then $sx \in \mathscr{I}$ (since $s \in \mathrm{SA}(x)$), and Lemma 2.1 implies that $sx < z$, since $x < sz < z$. So all the elements of C that appear have the form c_y with $y < z$. Writing coeff(y) for the coefficient of c_y, the aim is to show that $\mathrm{coeff}(y) = qq_{y,z}$.

Let $y \in \mathscr{I}$ with $y < z$, and suppose first that $s \in \mathrm{A}(y)$. Then $y < sy$, and so $y \leqslant sz$ by Lemma 2.1. So either $y = sz$ and $\mathrm{coeff}(y) = q$, or else $y \in \mathcal{T}$ and $\mathrm{coeff}(y) = q^2 q_{y,sz}$. In either case $\mathrm{coeff}(y) = qq_{y,z}$, by assumption (A4).

Now suppose that $s \in \mathrm{WD}(y)$. Then $y \notin \{\, sx \mid x \in \mathcal{T}_1 \,\}$, since $sy \notin \mathscr{I}$. So c_y occurs only in the the first sum in our expression and in the double sum. Hence

$$\mathrm{coeff}(y) = -(q_{y,sz} - \mu_{y,sz}) + \sum_x q\mu_{y,x} q_{x,sz}$$

where x runs through all elements of $\mathcal{T}$ such that $y \in \mathcal{R}(s,x)$. Again we see from assumption (A4) that $\mathrm{coeff}(y) = qq_{y,z}$.

Finally, suppose that $s \in \mathrm{SD}(y)$. In this case $y = sx$ with $x \in \mathcal{T}_1$, so that we obtain a term $q_{sy,sz} c_y$ in addition to the terms obtained in the case $s \in \mathrm{WD}(y)$. So again $\mathrm{coeff}(y) = qq_{y,z}$, as required. □

The following result completes the proof that Eq. (4.1) is satisfied.

Proposition 7.3 *Let $w \in \mathscr{I}$ and $s \in \mathrm{WD}(w)$. Then $T_s b_w = qb_w + \sum_{\{y \in \mathscr{I} \mid y < sw\}} r^s_{y,w} b_y$ for some polynomials $r^s_{y,w} \in q\mathcal{A}^+$.*

Proof Define $\mathcal{R} = \{\, y \in \mathscr{I} \mid y < w \text{ and } s \in \mathrm{D}(y) \,\}$, so that $\mathcal{R} = \mathcal{R}(s,w)$, and define also $\mathcal{T}_1 = \{\, y \in \mathscr{I} \mid y < w \text{ and } s \in \mathrm{SA}(y) \,\}$ and $\mathcal{T}_2 = \{\, y \in \mathscr{I} \mid y < w \text{ and } s \in \mathrm{WA}(y) \,\}$. In addition, let $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$. Since $b_w = c_w + q\sum_{y<w} q_{y,w} c_y$ we see from assumption (A2) that

$$\begin{aligned}
T_s b_w &= T_s c_w + \sum_{y \in \mathcal{R}} qq_{y,w} T_s c_y + \sum_{y \in \mathcal{T}_1} qq_{y,w} T_s c_y + \sum_{y \in \mathcal{T}_2} qq_{y,w} T_s c_y \\
&= (qc_w + \sum_{y \in \mathcal{R}} \mu_{y,w} c_y) - \sum_{y \in \mathcal{R}} q_{y,w} c_y + \sum_{y \in \mathcal{T}_1} qq_{y,w} c_{sy} + \sum_{y \in \mathcal{T}} qq_{y,w} \Big(qc_y + \sum_{x \in \mathcal{R}(s,y)} \mu_{x,y} c_x \Big) \\
&= qc_w - \sum_{y \in \mathcal{R}} (q_{y,w} - \mu_{y,w}) c_y + \sum_{y \in \mathcal{T}_1} qq_{y,w} c_{sy} + \sum_{y \in \mathcal{T}} qq_{y,w} \Big(qc_y + \sum_{x \in \mathcal{R}(s,y)} \mu_{x,y} c_x \Big).
\end{aligned}$$

Since $\mu_{y,w}$ is the constant term of $q_{y,w}$, every element of C appearing in the above expression has coefficient lying in $q\mathcal{A}^+$. So, using Lemma 2.1 and the fact that $w < sw$ (since $s \in \mathrm{WA}(w)$), it follows that

$$T_s b_w = \sum_{x < sw} t_{x,w} c_x \qquad \text{for some } t_{x,w} \in q\mathcal{A}^+. \tag{7.1}$$

Inverting the system of equations in Lemma 7.2 shows that for all $x \in \mathscr{I}$ there exist $p_{y,x} \in \mathcal{A}^+$ such that $c_x = b_x - q\sum_{y<x} p_{y,x} b_y$, and substituting this into Eq. (7.1) gives the required result, with $r^s_{y,w} = t_{y,w} - q\sum_{\{x \mid y<x<sw\}} p_{y,x} t_{x,w}$. □

We have now shown that all the requirements of Definition 4.2 are satisfied, and so $(\mathscr{I}, J)$ is a W-graph ideal. So we have proved the following theorem.

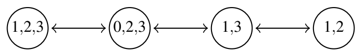
Theorem 7.4 *Let $\mathscr{I}$ be an ideal of $(W, \leqslant_{\mathsf{L}})$ and $J \subseteq \mathrm{Pos}(\mathscr{I})$. Then $(\mathscr{I}, J)$ is a W-graph ideal if and only if the construction described in Section 4 above produces a W-graph (C, μ, τ) such that Theorem 4.12 is satisfied.*

Remark 7.5 According to the construction, $C = \{ c_w \mid w \in \mathscr{I} \}$ and $\tau(w) = \mathrm{D}_J(\mathscr{I}, w)$ for all $w \in \mathscr{I}$. The function μ is defined as in Eq. (4.3), where $\mu_{y,w}$ is the constant term of $q_{y,w}$, and these polynomials satisfy the formulas in Corollary 4.8. In fact we showed that if (C, μ, τ) is a W-graph then the conclusion that $(\mathscr{I}, J)$ is a W-graph ideal needs only the weaker assumption that the $q_{y,w}$ are computed using (A4) above. Given that (C, μ, τ) is a W-graph, it is not hard to show that Theorem 4.12 is satisfied if and only if the statement of Corollary 4.15 holds.

To conclude this section we give an example of an ideal $\mathscr{I}$ of $(W, \leqslant_{\mathsf{L}})$ and a subset J of $\mathrm{Pos}(\mathscr{I})$ such that $(\mathscr{I}, J)$ is not a W-graph ideal, but nevertheless has the property that there exists a W-graph (C, μ, τ) with $C = \{ c_w \mid w \in \mathscr{I} \}$ and $\tau(c_w) = \mathrm{D}_J(\mathscr{I}, w)$ for all $w \in \mathscr{I}$.

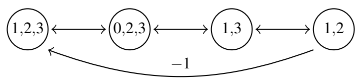
Example 7.6 Let (W, S) be the Coxeter system of type B_4, and let $S = \{s_0, s_1, s_2, s_3\}$, where $s_0 s_1$ has order 4 and $s_1 s_2$ and $s_2 s_3$ have order 3. Let $\mathscr{I} = \{1, s_0, s_1 s_0, s_2 s_1 s_0\}$ and note that $\mathscr{I} \subseteq D_J$, where $J = \{s_1, s_2, s_3\}$. We use Theorem 7.4 to determine whether or not $(\mathscr{I}, J)$ is a W-graph ideal. The first step is to compute the polynomials $q_{y,w}$, for all $y, w \in \mathscr{I}$ with $y < w$, using the formulas given in Corollary 4.8 (or (A4) above).

It is immediate that the three cases with $l(w) - l(y) = 1$ give $q_{y,w} = 1$. For the next case, let $(y, w) = (1, s_1 s_0)$, and observe that s_1 is the only strong descent of w. Since $s_1 \in \mathrm{WD}_J(y)$, the third formula of Corollary 4.8 applies, and gives $q_{1,s_1 s_0} = q^{-1}(q_{1,s_0} - \mu_{1,s_0}) = 0$. There are now two remaining possibilities for (y, w), both with $w = s_2 s_1 s_0$. Observe that s_2 is the only strong descent of w, and $s_2 \in \mathrm{WD}_J(y)$ for both values of y, namely $y = s_0$ and $y = 1$. Furthermore, in both cases $\{x \in \mathscr{I} \mid y < x < s_1 s_0 \text{ and } s_2 \notin \mathrm{D}(x)\}$ is empty, and so it follows that $q_{y,w} = q^{-1}(q_{y,s_1 s_0} - \mu_{y,s_1 s_0}) = 0$. So the graph obtained is



where the numbers in the circles give the values of $\mathrm{D}_J(w)$ for the various elements $w \in \mathscr{I}$, and the edges all have weight 1.

It is easily checked that the above graph is not a W-graph: the relation $T_{s_0} T_{s_3} = T_{s_3} T_{s_0}$ fails. So $(\mathscr{I}, J)$ is not a W-graph ideal. However, adding an edge of weight -1 joining the vertices 1 and $s_2 s_1 s_0$ gives



and it is easily checked that this is a W-graph for which the formulas in Theorem 4.12 hold.

8 Parabolic restriction

Let $(\mathscr{I}, J)$ be a W-graph ideal and let $K \subseteq S$. Let $\mathcal{H}_K$ be the subalgebra of $\mathcal{H}$ generated by $\{T_s \mid s \in K\}$. In this section we investigate the restriction of $\mathscr{S}(\mathscr{I}, J)$ to $\mathcal{H}_K$. (As we noted in Section 2 above, $\mathcal{H}_K$ can be identified with the Hecke algebra of the Coxeter system (W_K, K).) Let $\{ b_w \mid w \in \mathscr{I} \}$ be the standard basis of $\mathscr{S}(\mathscr{I}, J)$ and $\{ c_w \mid w \in \mathscr{I} \}$ the W-graph basis.

Each element $w \in W$ has a unique factorization $w = vd$ with $v \in W_K$ and $d \in D_K^{-1}$. Since $l(w) = l(v) + l(d)$ necessarily holds in this situation, it follows that $d \leqslant_{\mathsf{L}} w$. So $d \in \mathscr{I}$ whenever $w \in \mathscr{I}$. For each $d \in D_K^{-1}$ define $\mathscr{I}_d \subseteq W_K$ by $\mathscr{I}_d = \{ v \in W_K \mid vd \in \mathscr{I} \}$, so that

$$\mathscr{I} = \bigsqcup_{d \in D_K^{-1} \cap \mathscr{I}} \mathscr{I}_d d. \tag{8.1}$$

and $\mathscr{I}_d d = W_K d \cap \mathscr{I}$ in each case. Note that since $\mathscr{I} \subseteq D_J$, each d appearing in Eq. (8.1) is in $D_{K,J} = D_K^{-1} \cap D_J$, the set of minimal (W_K, W_J) double coset representatives.

Lemma 8.1 *Let $d \in D_K^{-1} \cap \mathscr{I}$. Then $\mathscr{I}_d$ is an ideal of $(W_K \leqslant_{\mathsf{L}})$, and $K \cap dJd^{-1} \subseteq \mathrm{Pos}(\mathscr{I}_d)$.*

Proof Let $w \in \mathscr{I}_d$ and let $v \in W_K$ with $v \leqslant_{\mathsf{L}} w$, so that $w = uv$ with $l(w) = l(u) + l(v)$. Since $v, w \in W_K$ and $d \in D_K^{-1}$ we have $l(wd) = l(w) + l(d)$ and $l(vd) = l(v) + l(d)$. Hence $wd = u(vd)$ and $l(wd) = l(w) + l(d) = l(u) + l(v) + l(d) = l(u) + l(vd)$. Since $wd \in \mathscr{I}$ (since $w \in \mathscr{I}_d$) it follows that $vd \in \mathscr{I}$, and hence that $v \in \mathscr{I}_d$. So $\mathscr{I}_d$ is an ideal of $(W_K \leqslant_{\mathsf{L}})$.

Now let $v \in \mathscr{I}_d$, so that $v \in W_K$ and $vd \in \mathscr{I}$, and let $s \in K \cap dJd^{-1}$, so that $s \in K$ and $sd = dr$ for some $r \in J$. Since $J \subseteq \mathrm{Pos}(\mathscr{I})$ it follows that $l((vd)r) > l(vd)$, and since $d \in D_K^{-1}$ and $v, vs \in W_K$ we find that $l(vs) + l(d) = l(vsd) = l(vdr) > l(vd) = l(v) + l(d)$. Hence $l(vs) > l(v)$, and we conclude that $K \cap dJd^{-1} \subseteq \mathrm{Pos}(\mathscr{I}_d)$. □

For each $d \in D_K^{-1} \cap \mathscr{I}$ let $\mathscr{J}_d \subseteq \mathscr{I}$ be defined by $\mathscr{J}_d = \bigcup_e \mathscr{I}_e e$, where e runs through the set $\{\, e \in D_K^{-1} \mid e \leqslant d \,\}$, and let $\mathscr{J}'_d = \mathscr{J}_d \setminus \mathscr{I}_d d$. Let $\mathscr{S}_d$ and $\mathscr{S}'_d$ be the $\mathscr{A}$-submodules of $\mathscr{S}(\mathscr{I}, J)$ spanned by $\{\, c_w \mid w \in \mathscr{J}_d \,\}$ and $\{\, c_w \mid w \in \mathscr{J}'_d \,\}$ respectively. Thus $\mathscr{S}'_d \subseteq \mathscr{S}_d$, and the quotient module $\mathscr{S} = \mathscr{S}_d / \mathscr{S}'_d$ has $\mathcal{A}$-basis $\{\, f(c_{wd}) \mid w \in \mathscr{I}_d \,\}$, where f is the natural homomorphism $\mathscr{S}_d \to \mathscr{S}$.

Clearly $\mathscr{S}_d$ and $\mathscr{S}'_d$ are both stable under the bar involution of $\mathscr{S}(\mathscr{I}, J)$, since $\overline{c_w} = c_w$ for all $w \in \mathscr{I}$. Hence $\mathscr{S}$ admits a bar involution such that $\overline{f(\alpha)} = f(\overline{\alpha})$ for all $\alpha \in \mathscr{S}_d$.

Lemma 8.2 *Let $y, w \in \mathscr{I}$ with $y \leqslant w$, and suppose that $d \in D_K^{-1} \cap \mathscr{I}$. If $w \in \mathscr{J}_d$ then $y \in \mathscr{J}_d$, and if $w \in \mathscr{J}'_d$ then $y \in \mathscr{J}'_d$.*

Proof Let $y \in W_K e$ and $w \in W_K e'$, where $e, e' \in D_K^{-1}$. Since $y \leqslant w$ it follows that $e \leqslant e'$, by Proposition 2.3. If $w \in \mathscr{J}_d$ then we have $e' \leqslant d$, by the definition of $\mathscr{J}_d$, so that $e \leqslant d$ and $y \in \mathscr{I}_e e \subseteq \mathscr{J}_d$. If $w \in \mathscr{J}'_d$ then $e' < d$, giving $e < d$ and $y \in \mathscr{J}'_d$. □

The following lemma is the key result in this section.

Lemma 8.3 *Let $d \in D_K^{-1} \cap \mathscr{I}$. Then $\mathscr{S}_d$ and $\mathscr{S}'_d$ are both $\mathcal{H}_K$-submodules of $\mathscr{S}(\mathscr{I}, J)$.*

Proof Let $w \in \mathscr{J}_d$, so that $w \in \mathscr{I}e = W_K d \cap \mathscr{I}$ for some $e \in D_K^{-1}$ with $e \leqslant d$, and let $s \in K$. If $sw \in \mathscr{I}$ then $sw \in \mathscr{I}e \subseteq \mathscr{J}_d$, since $sw \in sW_K d = W_K d$. If $y \in \mathscr{I}$ and $y < w$ then $y \in \mathscr{J}_d$, by Lemma 8.2. By Theorem 4.12 we see that $T_s c_w$ is an $\mathcal{A}$-linear combination of terms that all lie in $\{\, c_w \mid w \in \mathscr{J}_d \,\}$. So it follows that this set spans an $\mathcal{H}_K$-submodule of $\mathscr{S}(\mathscr{I}, J)$. The proof of the other part is the same, with $\mathscr{J}_d$ replaced by $\mathscr{J}'_d$. □

Observe that if $d \in D_K^{-1} \cap \mathscr{I}$ and $w \in \mathscr{J}_d$ then $b_w \in \mathscr{S}_d$, since $b_w = c_w + q\sum_{y<w} q_{y,w} c_y$, and Lemma 8.2 shows that each y involved is in $\mathscr{J}_d$. The same applies with $\mathscr{J}_d$ replaced by $\mathscr{J}'_d$ and $\mathscr{S}_d$ by $\mathscr{S}'_d$. It follows that the sets $\{\, b_w \mid w \in \mathscr{J}_d \,\}$ and $\{\, b_w \mid w \in \mathscr{J}'_d \,\}$ are $\mathcal{A}$-bases of $\mathscr{S}_d$ and $\mathscr{S}'_d$, and $\{\, f(b_{wd}) \mid w \in \mathscr{I}_d \,\}$ is an $\mathcal{A}$-basis of $\mathscr{S}$.

We are now able to prove the main result of this section.

Theorem 8.4 *Let $(\mathscr{I}, J)$ be a W-graph ideal. Suppose that $K \subseteq S$ and $d \in D_K^{-1} \cap \mathscr{I}$. Then $\mathscr{I}_d = \{v \in W_K \mid vd \in \mathscr{I}\}$ is a W_K-graph ideal with respect to $L = K \cap dJd^{-1}$.*

Proof It was proved in Lemma 8.1 that $\mathscr{I}_d$ is an ideal of $(W_K, \leqslant_{\mathsf{L}})$ and that $L \subseteq \mathrm{Pos}(\mathscr{I}_d)$. We proceed to show that Definition 4.2 is satisfied with $\mathscr{S}$ as $\mathscr{S}(\mathscr{I}_d, K \cap dJd^{-1})$ and with $\{\, f(b_{wd}) \mid w \in \mathscr{I}_d \,\}$ as its standard basis (where, as above, $\mathscr{S} = \mathscr{S}_d / \mathscr{S}'_d$ and $f \colon \mathscr{S}_d \to \mathscr{S}$ is the natural map).

Note that $f(b_d) = f(c_d)$, since $f(c_y) = 0$ for all $y \in \mathscr{I}$ with $y < d$. Hence $\overline{f(b_d)} = f(b_d)$, and since also

$$\overline{hf(\alpha)} = \overline{f(h\alpha)} = f(\overline{h\alpha}) = f(\overline{h}\overline{\alpha}) = \overline{h}f(\overline{\alpha}) = \overline{h}\,\overline{f(\alpha)}$$

for all $\alpha \in \mathcal{S}_d$ and $h \in \mathcal{H}_K$, it follows that condition (ii) in Definition 4.2 is satisfied. It remains to check that the generators T_s of $\mathcal{H}_K$ act on the basis elements $f(b_{wd})$ in accordance with the requirements of Eq. (4.1).

Let $s \in K$ and $w \in \mathscr{I}_d$, and suppose first that $s \in \mathrm{SA}(\mathscr{I}_d, w)$. Then $l(sw) > l(w)$ and $sw \in \mathscr{I}_d$. So $s(wd) = (sw)d \in \mathscr{I}$, and $l(s(wd)) = l(sw) + l(d) > l(w) + l(d) = l(wd)$. So $s \in \mathrm{SA}(\mathscr{I}, wd)$, and so $T_s b_{wd} = b_{s(wd)}$. Applying f to both sides gives $T_s f(b_{wd}) = f(b_{(sw)d})$, as required.

Suppose next that $s \in \mathrm{SD}(\mathscr{I}_d, w)$. Then $s \in \mathrm{SA}(\mathscr{I}_d, sw)$, and by the case just done we see that $T_s f(b_{wd}) = T_s^2 f(b_{(sw)d}) = (1 + (q - q^{-1})T_s) f(b_{(sw)d}) = f(b_{(sw)d}) + (q - q^{-1}) f(b_{wd})$, as required.

Now suppose that $s \in \mathrm{WD}_L(\mathscr{I}_d, w)$. This means that $sw \notin D_L$, whereas $w \in D_L$. So $l(sw) > l(w)$ and $sw = ws'$ for some $s' \in L$, by Lemma 2.4. Since the definition of L gives $s'd = dr$ for some $r \in J$ we see that $wd \in \mathscr{I} \subseteq D_J$ but $s(wd) = (wd)r \notin D_J$. So $s \in \mathrm{WD}(\mathscr{I}, wd)$, giving $T_s b_{wd} = -q^{-1} b_{wd}$, and applying f to this gives $T_s f(b_{wd}) = -q^{-1} f(b_{wd})$, as required.

Finally, suppose that $s \in \mathrm{WA}_L(\mathscr{I}_d, w)$, so that $sw \in D_L \setminus \mathscr{I}_d$. Since $sw \in W_K$ it follows that $swd \in (W_K \cap D_L)d = D^K_{K \cap dJd^{-1}} d \subseteq D_J$, by Lemma 2.5, since $d \in D_{K,J}$. Furthermore, since $sw \in W_K$ and $sw \notin \mathscr{I}_d$ it follows that $swd \notin \mathscr{I}$. So $s \in \mathrm{WA}_J(\mathscr{I}, wd)$, and therefore

$$T_s b_{wd} = q b_{wd} - \sum_{y < swd} r^s_{y,wd} b_y \qquad \text{for some } r^s_{y,wd} \in q\mathcal{A}^+. \tag{8.2}$$

Since $T_s b_{wd} \in \mathscr{S}_d$, if b_y has nonzero coefficient in the right hand side of Eq. (8.2) then $y \in \mathscr{J}_d$. But $f(b_y) = 0$ if $y \in \mathscr{J}'_d = \mathscr{J}_d \setminus \mathscr{I}_d d$. So applying f to Eq. (8.2) gives

$$T_s f(b_{wd}) = q f(b_{wd}) - \sum_y r^s_{yd,wd} f(b_y)$$

where the sum is over elements $y \in \mathscr{I}_d$ such that and $yd < swd$. Since $l(yd) = l(y) + l(d)$ and $l(swd) = l(sw) + l(d)$ it follows that $yd < swd$ if and only if $y < sw$ (by Lemma 2.2). So

$$T_s f(b_{wd}) = q f(b_{wd}) - \sum_{y \in \mathscr{I}_d,\, y < sw} r^s_{yd,wd} f(b_y)$$

which is of the required form. □

Corollary 8.5 *Let J and K be subsets of S and suppose that $w \in W$ is a W-graph determining element associated with J. If $w = vd$ with $v \in W_K$ and $d \in D_K^{-1}$ then v is a W_K-graph determining element associated with $K \cap dJd^{-1}$.*

Proof Let $\mathscr{I} = \{x \in W \mid x \leqslant_{\mathsf{L}} w\}$, so that $(\mathscr{I}, J)$ is a W-graph ideal. Clearly $d \in D_K^{-1} \cap \mathscr{I}$, since $d \leqslant_{\mathsf{L}} w$, and it follows from Theorem 8.4 that $(\mathscr{I}_d, K \cap dJd^{-1})$ is a W_K-graph ideal, where $\mathscr{I}_d = \{y \in W_K \mid yd \leqslant_{\mathsf{L}} w\}$. But $yd \leqslant_{\mathsf{L}} vd$ if and only if $y \leqslant_{\mathsf{L}} v$, since $y, v \in W_K$ and $d \in D_K^{-1}$. So $\mathscr{I}_d = \{y \in W_K \mid y \leqslant_{\mathsf{L}} v\}$, and the result follows. □

Remark 8.6 Let $\Gamma = (C, \mu, \tau) = \Gamma(\mathscr{I}, J)$, the W-graph obtained from the W-graph ideal $(\mathscr{I}, J)$, and let $K \subseteq S$. By Eq. (8.1) the vertex set $C = \{\, c_w \mid w \in \mathscr{I} \,\}$ is expressible as a disjoint union $\bigsqcup_d C_d$, where $C_d = \{\, c_{wd} \mid w \in \mathscr{I}_d \,\}$ and d runs through $D_K^{-1} \cap \mathscr{I}$. Let $\tau_K \colon C \to \mathcal{P}(K)$ be defined by $\tau_K(c) = \tau(c) \cap K$ for all $c \in C$, so that $\Delta = (C, \mu, \tau_K)$ is a W_K-graph, with M_Δ isomorphic to the restriction of M_Γ to $\mathcal{H}_K$. For each $d \in D_K^{-1} \cap \mathscr{I}$ let Δ_d be the full subgraph of Δ spanned by C_d. It is clear from the results in this section that Δ_d is a union of cells of Δ, and spans W_K-graph isomorphic to $\Gamma(\mathscr{I}_d, K \cap dJd^{-1})$.

In particular, it follows from Remark 8.6 that if V is a closed subset of C (so that V spans an $\mathcal{H}$-submodule of M_Γ) then $V \cap C_d$ is a closed subset of C_d. Hence we obtain the following result, which is, in a sense, dual to Theorem 5.9.

Theorem 8.7 *Let $(\mathscr{L}, J)$ be a strong W-graph subideal of the W-graph ideal $(\mathscr{I}, J)$, and let $K \subseteq S$. For each $d \in D_K^{-1} \cap \mathscr{L}$ let $\mathscr{L}_d = \{\, w \in W_K \mid wd \in \mathscr{L} \,\}$ and $\mathscr{I}_d = \{\, w \in W_K \mid wd \in \mathscr{I} \,\}$. Then $(\mathscr{L}_d, K \cap dJd^{-1})$ is a strong W_K-graph subideal of $(\mathscr{I}_d, K \cap dJd^{-1})$.*

Proof Definition 5.3 and Theorem 5.2 show that $\Gamma(\mathscr{L}, J)$ can be identified with the full subgraph of $\Gamma(\mathscr{I}, J)$ spanned by $\{\, c_w \mid w \in \mathscr{L} \,\}$, and that $V = \{\, c_w \mid w \in \mathscr{I} \setminus \mathscr{L} \,\}$ is a closed subset of C. Hence $V \cap C_d$ is a closed subset of C_d. Since $V \cap C_d = \{\, c_{wd} \mid w \in \mathscr{I}_d \setminus \mathscr{L}_d \,\}$, the result follows immediately from Definition 5.3 and Theorem 5.2. □

9 *W*-graph ideals for Coxeter groups of rank 2

Our main aim in this section is to determine all W-graph ideals for finite Coxeter groups of rank 2. Accordingly, we assume henceforth that W is the group generated by $S = \{s, t\}$ subject to the defining relations $s^2 = t^2 = (st)^m = 1$, where $m \geqslant 2$.

Notation. Whenever x and y are elements of a semigroup we define $[..xy]_k$ to be $(xy)^{k/2}$ if k is even and to be $y(xy)^{(k-1)/2}$ if k is odd.

Using this notation, $[..st]_m = [..ts]_m$ is the longest element of W, and every other element of W has a unique expression of the form $[..st]_l$ or $[..ts]_l$ with $l < m$. Note that

$$D_{\{s\}} = \{\, [..st]_l \mid l < m \,\},$$
$$D_{\{t\}} = \{\, [..ts]_l \mid l < m \,\}.$$

We assume henceforth that that $J \subseteq S$ and that $\emptyset \neq \mathscr{I} \subseteq D_J$ is an ideal of $(W, \leqslant_{\mathsf{L}})$. Recall from [6, Section 8] that $(\mathscr{I}, J)$ is a W-graph ideal if $\mathscr{I} = D_J$, and note that if $J = \{s, t\}$ then $D_{\{s,t\}} = \{1\}$, forcing $\mathscr{I} = D_J$.

Suppose now that $J = \{s\}$, and note that we must have

$$\mathscr{I} = \{\, [..st]_l \mid l \leqslant k \,\}$$

for some integer k with $0 \leqslant k \leqslant m-1$. Let w be an arbitrary element of $\mathscr{I}$ and let $l(w) = l$. If $l = 0$ then $sw = s \notin D_{\{s\}}$ and $w < tw = t \in \mathscr{I}$, giving $s \in \mathrm{WD}(w)$ and $t \in \mathrm{SA}(w)$. If $0 < l < k$ then $\{sw, tw\} = \{[..st]_{l-1}, [..st]_{l+1}\} \subset \mathscr{I}$; so $s \in \mathrm{SD}(w)$ and $t \in \mathrm{SA}(w)$ if l is even, $s \in \mathrm{SA}(w)$ and $t \in \mathrm{SD}(w)$ if l is odd. If $l = k < m-1$ the same conclusion holds with $\mathrm{SA}(w)$ replaced by $\mathrm{WA}(w)$, since in this case $[..st]_{l+1} \in D_{\{s\}} \setminus \mathscr{I}$. If $l = k = m-1$, which means that $\mathscr{I} = D_{\{s\}}$, then $s \in \mathrm{SD}(w)$ and $t \in \mathrm{WD}(w)$ if l is even, vice versa if l is odd.

It is now relatively straightforward to use (A3) and (A4) of Section 7 to compute the polynomials $q_{y,z}$ for $(\mathscr{I}, J) = (\mathscr{I}, \{s\})$.

Lemma 9.1 *With $\mathscr{I}$ and J as above, suppose that $y, z \in \mathscr{I}$ with $l(y) < l(z)$. Then*

$$q_{y,z} = \begin{cases} 1 & \text{if } l(z) - l(y) = 1, \\ 0 & \text{if } l(z) - l(y) > 1. \end{cases}$$

Proof The proof proceeds by induction on $l(z)$. If $l(z) = 1$ then $z = t$ and $y = 1$, and (A4) immediately gives $q_{y,z} = 1$, as required.

For the inductive step, suppose first that $l(z)$ is even. Then $s \in \mathrm{D}(z)$, and sz is the only element of $\mathscr{I}$ whose length is $l(z) - 1$. Since (A4) immediately gives $q_{sz,z} = 1$, it suffices to prove that $q_{y,z} = 0$ if $l(y) < l(z) - 1$.

If $l(y)$ is odd then $s \in \mathrm{A}(y)$, and $l(y) < l(z) - 1$ gives $l(y) \leqslant l(z) - 3 < l(sz) - 1$. So the inductive hypothesis gives $q_{y,sz} = 0$, and by (1) of (A4) it follows that $q_{y,z} = qq_{y,sz} = 0$.

Assume now that $l(y)$ is even, so that $s \in \mathrm{D}(y)$. Since $l(y) \leqslant l(z) - 2 < l(sz)$ the inductive hypothesis tells us that $q_{y,sz}$ is a constant, and so $q^{-1}(q_{y,sz} - \mu_{y,sz}) = 0$. If $s \in \mathrm{SD}(y)$ then $l(sy) = l(y) - 1 < l(z) - 1 = l(sz)$, and the inductive hypothesis gives $q_{sy,sz} = 0$. So whether $s \in \mathrm{SD}(w)$ or $s \in \mathrm{SA}(w)$ we have $q_{y,z} = \sum_x \mu_{y,x} q_{x,sz}$, where the sum extends over $x \in \mathscr{I}$ such that $y < x < sz$ and $s \notin \mathrm{D}(x)$. But $s \notin \mathrm{D}(x)$ implies that $l(x)$ is even, giving $l(x) < l(sz) - 1$, since $l(sz)$ is also even. Since this gives $q_{x,sz} = 0$ by the inductive hypothesis it follows that all the terms in the sum are 0, and $q_{y,z} = 0$, as required.

If $l(z)$ odd then the same proof applies, with odd and even swapped and with s replaced by t. This completes the induction. □

It follows from Lemma 9.1 and the discussion preceding it that if $k < m - 1$ then the construction produces a graph of the form

$$\textcircled{s} \longleftrightarrow \textcircled{t} \longleftrightarrow \textcircled{s} \longleftrightarrow \textcircled{t} \longleftrightarrow \cdots$$

where the number of vertices is $k + 1$ and all edges have weight 1. In other words, if we let $V = \{v_1, v_2, \ldots, v_{k+1}\}$ be the vertex set, then $\tau\colon V \to \mathcal{P}(S)$ is given by

$$\tau(v_i) = \begin{cases} \{s\} & \text{if } i \text{ is odd,} \\ \{t\} & \text{if } i \text{ is even,} \end{cases}$$

and the integer $\mu(v_i, v_j)$ is 1 whenever $|i - j| = 1$ and is 0 whenever $|i - j| > 1$. It follows from Theorem 7.4 that $(\mathscr{I}, J)$ is a W-graph ideal if and only if $\Gamma = (V, \mu, \tau)$ is a W-graph.

Note that if $k = m - 2$ then $\mathscr{I} = D_J \setminus \{[..st]_{m-1}\}$. In this case it follows from results already obtained $(\mathscr{I}, J)$ is a W-graph ideal. Indeed, we saw in Section 5 that (D_J, J) is a W-graph ideal, and since $\mathrm{D}([..st]_{m-1}) = \{s, t\}$ (as noted in the discussion above), it follows that the set $\{[..st]_{m-1}\}$ is (D_J, J)-closed. Hence $(D_J \setminus \{[..st]_{m-1}\}, J)$ is a strong W-graph subideal of (D_J, J).

The next lemma shows that (V, μ, τ) is a W-graph if and only if $k + 2$ is a divisor of m.

Lemma 9.2 *Let M be a free $\mathcal{A}$-module with $\mathcal{A}$-basis $V = \{v_1, \ldots, v_{k+1}\}$, where $k \geqslant 0$, and for each $r \in \{s, t\}$ let $\phi_r\colon M \to M$ be the $\mathcal{A}$-homomorphism satisfying*

$$\phi_r(v_i) = \begin{cases} -q^{-1} v_i & \text{if } \tau(v_i) = \{r\} \\ q v_i + \sum\limits_{j \in \mathcal{R}_i} v_j & \text{if } \tau(v_i) \neq \{r\} \end{cases}$$

where $\mathcal{R}_i = \{i-1, i+1\} \cap \{1, 2, \ldots, k+1\}$. Then the relation $\phi_r^2 = 1 + (q - q^{-1})\phi_r$ is satisfied for both values of $r \in \{s, t\}$, and $[..\phi_s\phi_t]_n = [..\phi_t\phi_s]_n$ if and only if n is a multiple of $k + 2$.

Proof Observe that if $\tau(v_i) \neq \{r\}$ then $\tau(v_j) = \{r\}$ for all $j \in \mathcal{R}_i$. It follows by a trivial calculation that $\phi_r^2 = 1 + (q - q^{-1})\phi_r$.

If $m = k+2$ then M is isomorphic to the $\mathcal{H}$-module M_Γ, where $\Gamma = \Gamma(D_J \setminus \{[..st]_{m-1}\}, J)$, with T_s acting via ϕ_s and T_t acting via ϕ_t. Hence $[..\phi_s\phi_t]_{k+2} = [..\phi_t\phi_s]_{k+2}$. It follows from this that also $[..\phi_s\phi_t]_n = [..\phi_t\phi_s]_n$ whenever n is a multiple of $k+2$. It remains to prove the converse: if $[..\phi_s\phi_t]_n = [..\phi_t\phi_s]_n$ then n is a multiple of $k+2$.

So assume that $[..\phi_s\phi_t]_n = [..\phi_t\phi_s]_n$. If $k = 0$ then $\phi_s(v_1) = -q^{-1}v_1$ and $\phi_t(v_1) = qv_1$, and it follows that if $n = 2l+1$ is odd then $[..\phi_s\phi_t]_n = (-1)^l\phi_s \neq (-1)^l\phi_t = [..\phi_t\phi_s]_n$, contrary to our hypothesis. So n is even, as required.

Assume now that $k \geqslant 1$. It is convenient to regard M as embedded in a $\mathbb{C}[q,q^{-1}]$-module with basis V, and extend ϕ_s and ϕ_t to $\mathbb{C}[q,q^{-1}]$-endomorphisms of this module. Let ζ be a primitive $2(k+2)$-th root of unity, and write $\theta_k = \zeta^k - \zeta^{-k}$ for all integers k.

Define $u_1 = \sum_{i\in O}\theta_i v_i$ and $u_2 = \sum_{i\in E}\theta_i v_i$, where O and E are respectively the set of odd integers in $\{1,2,\ldots,k+1\}$ and the set of even integers in $\{1,2,\ldots,k+1\}$. It is easily seen that $\phi_s(u_1) = -q^{-1}u_1$ and $\phi_t(u_2) = -q^{-1}u_2$, while

$$\phi_s(u_2) = qu_2 + \sum_{i\in O}(\theta_{i+1} - \theta_{i-1})v_i$$
$$\phi_t(u_1) = qu_1 + \sum_{i\in E}(\theta_{i+1} - \theta_{i-1})v_i$$

since $\theta_0 = \theta_{k+2} = 0$. Now since $\theta_{i+1} - \theta_{i-1} = (\zeta + \zeta^{-1})\theta_i$ it follows that the two-dimensional submodule spanned by $\{u_1, u_2\}$ is preserved by both ϕ_s and ϕ_t, which act via the the following two matrices:

$$F_s = \begin{pmatrix} -q^{-1} & \zeta + \zeta^{-1} \\ 0 & q \end{pmatrix}, \qquad F_t = \begin{pmatrix} q & 0 \\ \zeta + \zeta^{-1} & -q^{-1} \end{pmatrix}.$$

Since $[..\phi_s\phi_t]_n = [..\phi_t\phi_s]_n$ it follows that $[..F_sF_t]_n = [..F_tF_s]_n$. This must remain valid on specializing to $q = 1$, in which case $F_s^2 = F_t^2 = 1$ and $(F_sF_t)^n = ([..F_tF_s]_n)^{-1}[..F_sF_t]_n = 1$. But since

$$F_sF_t = \begin{pmatrix} \zeta^2 + \zeta^{-2} + 1 & -(\zeta + \zeta^{-1})q^{-1} \\ (\zeta + \zeta^{-1})q & -1 \end{pmatrix}$$

and the eigenvalues of this are ζ^2 and ζ^{-2}, it follows that $(\zeta^2)^n = 1$. Since ζ^2 is a primitive $(k+2)$-th root of 1 we conclude that $k+2$ is a divisor of n, as required. □

Suppose now that $J = \emptyset$, so that $D_J = W$. Since we know that $(W, \emptyset)$ is a W-graph ideal, we assume that $\mathscr{I}$ is an ideal of $(W, \leqslant_{\mathsf{L}})$ such that $\mathscr{I} \neq W$. Then

$$\mathscr{I} = \mathscr{I}_{h,k} = \{\, [..st]_l \mid l \leqslant h \,\} \cup \{\, [..ts]_l \mid l \leqslant k \,\}$$

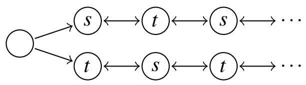
for some $h, k \in \{0,1,2,\ldots,m-1\}$. Since $D_J = W$ there are no weak descents. So $\mathrm{D}(1) = \emptyset$, and for every other $w \in \mathscr{I}$ we have either $\mathrm{D}(w) = \{s\}$ (if the reduced expression for w starts with s) or $\mathrm{D}(w) = \{t\}$ (if it starts with t).

For the purposes of applying Theorem 7.4 we need to find the integers $\mu_{y,w}$ that appear in (A2) of Section 7. This means that $y < w$ and $\mathrm{D}(y) \nsubseteq \mathrm{D}(w)$. Clearly we may as well assume that $\mathrm{D}(w) = \{s\}$ and $\mathrm{D}(y) = \{t\}$.

Lemma 9.3 *Let $\mathscr{I} = \mathscr{I}_{k,h}$ (as defined above) and let $J = \emptyset$. Let y, w be elements of $\mathscr{I}$ with $\mathrm{D}_J(\mathscr{I}, w) = \{s\}$ and $\mathrm{D}_J(\mathscr{I}, y) = \{t\}$, and $0 < l(y) < l(w)$. Then $\mu_{y,w} = 1$ if $l(w) - l(y) = 1$, and $\mu_{y,w} = 0$ otherwise.*

Proof If $l(w)-l(y)=1$ then $y=sw$, and it is immediate from (A4) of Section 7 that $\mu_{y,w}=q_{y,w}=1$. If $l(w)-l(y)>1$ then case (1) of (A4) applies, since $s\in \mathrm{A}(y)$, and so $q_{y,w}=qq_{y,sw}$. So the constant term of $q_{y,w}$ is zero, as required. □

So, after removing superfluous edges, the graph produced by application of our algorithm to $(\mathscr{I}_{h,k},\emptyset)$ has the form



where there are $h+k+1$ vertices, k in the top row and h in the bottom row, and all edges have weight 1. In other words, if we let $V=\{\,v_i \mid 1\leqslant i\leqslant k\,\}\cup\{x\}\cup\{\,u_i \mid 1\leqslant i\leqslant h\,\}$ be the vertex set, where the v_i correspond to the top row and the u_i to the bottom row, and temporarily let $v_0=x$ and $v_{-i}=u_i$ for $1\leqslant i\leqslant h$, then $\tau\colon V\to\mathcal{P}(S)$ is given by $\tau(v_0)=\emptyset$ and

$$\tau(v_i)=\begin{cases}\{s\} & \text{if } i \text{ is odd and positive or even and negative,}\\ \{t\} & \text{if } i \text{ is even and positive or odd and negative,}\end{cases}$$

and the integer $\mu(v_i,v_j)$ is 1 whenever $|i-j|=1$ and is 0 whenever $|i-j|>1$. It follows from Theorem 7.4 that $(\mathscr{I}_{h,k},\emptyset)$ is a W-graph ideal if and only if $\Gamma=(V,\mu,\tau)$ is a W-graph.

Note that in the particular case $h=0$ and $k=m-1$ we have $\mathscr{I}_{h,k}=D_{\{s\}}$, and it follows from [6, Proposition 8.3] that $(\mathscr{I}_{h,k},\emptyset)$ is a W-graph ideal.

Our next lemma shows that, in the general case, $(\mathscr{I}_{h,k},\emptyset)$ is a W-graph ideal if and only if $h+1$ and $k+1$ are both divisors of m.

Lemma 9.4 *Let M be a free $\mathcal{A}$-module with $\mathcal{A}$-basis $\{x\}\sqcup\{u_1,u_2,\ldots,u_h\}\sqcup\{v_1,v_2,\ldots,v_k\}$, and put $u_0=v_0=u_{h+1}=v_{k+1}=0$. Let ϕ_s and ϕ_t be $\mathcal{A}$-endomorphisms of M satisfying the following rules:*

(i) $\phi_s(x)=qx+v_1$ *and* $\phi_t(x)=qx+u_1$,

(ii) $\phi_s(v_i)=-q^{-1}v_i$ *if i is odd, and* $\phi_s(v_i)=qv_i+v_{i-1}+v_{i+1}$ *if i is even,*

(iii) $\phi_s(u_i)=-q^{-1}u_i$ *if i is even, and* $\phi_s(u_i)=qu_i+u_{i-1}+u_{i+1}$ *if i is odd,*

(iv) $\phi_t(v_i)=-q^{-1}v_i$ *if i is even, and* $\phi_t(v_i)=qv_i+v_{i-1}+v_{i+1}$ *if i is odd,*

(v) $\phi_t(u_i)=-q^{-1}u_i$ *if i is odd, and* $\phi_t(u_i)=qu_i+u_{i-1}+u_{i+1}$ *if i is even.*

Then $\phi_s^2=1+(q-q^{-1})\phi_s$ and $\phi_t^2=1+(q-q^{-1})\phi_t$, and $[..\phi_s\phi_t]_n=[..\phi_t\phi_s]_n$ if and only if $h+1$ and $k+1$ are both divisors of n.

Proof Checking that $\phi_s^2=1+(q-q^{-1})\phi_s$ and $\phi_t^2=1+(q-q^{-1})\phi_t$ is straightforward.

If $h=0$ and $m=k+1$ then M is isomorphic to the $\mathcal{H}$-module M_Γ, where $\Gamma=\Gamma(D_{\{s\}},\emptyset)$, with T_s acting via ϕ_s and T_t acting via ϕ_t. Hence $[..\phi_s\phi_t]_{k+1}=[..\phi_t\phi_s]_{k+1}$ if $h=0$. It follows from this that also $[..\phi_s\phi_t]_n=[..\phi_t\phi_s]_n$ whenever $h=0$ and n is a multiple of $k+1$. Similarly, $[..\phi_s\phi_t]_n=[..\phi_t\phi_s]_n$ whenever $k=0$ and n is a multiple of $h+1$.

Turning to the general case, let M_U be the $\mathcal{A}$-submodule of M spanned by $\{u_1,u_2,\ldots,u_h\}$ and let M_V be the $\mathcal{A}$-submodule of M spanned by $\{v_1,v_2,\ldots,v_k\}$. Note that M_U and M_V are both invariant under ϕ_s and ϕ_t. Let G_s and G_t be the matrices of ϕ_s and ϕ_t on M_U, relative to the ordered basis $(u_h,u_{h-1},\ldots,u_1)$, and let F_s and F_t be the matrices of ϕ_s and ϕ_t on M_V, relative to the ordered basis $(v_1,v_2,\ldots,v_k)$. Then the matrices of ϕ_s and ϕ_t on M relative to the ordered basis $(u_h,u_{h-1},\ldots,u_1,x,v_1,\ldots,v_{k-1},v_k)$ are

$$H_s=\begin{bmatrix}G_s&0&0\\0&q&0\\0&v&F_s\end{bmatrix}\qquad\text{and}\qquad H_t=\begin{bmatrix}G_t&u&0\\0&q&0\\0&0&F_t\end{bmatrix}$$

where all entries of the columns u and v are zero, except for the last entry of u and the first entry of v, which are both 1.

If $[..\phi_s\phi_t]_n = [..\phi_t\phi_s]_n$ then $[..G_sG_t]_n = [..G_tG_s]_n$, and it follows by Lemma 9.2 that $h+1$ must be a divisor of n. Similarly also $[..F_sF_t]_n = [..F_tF_s]_n$, and it follows by Lemma 9.2 that $k+1$ must be a divisor of n. It remains to prove that if $h+1$ and $k+1$ are divisors of n then $[..H_sH_t]_n = [..H_tH_s]_n$.

Assume that $h+1$ and $k+1$ are divisors of n. Observe that ϕ_s and ϕ_t act on the quotient module M/M_U via the following two matrices,

$$H'_s = \begin{bmatrix} q & 0 \\ v & F_s \end{bmatrix} \qquad \text{and} \qquad H'_t = \begin{bmatrix} q & 0 \\ 0 & F_t \end{bmatrix}$$

which are also the matrices of ϕ_s and ϕ_t on M in the case $h = 0$. Since $[..\phi_s\phi_t]_n = [..\phi_t\phi_s]_n$ in this case, it follows that $[..H'_sH'_t]_n = [..H'_tH'_s]_n$. Similarly the matrices

$$H''_s = \begin{bmatrix} G_s & 0 \\ 0 & q \end{bmatrix} \qquad \text{and} \qquad H''_t = \begin{bmatrix} G_t & u \\ 0 & q \end{bmatrix}$$

satisfy $[..H''_sH''_t]_n = [..H''_tH''_s]_n$. But it is clear that

$$[..H_sH_t]_n = \begin{bmatrix} [..G_sG_t]_n & * & 0 \\ 0 & q^n & 0 \\ 0 & * & [..F_sF_t]_n \end{bmatrix} = \begin{bmatrix} [..H''_sH''_t]_n & 0 \\ * & [..F_sF_t]_n \end{bmatrix} = \begin{bmatrix} [..G_sG_t]_n & * \\ 0 & [..H'_sH'_t]_n \end{bmatrix}$$

where the asterisks mark entries whose values are irrelevant to our argument. Moreover

$$[..H_tH_s]_n = \begin{bmatrix} [..G_tG_s]_n & * & 0 \\ 0 & q^n & 0 \\ 0 & * & [..F_tF_s]_n \end{bmatrix} = \begin{bmatrix} [..H''_tH''_s]_n & 0 \\ * & [..F_tF_s]_n \end{bmatrix} = \begin{bmatrix} [..G_tG_s]_n & * \\ 0 & [..H'_tH'_s]_n \end{bmatrix}$$

by similar calculations, and since $[..H'_sH'_t]_n = [..H'_tH'_s]_n$ and $[..H''_sH''_t]_n = [..H''_tH''_s]_n$ it follows that $[..H_sH_t]_n = [..H_tH_s]_n$, as required. □

The following theorem gathers together the various results proved above, and their obvious analogues obtained by swapping s and t.

Theorem 9.5 *Let (W,S) be a Coxeter system of type $I_2(m)$, and let $S = \{s,t\}$. Then $(\mathscr{I}, J)$ is a W-graph ideal if and only if one of the following alternatives is satisfied:*

(i) $(\mathscr{I}, J) = (\{1\}, S)$,
(ii) $(\mathscr{I}, J) = (D_{\{s\}}, \{s\})$,
(iii) $(\mathscr{I}, J) = (\{\, [..st]_l \mid l \leqslant k\,\}, \{s\})$, *where $k+2$ divides m,*
(iv) $(\mathscr{I}, J) = (D_{\{t\}}, \{t\})$,
(v) $(\mathscr{I}, J) = (\{\, [..ts]_l \mid l \leqslant k\,\}, \{t\})$, *where $k+2$ divides m,*
(vi) $(\mathscr{I}, J) = (W, \emptyset)$,
(vii) $(\mathscr{I}, J) = (\{\, [..st]_l \mid l \leqslant h\,\} \cup \{\, [..ts]_l \mid l \leqslant k\,\}, \emptyset)$, *where $h+1$ and $k+1$ divide m.*

Our final objective is to determine all the W-graph biideals in type $I_2(m)$. We need the following lemma.

Lemma 9.6 *With (W,S) as above, let $\mathscr{I} = \{\, [..st]_l \mid l \leqslant h\,\} \cup \{\, [..ts]_l \mid l \leqslant k\,\}$, where h and k are nonnegative integers, and assume that $(\mathscr{I}, \emptyset)$ is a W-graph ideal. Let $C = \{\, c_w \mid w \in \mathscr{I}\,\}$ be the W-graph basis of the $\mathcal{H}$-module $\mathscr{S}(\mathscr{I}, \emptyset)$, and let $w \in \mathscr{I}$ with $l(w) \leqslant \min(h,k)+1$. Then $T_wc_1 = c_w + \sum_x q^{l(w)-l(x)}c_x$, where x runs through the set $\{\, x \in W \mid l(x) < l(w)\,\}$.*

Proof Note first that $\mathscr{I}$ contains all elements of W such that $l(w) \leqslant \min(h,k)$, and hence contains all x such that $l(x) < l(w)$.

We use induction on $l(w)$. If $l(w) = 0$ the statement becomes $T_1 c_1 = c_1$, which is true since T_1 is the identity element of $\mathcal{H}$. So assume that $l(w) = l > 0$, and let $w = rv$ with $r \in \{s,t\}$ and $l(v) = l-1$. Since the proofs for the two cases are essentially the same, we shall only do the case $r = s$.

Recall that the edge weights for $\Gamma(\mathscr{I},\emptyset)$ were found in Lemma 9.3. This makes it easy to evaluate $T_s c_x$ for all $x \in \mathscr{I}$. In particular, $T_s c_1 = qc_1 + c_s$. This shows that the desired formula holds when $w = sv$ and $v = 1$. So henceforth we assume that $v \neq 1$. Note that since $l(sv) > l(v)$ it follows that $l(tv) < l(v)$.

Observe that $\{v\} \cup \{\, x \in W \mid l(x) < l(v) \,\}$ is a union of right cosets of the group $\{1,t\}$, namely those cosets whose minimal element has length $l-2$ or less. So the inductive hypothesis can be written as

$$T_v c_1 = \sum_{x \in \mathcal{E}} q^{l(v)-l(x)-1}(qc_x + c_{tx}),$$

where $\mathcal{E} = \{\, x \in D_{\{t\}}^{-1} \mid l(x) \leqslant l-2 \,\}$. Similarly, the set $\{w\} \cup \{\, x \in W \mid l(x) < l(w) \,\}$ is a union of right cosets of $\{1,s\}$. Writing $\mathcal{F} = \{\, x \in D_{\{s\}}^{-1} \mid l(x) \leqslant l-1 \,\}$, our aim is to show that

$$T_w c_1 = \sum_{x \in \mathcal{F}} q^{l(w)-l(x)-1}(qc_x + c_{sx}).$$

Observe that $\{\, tx \mid x \in \mathcal{E} \,\} = \mathcal{F} \setminus \{1\}$.

If $x \in \mathcal{E}$ and $x \neq 1$ then $\mathrm{D}(x) = \{s\}$ and $\mathrm{D}(tx) = \{t\}$. Note also that $stx \in \mathscr{I}$, since either $l(stx) < l(w)$ or $stx = w$. So

$$\begin{aligned} T_s(qc_x + c_{tx}) &= -c_x + (qc_{tx} + c_{stx} + c_x) \\ &= qc_{tx} + c_{stx}. \end{aligned}$$

When $x = 1$ we get $T_s(qc_x + c_{tx}) = T_s(qc_1 + c_t) = q^2 c_1 + qc_s + qc_t + c_{st}$. So

$$\begin{aligned} T_w c_1 = T_s(T_v c_1) &= q^{l(v)-1}(q^2 c_1 + qc_s + qc_t + c_{st}) + \sum_{x \in \mathcal{E} \setminus \{1\}} q^{l(v)-l(x)-1}(qc_{tx} + c_{stx}) \\ &= q^{l(w)-1}(qc_1 + c_s) + q^{l(w)-2}(qc_t + c_{st}) + \sum_{y \in \mathcal{F} \setminus \{1,t\}} q^{l(v)-l(y)}(qc_y + c_{sy}) \\ &= \sum_{y \in \mathcal{F}} q^{l(w)-l(y)-1}(qc_y + c_{sy}) \end{aligned}$$

as required. □

Proposition 9.7 *Let (W,S) be a Coxeter system of type $I_2(m)$, with $S = \{s,t\}$. Let k be a nonnegative integer such that $k+1$ divides m, and let $\mathscr{I} = \{\, w \in W \mid l(w) \leqslant k \,\}$. Then $(\mathscr{I},\emptyset,\emptyset)$ is a W-graph biideal.*

Proof By case (vii) in Theorem 9.5 we know that $(\mathscr{I},\emptyset)$ is a W-graph ideal, and since $\mathscr{I} = \mathscr{I}^{-1}$ it follows that $(\mathscr{I},\emptyset)$ is also a W-graph right ideal. Identifying $\mathscr{S}^{\mathrm{o}}(\mathscr{I},\emptyset)$ with $\mathscr{S}(\mathscr{I},\emptyset)$ by putting $b_w^{\mathrm{o}} = b_w$ for all $w \in \mathscr{I}$, the task is to show that the left and right actions of $\mathcal{H}$ commute.

Note that if $k = m-1$ then $\mathscr{I} = W \setminus \{w_S\}$, where $w_S = [..st]_m$ is the longest element of W. But $(W,\emptyset,\emptyset)$ is a W-graph biideal, by Remark 6.7, and $\{c_{w_S}\}$ is closed for both the left and right actions. So it follows from Theorem 6.12 that $(\mathscr{I},\emptyset,\emptyset)$ is a W-graph biideal in this case.

Since the standard basis and W-graph basis of $\mathscr{S}(\mathscr{I},\emptyset)$ are related by the rule that $b_w = T_w c_1$ for all $w \in \mathscr{I}$, it follows from Proposition 9.6 that $b_w = c_w + \sum_{v<w} q^{l(w)-l(v)} c_v$ for all $w \in \mathscr{I}$. The right ideal analogue of Proposition 9.6 gives $b^{\mathrm{o}}_w = c^{\mathrm{o}}_w + \sum_{v<w} q^{l(w)-l(v)} c^{\mathrm{o}}_v$ for all $w \in \mathscr{I}$. Since $b^{\mathrm{o}}_w = b_w$, we must have $c^{\mathrm{o}}_w = c_w$ for all $w \in \mathscr{I}$.

The left and right actions of T_s and T_t are given by rules that are independent of the value of m. For example, for all $w \in \mathscr{I}$,

$$T_s c_w = \begin{cases} -q^{-1} c_w & \text{if the reduced expression for } w \text{ starts with } s, \\ q c_1 + c_s & \text{if } w = 1, \\ q c_t + c_{st} & \text{if } w = t, \\ q c_w + c_{sw} + c_{tw} & \text{if the reduced expression for } w \text{ starts with } t \text{ and } 1 < l(w) < k, \\ q c_w + c_{tw} & \text{if the reduced expression for } w \text{ starts with } t \text{ and } l(w) = k. \end{cases}$$

If it happens that $m = k+1$ then, as we have seen, $(\mathscr{I},\emptyset,\emptyset)$ is a W-graph biideal, and so the left and right actions commute. Since the value of m is irrelevant, the left and right actions always commute. □

Proposition 9.8 *Let (W,S) be a Coxeter system of type $I_2(m)$, with $S = \{s,t\}$. Let h and k be integers in $\{1,2,\ldots,m-1\}$ with $|h-k| = 1$. Let $\mathscr{I} = \{\,[..st]_l \mid l \leqslant h\,\} \cup \{\,[..ts]_l \mid l \leqslant k\,\}$. Then $(\mathscr{I},\emptyset,\emptyset)$ is not a W-graph biideal.*

Proof Suppose, for a contradiction, that $(\mathscr{I},\emptyset,\emptyset)$ is a W-graph biideal. It is obvious that essentially the same proof will apply whether $h = k-1$ or $k = h-1$. So we assume that $h = k-1$, which means that $[..st]_k$ is not in $\mathscr{I}$ and $[..ts]_k$ is in $\mathscr{I}$. Let $\{\,c_w \mid w \in \mathscr{I}\,\}$ be the $(W \times W^{\mathrm{o}})$-graph basis of the $(\mathcal{H},\mathcal{H})$-bimodule $M = \mathscr{S}(\mathscr{I},\emptyset,\emptyset)$.

Put $w = [..st]_{k-1}$, and suppose first that k is even. We shall show that $(T_s c_w)T_s \neq T_s(c_w T_s)$, contradicting the fact that M is a bimodule. In the first instance we assume that $k > 2$, although the calculations are much the same in the case $k = 2$. Given that $k > 2$ the reduced expression for w starts with t and ends with t, and there is at least one s in between. Observe that $c_w T_s = q c_w + c_{wt} + c_{ws}$ but $T_s c_w = q c_w + c_{tw}$, since $sw \notin \mathscr{I}$. Note also that ws is the longest element of $\mathscr{I}$. So we find that

$$(T_s c_w)T_s = q c_w T_s + c_{tw} T_s = q(q c_w + c_{ws} + c_{wt}) + (q c_{tw} + c_{tws} + c_{twt}),$$

whereas

$$T_s(c_w T_s) = q T_s c_w + T_s c_{wt} + T_s c_{ws} = q(q c_w + c_{tw}) + (q c_{wt} + c_{swt} + c_{twt}) + (q c_{ws} + c_{tws}).$$

The two expressions are not equal: the second features a c_{swt} that does not appear in the first. If $k = 2$ then we find that

$$(T_s c_t)T_s = (q c_t + c_1)T_s = q(q c_t + c_{ts}) + q c_1 + c_s,$$

whereas

$$T_s(c_t T_s) = T_s(q c_t + c_1 + cts) = q(q c_t + c_{st}) + (q c_1 + c_s) + (q c_{ts} + c_s),$$

and again the two expressions are not equal.

When k is odd similar calculations show that $(T_t c_w)T_s \neq T_t(c_w T_s)$. If $k = 3$ then

$$(T_t c_{st})T_s = (qc_{st} + c_t)T_s = q(qc_{st} + c_{sts} + c_s) + (qc_t + c_{ts})$$

whereas

$$T_t(c_{st}T_s) = T_t(qc_{st} + c_{sts} + c_s) = q(qc_{st} + c_t) + (qc_{sts} + c_{ts}) + (qc_s + c_{ts}),$$

and if $k \geqslant 5$ then

$$(T_t c_w)T_s = (qc_w + c_{sw})T_s = q(qc_w + c_{ws} + c_{wt}) + (qc_{sw} + c_{swt} + c_{sws})$$

whereas

$$T_t(c_w T_s) = T_t(qc_w + c_{ws} + c_{wt}) = q(qc_w + c_{sw}) + (qc_{ws} + c_{sws}) + (qc_{wt} + c_{twt} + c_{swt}).$$

A contradiction has been obtained in all cases. □

Theorem 9.9 *Let (W,S) be a Coxeter system of type $I_2(m)$, and let $S = \{s,t\}$. Then $(\mathscr{I}, J, K)$ is a W-graph biideal if and only if one of the following alternatives is satisfied:*

(i) $(\mathscr{I}, J, K) = (W, \emptyset, \emptyset)$,
(ii) $(\mathscr{I}, J, K) = (\{\, w \in W \mid l(w) \leqslant k \,\}, \emptyset, \emptyset)$, *where $k+1$ divides m,*
(iii) $(\mathscr{I}, J, K) = (\{1,t\}, \emptyset, \emptyset)$ *and m is even,*
(iv) $(\mathscr{I}, J, K) = (\{1,s\}, \emptyset, \emptyset)$ *and m is even,*
(v) $\mathscr{I} = \{1\}$ *and m is even, and J, K are any subsets of S,*
(vi) $\mathscr{I} = \{1\}$ *and m is odd, and $J, K \in \{\emptyset, S\}$.*

Proof Let us first check that $(\mathscr{I}, J, K)$ is a W-graph biideal if it is in the list. For case (i) Remark 6.7 applies, and for case (ii) Proposition 9.7 applies. For case (iii), observe that $(\mathscr{I}, J) = (\{1,t\}, \emptyset)$ is a W-graph ideal by case (vii) of Theorem 9.5, since m is even. Since $\mathscr{I} = \mathscr{I}^{-1}$, it is also a W-graph right ideal. Observe that T_s acts as scalar multiplication by q, in both the left action and the right action. Moreover, the left action of T_t is the same as the right action. So the left and right $\mathcal{H}$-actions commute, as required. Case (iv) is the same as case (iii), and cases (v) and (vi) are trivial.

It remains to prove that there are no others. So assume that $(\mathscr{I}, J, K)$ is a W-graph biideal. Since $\mathscr{I}$ has to be an ideal of $(W, \leqslant_{\mathsf{L}})$ and of $(W, \leqslant_{\mathsf{R}})$ we see that if $\mathscr{I}$ contains some element of length l then it must contain all $2l-1$ elements of length less than l. So clearly we must have $\mathscr{I} = \{\, [..st]_l \mid l \leqslant h \,\} \cup \{\, [..ts]_l \mid l \leqslant k \,\}$ for some integers h and k, with either $h = k$ or $|h-k| = 1$.

Assume first that $\min(h,k) \geqslant 1$. Then both s and t are in $\mathscr{I}$, and Remark 6.2 shows that $J = K = \emptyset$. So Proposition 9.8 shows that $h = k$, and since $(\mathscr{I}, J)$ is a W-graph ideal it follows from Theorem 9.5 that either $\mathscr{I} = W$ or $k+1$ is a divisor of m. So the only possibilities correspond to case (i) and case (ii) in the theorem statement.

Obviously $h = k = 0$ gives case (v) or case (vi) of the theorem statement. So it remains to consider the possibilities that $h = 0$ and $k = 1$, giving $\mathscr{I} = \{1,s\}$, or $h = 1$ and $k = 0$, giving $\mathscr{I} = \{1,t\}$. Since $h+1$ and $k+1$ have to be divisors of m, it follows that m must be even. If $J = K = \emptyset$ then we obtain cases (iii) and (iv) of the theorem statement. We must show that all other cases lead to contradictions.

Suppose first that $\mathscr{I} = \{1,s\}$. Then $s \notin J$ and $s \notin K$, and since J and K are not both empty, one or other must be $\{t\}$. Let $\{c_1, c_s\}$ be the $(W \times W^{\circ})$-graph basis of the bimodule $\mathscr{S}(\mathscr{I}, J, K)$. If $J = \{t\}$ then

$$(T_t c_1)T_s = (-q^{-1}c_1)T_s = -q^{-1}(qc_1 + c_s) \neq -c_1 + qc_s = T_t(qc_1 + c_s) = T_t(c_1 T_s),$$

while if $K = \{t\}$ then

$$T_s(c_1 T_t) = T_s(-q^{-1}c_1) = -q^{-1}(qc_1 + c_s) \neq -c_1 + qc_s = (qc_1 + c_s)T_t = (T_s c_1)T_t.$$

So in either case we have a contradiction. A similar argument disposes of $\mathscr{I} = \{1, t\}$. □

References

1. Vinay V. Deodhar, Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function. *Invent. Math.* **39** (1977), no. 2, 187–198.
2. Vinay V. Deodhar, On some geometric aspects of Bruhat orderings II. Parabolic analogue of Kazhdan–Lusztig polynomials. *Journal of Algebra* **111** (1997), no. 2, 483–506.
3. Meinolf Geck, Kazhdan–Lusztig cells and the Murphy basis. *Proc. London Math. Soc. (3)* **93** (2006), 635–665.
4. Meinolf Geck, On the induction of Kahzdan–Lusztig cells. *Bull. London Math. Soc.* **35** (2003), 608–614.
5. Meinolf Geck and Götz Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras.* London Mathematical Society Monographs. New Series, **21**. *The Clarendon Press, Oxford University Press, New York*, 2000.
6. Robert B. Howlett and Van Minh Nguyen, W-graph ideals. *Journal of Algebra* **361** (2012), 188–212.
7. Robert B. Howlett and Yunchuan Yin, Inducing W-graphs. *Math. Zeit.* **244** (2003), 415–431.
8. David Kazhdan and George Lusztig, Representations of Coxeter groups and Hecke algebras. *Invent. Math.* **53** (1979), 165–184.
9. G. Lusztig, *Hecke algebras with unequal parameters.* CRM Monograph Series **18**, *American Mathematical Society, Providence, RI*, 2003.
10. Van Minh Nguyen, W-graph ideals II. *Journal of Algebra* **361** (2012), 248–263.
11. Van Minh Nguyen, W-graph determining elements in type A. (2015), submitted.