WEAK MARTINGALE SOLUTIONS TO THE STOCHASTIC LANDAU–LIFSHITZ–GILBERT EQUATION WITH MULTI-DIMENSIONAL NOISE VIA A CONVERGENT FINITE-ELEMENT SCHEME

BENIAMIN GOLDYS, JOSEPH GROTOWSKI, AND KIM-NGAN LE

ABSTRACT. We propose an unconditionally convergent linear finite element scheme for the stochastic Landau–Lifshitz–Gilbert (LLG) equation with multi-dimensional noise. By using the Doss-Sussmann technique, we first transform the stochastic LLG equation into a partial differential equation that depends on the solution of the auxiliary equation for the diffusion part. The resulting equation has solutions absolutely continuous with respect to time. We then propose a convergent θ-linear scheme for the numerical solution of the reformulated equation. As a consequence, we are able to show the existence of weak martingale solutions to the stochastic LLG equation.

CONTENTS

1. INTRODUCTION

The deterministic Landau-Lifschitz-Gilbert (LLG) equation provides a basis for the theory and applications of ferromagnetic materials and fabrication of magnetic memories in particular, see for example [15, 9, 12, 17]. The setting is described as follows. Let D be a bounded domain in $\mathbb{R}^3$ with a smooth boundary ∂D, occupied by a ferromagnetic material. A configuration of magnetic moments across the domain D can be represented by a function on D taking values in $\mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$. According to the Landau and Lifschitz

Date: April 12, 2017.

2000 *Mathematics Subject Classification.* Primary 35Q40, 35K55, 35R60, 60H15, 65L60, 65L20, 65C30; Secondary 82D45.

Key words and phrases. stochastic partial differential equation, Landau–Lifshitz–Gilbert equation, finite element, ferromagnetism.

theory of ferrormagnetism [17], modified later by Gilbert [12], the time evolution of magnetic moments $\boldsymbol{M}(t,x)$ is described, in the simplest case, by the Landau-Lifschitz-Gilbert (LLG) equation

$$\begin{cases} \dfrac{\partial \boldsymbol{M}}{\partial t} = \lambda_1 \boldsymbol{M} \times \Delta \boldsymbol{M} - \lambda_2 \boldsymbol{M} \times (\boldsymbol{M} \times \Delta \boldsymbol{M}) \quad \text{in } (0,T)\times D, \\ \frac{\partial \boldsymbol{M}}{\partial \boldsymbol{n}} = 0 \quad \text{in } (0,T)\times \partial D, \\ \boldsymbol{M}(0,\cdot) = \boldsymbol{M}_0(\cdot) \quad \text{in } D, \end{cases} \tag{1.1}$$

where $\lambda_1 \neq 0$ and $\lambda_2 > 0$ are constants, and $\boldsymbol{n}$ stands for the outward normal vector on ∂D; see e.g. [9]. Assuming that $\boldsymbol{M}_0 \in H^{1,2}\left(D, \mathbb{S}^2\right)$, one can show that

$$|\boldsymbol{M}(t,x)| = 1, \quad t \in [0,T],\ x \in D \tag{1.2}$$

In this paper we are concerned with a stochastic version of the LLG equation. Randomly fluctuating fields were originally introduced in physics by Néel in [**?**] as formal quantities responsible for magnetization fluctuations. The necessity of being able to describe deviations from the average magnetization trajectory in an ensemble of noninteracting nanoparticles was later emphasised by Brown in [6, 7]. According to a non-rigorous argument of Brown, the magnetisation $\boldsymbol{M}$ evolves randomly according to a stochastic version of (1.1) that takes the form, (see [8] for more details about the physical background and derivation of this equation)

$$\begin{cases} d\boldsymbol{M} = \big(\lambda_1 \boldsymbol{M} \times \Delta \boldsymbol{M} - \lambda_2 \boldsymbol{M} \times (\boldsymbol{M} \times \Delta \boldsymbol{M})\big)dt + \sum_{i=1}^{q} (\boldsymbol{M} \times \boldsymbol{g}_i) \circ dW_i(t), \\ \dfrac{\partial \boldsymbol{M}}{\partial \boldsymbol{n}} = 0 \quad \text{on } (0,T)\times \partial D, \\ \boldsymbol{M}(0,\cdot) = \boldsymbol{M}_0 \quad \text{in } D, \end{cases} \tag{1.3}$$

where $\boldsymbol{g}_i \in \mathbb{W}^{2,\infty}(D)$, $i = 1, \cdots, q$, satisfy the homogeneous Neumann boundary conditions and $(W_i)_{i=1}^q$ is a q-dimensional Wiener process. In view of the property (1.2) for the deterministic system, we require that $\boldsymbol{M}$ also satisfies (1.2). To this end, we are forced to use the Stratonovich differential $\circ dW_i(t)$ in equation (1.3). The mathematical theory of equation (1.3) has been initiated only recently, in [8], where the existence of weak martingale solutions to (1.3) was proved for the case $q = 1$ using the Galerkin-Faedo approximations. We note that the Galerkin-Faedo approximations do not usually provide a useful computational tool for solving an equation exactly.

The aim of this paper is two-fold. We will prove the existence of solutions to the stochastic LLG equation (1.3) and at the same time will provide an efficient and flexible algorithm for solving numerically this equation. To this end we will use the finite element method and a new transformation of the Stratonovich type equation (1.3) to a deterministic PDE (4.2) with coefficients determined by a stochastic ODE (3.6) that can be solved separately. The deterministic PDE we obtain has solutions that are absolutely continuous with respect to time, and hence is suitable for the construction of a convergent finite element scheme. Our approach is based on the Doss-Sussmann technique [11, 18]. This transformation was introduced in [13] to study the stochastic LLG equation with a single Wiener process

($q = 1$), in which case the auxiliary ODE is deterministic. Since the vector fields $\boldsymbol{u} \times \boldsymbol{g}_i$ are non-commuting, the case of $q > 1$ is more difficult and requires new arguments.

We apply the finite element method to the PDE resulting from this transformation and prove the convergence of linear finite element scheme to a weak martingale solution to (1.3) (after taking an inverse transformation). Our proof is simpler than the proof in [8] and covers the case of $q > 1$. We note here that under appropriate assumptions even the case of infinite-dimensional noise ($q = \infty$) can be handled in exactly the same way.

Let us recall that the first convergent finite element scheme for the stochastic LLG equation was studied in [5] and is based on a Crank–Nicolson type time-marching evolution, relying on a nonlinear iteration solved by a fixed point method. On the other hand, there has been an intensive development of a new class of numerical methods for the LLG equation (1.1) based on a linear iterations, yielding unconditional convergence and stability [1, 3]. The ideas developed there are extended and generalized in [13, 2] in order to take into account the stochastic term. A fully linear discrete scheme for (1.3) is studied in [13] but with one-dimensional noise. The method is based on the so–called Doss-Sussmann technique [11, 18], which allows one to replace the stochastic partial differential equation (PDE) by an equivalent PDE with random coefficients. In contrast, [2] considers, for a more general noise, a projection scheme applied directly to the original stochastic equation (1.3). However, this approach requires a quite specific and complicated treatment of the stochastic term. In the current paper, we propose a convergent θ-linear scheme for the numerical solution of the tranformed equation and prove unconditional stability and convergence for the scheme when $\theta > 1/2$. To the best of our knowledge this is a new result for this problem.

The paper is organised as follows. In Section 2 we define the notion of weak martingale solutions to (1.3) and state our main result. In Section 3, we introduce an auxiliary stochastic ODE and prove some properties of solution necessary for the transformation of equation (1.3) to a deterministic PDE with random coefficients. Details of this transformation are presented in Section 4. We also show in this section how a weak solution to (1.3) can be obtained from a weak solution of the reformulated form. In Section 5 we introduce our finite element scheme and present a proof for the stability of approximate solutions. Section 6 is devoted to the proof of the main theorem, namely the convergence of finite element solutions to a weak solution of the reformulated equation. Finally, in the Appendix we collect, for the reader's convenience, a number of facts that are used in the course of the proof.

Throughout this paper, c denotes a generic constant that may take different values at different occurrences. In what follows we will also use the notation $D_T = (0, T) \times D$.

2. Definition of a weak solution and the main result

In this section we state the definition of a weak solution to (1.3) and present our main result. Before doing so, we introduce some suitable Sobolev spaces, and fix some notation. The standing assumption for the rest of the paper is that D is a bounded open domain in $\mathbb{R}^3$ with a smooth boundary.

For any $U \subset \mathbb{R}^d$, $d \geq 1$, we denote by $\mathbb{L}^2(U)$ the space of Lebesgue square-integrable functions defined on U and taking values in $\mathbb{R}^3$. The function space $\mathbb{H}^1(U)$ is defined as:

$$\mathbb{H}^1(U) = \left\{ \boldsymbol{u} \in \mathbb{L}^2(U) : \frac{\partial \boldsymbol{u}}{\partial x_i} \in \mathbb{L}^2(U) \quad \text{for } i \leq d. \right\}.$$

Remark 2.1. *For $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{H}^1(D)$ we denote*

$$\boldsymbol{u} \times \nabla \boldsymbol{v} := \left(\boldsymbol{u} \times \frac{\partial \boldsymbol{v}}{\partial x_1}, \boldsymbol{u} \times \frac{\partial \boldsymbol{v}}{\partial x_2}, \boldsymbol{u} \times \frac{\partial \boldsymbol{v}}{\partial x_3} \right)$$

$$\nabla \boldsymbol{u} \times \nabla \boldsymbol{v} := \sum_{i=1}^{3} \frac{\partial \boldsymbol{u}}{\partial x_i} \times \frac{\partial \boldsymbol{v}}{\partial x_i}$$

$$\langle \boldsymbol{w} \times \nabla \boldsymbol{v}, \nabla \boldsymbol{u} \rangle_{\mathbb{L}^2(D)} := \sum_{i=1}^{3} \left\langle \boldsymbol{w} \times \frac{\partial \boldsymbol{v}}{\partial x_i}, \frac{\partial \boldsymbol{u}}{\partial x_i} \right\rangle_{\mathbb{L}^2(D)} \qquad \forall\, \boldsymbol{w} \in \mathbb{L}^\infty(D).$$

Definition 2.2. *Given $T \in (0, \infty)$ and a family of functions $\{g_i : i = 1, \dots, q\} \subset \mathbb{L}^\infty(D)$, a weak martingale solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}, W, \boldsymbol{M})$ to (1.3), for the time interval $[0, T]$, consists of*

(a) *a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ with the filtration satisfying the usual conditions,*
(b) *a q-dimensional $(\mathcal{F}_t)$-adapted Wiener process $W = (W_t)_{t\in[0,T]}$,*
(c) *a progressively measurable process $\boldsymbol{M} : [0, T] \times \Omega \to \mathbb{L}^2(D)$*

such that

(1) $\boldsymbol{M}(\cdot, \omega) \in C([0, T]; \mathbb{H}^{-1}(D))$ *for $\mathbb{P}$-a.e. $\omega \in \Omega$;*
(2) $\mathbb{E}\left(\operatorname{ess\,sup}_{t\in[0,T]} \|\nabla \boldsymbol{M}(t)\|^2_{\mathbb{L}^2(D)}\right) < \infty$;
(3) $|\boldsymbol{M}(t, x)| = 1$ *for each $t \in [0, T]$, a.e. $x \in D$, and $\mathbb{P}$-a.s.;*
(4) *for every $t \in [0, T]$, for all $\boldsymbol{\psi} \in \mathbb{C}_0^\infty(D)$, $\mathbb{P}$-a.s.:*

$$\begin{aligned} \langle \boldsymbol{M}(t), \boldsymbol{\psi} \rangle_{\mathbb{L}^2(D)} - \langle \boldsymbol{M}_0, \boldsymbol{\psi} \rangle_{\mathbb{L}^2(D)} &= -\lambda_1 \int_0^t \langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla \boldsymbol{\psi} \rangle_{\mathbb{L}^2(D)}\, ds \\ &\quad - \lambda_2 \int_0^t \langle \boldsymbol{M} \times \nabla \boldsymbol{M}, \nabla(\boldsymbol{M} \times \boldsymbol{\psi}) \rangle_{\mathbb{L}^2(D)}\, ds \\ &\quad + \sum_{i=1}^{q} \int_0^t \langle \boldsymbol{M} \times \boldsymbol{g}_i, \boldsymbol{\psi} \rangle_{\mathbb{L}^2(D)} \circ dW_i(s). \end{aligned} \tag{2.1}$$

As the main result of this paper, we will establish a finite element scheme defined via a sequence of functions which are piecewise linear in both the space and time variables. We also prove that this sequence contains a subsequence converging to a weak martingale solution in the sense of Definition 2.2. A precise statement will be given in Theorem 6.9.

3. The auxiliary equation for the diffusion part

In this section we introduce the auxiliary equation (3.12) that will be used in the next section to define a new variable from $\boldsymbol{M}$, and establish some properties of its solution.
Let $\boldsymbol{g}_1, \dots, \boldsymbol{g}_q \in C\left(\overline{D}, \mathbb{R}^3\right)$, be fixed. For $i = 1, \dots, q$, and $x \in \overline{D}$ we define linear operators $G_i(x) : \mathbb{R}^3 \to \mathbb{R}^3$ by $G_i(x)\boldsymbol{u} = \boldsymbol{u} \times \boldsymbol{g}_i(x)$. In what follows we suppress the argument x. It is easy to check that

$$G_i^\star = -G_i\,, \tag{3.1}$$

$$\text{and} \quad \left(G_i^2\right)^\star = G_i^2\,. \tag{3.2}$$

We will consider a stochastic Stratonovitch equation on the algebra $\mathcal{L}\left(\mathbb{R}^3\right)$ of linear operators in $\mathbb{R}^3$:

$$Z_t = I + \sum_{i=1}^{q} \int_0^t G_i Z_s \circ dW_i(s), \quad t \geq 0\,. \tag{3.3}$$

Lemma 3.1. *Let $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_q \in C\left(\overline{D}, \mathbb{R}^3\right)$. Then the following holds.*
(a) For every $x \in \overline{D}$ equation (3.3) has a unique strong solution, which has a t-continuous version in $\mathcal{L}\left(\mathbb{R}^3\right)$.
(b) For every $t \geq 0$ and $x \in \overline{D}$

$$|Z_t \boldsymbol{u}| = |\boldsymbol{u}| \quad \mathbb{P}-a.s \quad \text{for every } \boldsymbol{u} \in \mathbb{R}^3. \tag{3.4}$$

In particular, for every $t \geq 0$ the operator Z_t is invertible and $Z_t^{-1} = Z_t^\star$.
(c) If moreover $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_q \in C^\alpha\left(\overline{D}, \mathbb{R}^3\right)$ for a certain $\alpha \in (0,1)$ then the mapping $(t,x) \to Z_t(x)$ has a continuous version in $\mathcal{L}\left(\mathbb{R}^3\right)$.

Proof. Equation (3.3) can be equivalently written as an Itô equation

$$Z_t = I + \frac{1}{2}\sum_{i=1}^{q} \int_0^t G_i^2 Z_s\, ds + \sum_{i=1}^{q} \int_0^t G_i Z_s\, dW_i(s), \quad t \geq 0\,. \tag{3.5}$$

Since the coefficients of equation (3.5) are Lipschitz, the existence and uniqueness of strong solutions to equation (3.5), and the existence of its continuous version is standard, see for example Theorem 18.3 in [16]. Hence, the same result holds for (3.3).
To prove (b) we fix $x \in \overline{D}$, $t \geq 0$ and $\boldsymbol{u} \in \mathbb{R}^3$ and put $Z^{\boldsymbol{u}} = Z\boldsymbol{u}$. Then equation (3.5) yields

$$Z_t^{\boldsymbol{u}} = \boldsymbol{u} + \frac{1}{2}\sum_{i=1}^{q} \int_0^t G_i^2 Z_s^{\boldsymbol{u}}\, ds + \sum_{i=1}^{q} \int_0^t G_i Z_s^{\boldsymbol{u}}\, dW_i(s)\,. \tag{3.6}$$

Applying the Itô formula to the process $|Z_t^{\boldsymbol{u}}|^2$ and invoking (3.1) we obtain

$$\begin{aligned} d|Z_t^{\boldsymbol{u}}|^2 &= 2\left\langle Z_t^{\boldsymbol{u}}, dZ_t^{\boldsymbol{u}}\right\rangle + \sum_{i=1}^{q} |G_i Z_t^{\boldsymbol{u}}|^2\, dt \\ &= \sum_{i=1}^{q} \left\langle Z_t^{\boldsymbol{u}}, G_i^2 Z_t^{\boldsymbol{u}}\right\rangle\, dt + 2\sum_{i=1}^{q} \left\langle Z_t^{\boldsymbol{u}}, G_i Z_t^{\boldsymbol{u}}\right\rangle\, dW_i(t) + \sum_{i=1}^{q} |G_i Z_t^{\boldsymbol{u}}|^2\, dt \\ &= 0, \end{aligned}$$

or equivalently

$$|Z_t^{\boldsymbol{u}}|^2 = |\boldsymbol{u}|^2, \quad \text{for all } t \geq 0, \quad \mathbb{P}-a.s..$$

To prove (c), we begin by letting $0 \leq s < t \leq T$ and $x, y \in \overline{D}$. For any $p \geq 1$ we have

$$\mathbb{E}\,|Z_t(y) - Z_s(x)|^p \leq 2^{p-1}\mathbb{E}\,|Z_t(y) - Z_s(y)|^p + 2^{p-1}\mathbb{E}\,|Z_s(y) - Z_s(x)|^p\,. \tag{3.7}$$

It is well known that there exists $C_1 > 0$ such that

$$\mathbb{E}\,|Z_t(y) - Z_s(y)|^p \leq C_1 |t-s|^{\frac{p}{2}}. \tag{3.8}$$

If there exists $\alpha \in (0,1]$ such that

$$|g_i(x) - g_i(y)| \leq c_i |x-y|^\alpha, \quad x, y \in \overline{D},\ i = 1, \ldots, q$$

then for a certain $C > 0$, for any $h \in \mathbb{R}^3$ there holds

$$\begin{aligned} |G_i(x)h| &\leq C|h|\,, \\ |(G_i(x) - G_i(y))h| &\leq C|x-y|^\alpha |h|\,, \\ \left|\left(G_i^2(x) - G_i^2(y)\right)h\right| &\leq C|x-y|^\alpha |h|. \end{aligned} \tag{3.9}$$

Then

$$\begin{aligned} Z_s(y) - Z_s(x) &= \frac{1}{2}\sum_{i=1}^q \int_0^s G_i^2(y)Z_r(y)\,dr + \sum_{i=1}^q \int_0^s G_i(y)Z_r(y)dW_i(r) \\ &\quad - \frac{1}{2}\sum_{i=1}^q \int_0^s G_i^2(x)Z_r(x)\,dr - \sum_{i=1}^q \int_0^s G_i(x)Z_r(x)dW_i(r) \\ &= \frac{1}{2}\sum_{i=1}^q \int_0^s \left(G_i^2(y) - G_i^2(x)\right) Z_r(y)\,dr + \frac{1}{2}\sum_{i=1}^q \int_0^s G_i^2(x)\left(Z_r(y) - Z_r(x)\right)\,dr \\ &\quad + \sum_{i=1}^q \int_0^s \left(G_i(y) - G_i(x)\right) Z_r(y)dW_i(r) + \sum_{i=1}^q \int_0^s G_i(x)\left(Z_r(y) - Z_r(x)\right)dW_i(r). \end{aligned}$$

Using (3.9) we obtain

$$\mathbb{E}\,|Z_s(y) - Z_s(x)|^p \leq \widetilde{C}|x-y|^{\alpha p} + \widetilde{C}\int_0^s \mathbb{E}\,|Z_r(y) - Z_r(x)|^p\,dr.$$

Therefore, invoking the Gronwall Lemma we obtain

$$\mathbb{E}\,|Z_s(y) - Z_s(x)|^p \leq \widetilde{C}e^{\widetilde{C}T}|x-y|^{\alpha p}. \tag{3.10}$$

Combining (3.7), (3.8) and (3.10) we obtain

$$\mathbb{E}\,|Z_t(y) - Z_s(x)|^p \leq c_1|t-s|^{\frac{p}{2}} + c_2|x-y|^{\alpha p}. \tag{3.11}$$

Let $\beta > 0$ and $r = d + 1 + \beta$. Let p be chosen in such a way that

$$\frac{p}{2} \geq r \quad \text{and} \quad p\alpha \geq r\,.$$

The set $[0,T] \times D$ can be covered by a finite number of open sets B_k with the property $|t-s|^r + |x-y|^r < 1$ on every set B_k. In each B_k, (3.11) then yields

$$\mathbb{E}\,|Z_t(y) - Z_s(x)|^p \leq c\left(|t-s|^r + |x-y|^r\right)\,,$$

and the result then follows by the Kolmogorv-Chentsov theorem, see p. 57 of [16]. □

Lemma 3.2. *Assume that $\boldsymbol{g}_i \in C_b^{1+\alpha}(D, \mathbb{R}^3)$. Then the following holds.*
(a) For every $t \geq 0$ we have $Z_t \in C_b^1(D, \mathcal{L}(\mathbb{R}^3))$ $\mathbb{P}$-a.s.
(b) For every $x \in D$ the process $\xi_t(x) = \nabla Z_t(x)$ is the unique solution of the linear Itô equation

$$d\xi_t(x) = \frac{1}{2}\sum_{i=1}^q \left(G_i^2\xi_t(x) + H_iZ_t(x)\right)dt + \sum_{i=1}^q \left(G_i\xi_t(x) + I_iZ_t(x)\right)dW_i(t),$$

with $\xi_0(x) = 0$ and the operators $H_i, I_i \in \mathcal{L}\left(\mathbb{R}^3\right)$ defined as

$$I_i\boldsymbol{u} = \boldsymbol{u} \times \nabla \boldsymbol{g}_i \quad \text{and} \quad H_i\boldsymbol{u} = G_i\boldsymbol{u} \times \nabla \boldsymbol{g}_i + G_i(\boldsymbol{u} \times \nabla \boldsymbol{g}_i).$$

(c) For every $\gamma < \min\left(\alpha, \frac{1}{2}\right)$ *the mapping* $(t,x) \to \nabla Z_t(x)$ *is* γ*-Hölder continuous.*
(d) We have

$$\mathbb{E} \sup_{t\le T} \sup_{x\in\overline{D}} |\nabla Z_t|^2 < \infty\,.$$

Proof. (a) Let E denote the Banach space of continuous and adapted processes Z taking values in the space of linear operators $\mathcal{L}\left(\mathbb{R}^3\right)$ and endowed with the norm

$$\|Z\|_E = \left(\mathbb{E} \sup_{t\le T} |Z_t|^2\right)^{1/2}.$$

For every $x \in D$ we define a mapping

$$\mathcal{K} : D \times E \to D \times E, \quad \mathcal{K}(x,Z)(t) = I + \sum_{i=1}^{q} \int_0^t G_i(x) Z_s \circ dW_i(s)\,.$$

It is easy to check that the assumptions of Lemma 9.2, p. 238 in [10] are satisfied and therefore (a) holds.
(b) The proof is completely analogous to the proof of Theorem 9.8 in [10], and is hence omitted.
(c) The proof is analogous to the proof of part (c) of Lemma 3.1.
(d) The estimate follows easily from (c). □

For every $\boldsymbol{u} \in \mathbb{L}^2(D)$ we will consider the $L^2(D)$-valued process $Z_t(u)$ defined by

$$[Z_t(u)](x) = Z_t(x)u(x) \quad x - a.e.$$

Clearly,

$$Z_t(\boldsymbol{u}) = \boldsymbol{u} + \sum_{i=1}^{q} \int_0^t Z_s(\boldsymbol{u}) \times \boldsymbol{g}_i \circ dW_i(s), \quad t \ge 0\,, \tag{3.12}$$

where the equality holds in $\mathbb{L}^2(D)$. The process Z_t is now an operator-valued process taking values $\mathcal{L}\left(\mathbb{L}^2(D)\right)$ and it will still be denoted by Z_t. The next lemmas follow immediately from the properties of the matrix-valued process considered above.

Lemma 3.3. *Assume that* $\{g_i : i = 1,\dots,q\} \subset C_b^{1+\alpha}(D)$*. Then for every* $\boldsymbol{u} \in \mathbb{L}^2(D)$ *the stochastic differential equation* (3.12) *has a unique strong continuous solution in* $\mathbb{L}^2(D)$*. Moreover, there exists* $\Omega_0 \subset \Omega$ *such that* $\mathbb{P}(\Omega_0) = 1$ *and for every* $\omega \in \Omega_0$ *the following holds.*
(a) For all $t \ge 0$ *and every* $\boldsymbol{u} \in \mathbb{L}^2(D)$*,*

$$|Z_t(\omega, \boldsymbol{u})| = |\boldsymbol{u}|\,.$$

(b) For every $t \ge 0$ *the mapping* $\boldsymbol{u} \to Z_t(\omega, \boldsymbol{u})$ *defines a linear bounded operator* $Z_t(\omega)$ *on* $\mathbb{L}^2(D)$*. In particular,*

$$Z_t(\omega, \boldsymbol{u} + \boldsymbol{v}) = Z_t(\omega, \boldsymbol{u}) + Z_t(\omega, \boldsymbol{v})\,. \tag{3.13}$$

Moreover, for every $T > 0$ *there exists a constant* $C_T > 0$ *such that*

$$\mathbb{E} \sup_{t\le T} |Z_t(\boldsymbol{u})|^2_{\mathbb{L}^2(D)} \le C_T |\boldsymbol{u}|^2_{\mathbb{L}^2(D)}\,. \tag{3.14}$$

(c) For every $t \geq 0$ the operator $Z_t(\omega)$ is invertible and the inverse operator is the unique solution of the stochastic differential equation on $\mathbb{L}^2(D)$:

$$Z_t^{-1}(\boldsymbol{u}) = \boldsymbol{u} - \sum_{i=1}^{q} \int_0^t Z_s^{-1} G_i(\boldsymbol{u}) \circ dW_i(s), \quad \boldsymbol{u} \in \mathbb{L}^2(D)\,. \tag{3.15}$$

Finally,

$$Z_t^{-1}(\omega) = Z_t^{\star}(\omega). \tag{3.16}$$

Lemma 3.4. *Assume that $\boldsymbol{g}_i \in C_b^{1+\alpha}\left(D, \mathbb{R}^3\right)$ for $i = 1, \cdots, q$. Then, for every $\boldsymbol{u} \in \mathbb{H}^1(D)$ $Z_t(\boldsymbol{u}) \in \mathbb{H}^1$ $\mathbb{P}$-a.s. Furthermore, the process $\xi_t(\boldsymbol{u}) := \nabla Z_t(\boldsymbol{u})$, is the unique solution of the linear equation*

$$d\xi_t(\boldsymbol{u}) = \frac{1}{2}\sum_{i=1}^{q}\left(G_i^2\xi_t(\boldsymbol{u}) + H_i Z_t(\boldsymbol{u})\right) dt + \sum_{i=1}^{q}\left(G_i\xi_t(\boldsymbol{u}) + I_i Z_t(\boldsymbol{u})\right) dW_i(t),$$

with $\xi_0(\boldsymbol{u}) = \nabla \boldsymbol{u}$.

Lemma 3.5. *For any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{L}^2(D)$, there holds for all $t \geq 0$ and $\mathbb{P}$-a.s.:*

$$Z_t(\boldsymbol{u} \times \boldsymbol{v}) = Z_t(\boldsymbol{u}) \times Z_t(\boldsymbol{v}), \tag{3.17}$$

Proof. Let $Z_t^{\boldsymbol{u}} := Z_t(\boldsymbol{u})$ and $Z_t^{\boldsymbol{v}} := Z_t(\boldsymbol{v})$ for all $t \geq 0$. We now prove (3.17); the property (3.13) can be obtained in the same manner. Using the Itô formula for $Z_t^{\boldsymbol{u}} \times Z_t^{\boldsymbol{v}}$ and (3.6), we obtain

$$\begin{aligned}
d(Z_t^{\boldsymbol{u}} \times Z_t^{\boldsymbol{v}}) &= dZ_t^{\boldsymbol{u}} \times Z_t^{\boldsymbol{v}} + Z_t^{\boldsymbol{u}} \times dZ_t^{\boldsymbol{v}} + \sum_{i=1}^{q} (Z_t^{\boldsymbol{u}} \times \boldsymbol{g}_i) \times (Z_t^{\boldsymbol{v}} \times \boldsymbol{g}_i)\, dt \\
&= \sum_{i=1}^{q} \left(Z_t^{\boldsymbol{u}} \times (Z_t^{\boldsymbol{v}} \times \boldsymbol{g}_i) - Z_t^{\boldsymbol{v}} \times (Z_t^{\boldsymbol{u}} \times \boldsymbol{g}_i)\right) dW_i(t) \\
&\quad + \frac{1}{2}\sum_{i=1}^{q} \left(Z_t^{\boldsymbol{u}} \times \left((Z_t^{\boldsymbol{v}} \times \boldsymbol{g}_i) \times \boldsymbol{g}_i\right) - Z_t^{\boldsymbol{v}} \times \left(Z_t^{\boldsymbol{u}} \times \boldsymbol{g}_i) \times \boldsymbol{g}_i\right)\right) dt \\
&\quad + \sum_{i=1}^{q} (Z_t^{\boldsymbol{u}} \times \boldsymbol{g}_i) \times (Z_t^{\boldsymbol{v}} \times \boldsymbol{g}_i)\, dt.
\end{aligned} \tag{3.18}$$

Using an elementary identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \langle \boldsymbol{a}, \boldsymbol{c} \rangle\, \boldsymbol{b} - \langle \boldsymbol{a}, \boldsymbol{b} \rangle\, \boldsymbol{c}\,, \quad \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3\,, \tag{3.19}$$

we find that

$$Z_t^{\boldsymbol{u}} \times (Z_t^{\boldsymbol{v}} \times \boldsymbol{g}_i) - Z_t^{\boldsymbol{v}} \times (Z_t^{\boldsymbol{u}} \times \boldsymbol{g}_i) = (Z_t^{\boldsymbol{u}} \times Z_t^{\boldsymbol{v}}) \times \boldsymbol{g}_i \tag{3.20}$$

and

$$\begin{aligned}
&Z_t^{\boldsymbol{u}} \times \left((Z_t^{\boldsymbol{v}} \times \boldsymbol{g}_i) \times \boldsymbol{g}_i\right) - Z_t^{\boldsymbol{v}} \times \left(Z_t^{\boldsymbol{u}} \times \boldsymbol{g}_i) \times \boldsymbol{g}_i\right) \\
&\quad = \langle Z_t^{\boldsymbol{u}}, \boldsymbol{g}_i \rangle\, (Z_t^{\boldsymbol{v}} \times \boldsymbol{g}_i) - \langle Z_t^{\boldsymbol{v}}, \boldsymbol{g}_i \rangle\, (Z_t^{\boldsymbol{u}} \times \boldsymbol{g}_i) - 2(Z_t^{\boldsymbol{u}} \times \boldsymbol{g}_i) \times (Z_t^{\boldsymbol{v}} \times \boldsymbol{g}_i) \\
&\quad = \left((Z_t^{\boldsymbol{u}} \times Z_t^{\boldsymbol{v}}) \times \boldsymbol{g}_i\right) \times \boldsymbol{g}_i - 2(Z_t^{\boldsymbol{u}} \times \boldsymbol{g}_i) \times (Z_t^{\boldsymbol{v}} \times \boldsymbol{g}_i).
\end{aligned} \tag{3.21}$$

Invoking (3.20) and (3.21), equation (3.18) we obtain

$$d(Z_t^{\boldsymbol{u}} \times Z_t^{\boldsymbol{v}}) = \frac{1}{2}\sum_{i=1}^{q}\big((Z_t^{\boldsymbol{u}} \times Z_t^{\boldsymbol{v}}) \times \boldsymbol{g}_i\big) \times \boldsymbol{g}_i\, dt + \sum_{i=1}^{q}\big((Z_t^{\boldsymbol{u}} \times Z_t^{\boldsymbol{v}}) \times \boldsymbol{g}_i\big) dW_i(t).$$

Therefore, the process $V_t := Z_t^{\boldsymbol{u}} \times Z_t^{\boldsymbol{v}}$ is a solution of the following stochastic differential equation:

$$\begin{cases} dV_t = \frac{1}{2}\sum_{i=1}^{q}\big(V_t \times \boldsymbol{g}_i\big) \times \boldsymbol{g}_i\, dt + \sum_{i=1}^{q}\big(V_t \times \boldsymbol{g}_i\big) dW_i(t) \\ V_0 = \boldsymbol{u} \times \boldsymbol{v}. \end{cases}$$

On the other hand, it follows from (3.6) that the process $Z_t(\boldsymbol{u} \times \boldsymbol{v})$ satisfies the same equation. Hence, (3.17) follows from the uniqueness of solutions to (3.18).

□

Lemma 3.6. *For any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{H}^1(D)$, there holds for all $t \geq 0$ and $\mathbb{P}$-a.s.:*

$$\langle \nabla Z_t(\boldsymbol{u}), \nabla Z_t(\boldsymbol{v})\rangle_{\mathbb{L}^2(D)} = \langle \nabla \boldsymbol{u}, \nabla \boldsymbol{v}\rangle_{\mathbb{L}^2(D)} + F(t, \boldsymbol{u}, \boldsymbol{v}), \tag{3.22}$$

with

$$F(t, \boldsymbol{u}, \boldsymbol{v}) := \sum_{i=1}^{q}\int_0^t F_{1,i}(s, \boldsymbol{u}, \boldsymbol{v})\, ds + \sum_{i=1}^{q}\int_0^t F_{2,i}(s, \boldsymbol{u}, \boldsymbol{v})\, dW_i(s)$$

where

$$\begin{aligned} F_{1,i}(t, \boldsymbol{u}, \boldsymbol{v}) := &\big\langle \nabla Z_t(\boldsymbol{u}), (\tfrac{1}{2}H_i - G_iI_i)Z_t(\boldsymbol{v})\big\rangle_{\mathbb{L}^2(D)} - \big\langle \nabla(\tfrac{1}{2}H_i - G_iI_i)Z_t(\boldsymbol{u}), Z_t(\boldsymbol{v})\big\rangle_{\mathbb{L}^2(D)} \\ &+ \langle I_iZ_t(\boldsymbol{u}), I_iZ_t(\boldsymbol{v})\rangle_{\mathbb{L}^2(D)}; \end{aligned}$$

and

$$F_{2,i}(t, \boldsymbol{u}, \boldsymbol{v}) := \langle \nabla Z_t(\boldsymbol{u}), I_iZ_t(\boldsymbol{v})\rangle_{\mathbb{L}^2(D)} - \langle \nabla(I_iZ_t(\boldsymbol{u})), Z_t(\boldsymbol{v})\rangle_{\mathbb{L}^2(D)}$$

Proof. Let $\xi_t^{\boldsymbol{u}} := \xi_t(\boldsymbol{u})$ and $\xi_t^{\boldsymbol{v}} := \xi_t(\boldsymbol{v})$ for all $t \geq 0$. In addition, we consider a C^∞ function $\phi : \big(\mathbb{L}^2(D)\big)^2 \to \mathbb{R}$ defined by $\phi(\boldsymbol{x}, \boldsymbol{y}) = \langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\mathbb{L}^2(D)}$. By using the Itô Lemma we obtain

$$\begin{aligned} d\,\langle \xi_t^{\boldsymbol{u}}, \xi_t^{\boldsymbol{v}}\rangle_{\mathbb{L}^2(D)} &= \langle d\xi_t^{\boldsymbol{u}}, \xi_t^{\boldsymbol{v}}\rangle_{\mathbb{L}^2(D)} + \langle \xi_t^{\boldsymbol{u}}, d\xi_t^{\boldsymbol{v}}\rangle_{\mathbb{L}^2(D)} + \langle d\xi_t^{\boldsymbol{u}}, d\xi_t^{\boldsymbol{v}}\rangle_{\mathbb{L}^2(D)} \\ &= \sum_{i=1}^{q}\Big(\tfrac{1}{2}\,\langle G_i^2\xi_t^{\boldsymbol{u}} + H_iZ_t^{\boldsymbol{u}}, \xi_t^{\boldsymbol{v}}\rangle_{\mathbb{L}^2(D)} + \tfrac{1}{2}\,\langle \xi_t^{\boldsymbol{u}}, G_i^2\xi_t^{\boldsymbol{v}} + H_iZ_t^{\boldsymbol{v}}\rangle_{\mathbb{L}^2(D)} \\ &\qquad + \langle G_i\xi_t^{\boldsymbol{u}} + I_iZ_t^{\boldsymbol{u}}, G_i\xi_t^{\boldsymbol{v}} + I_iZ_t^{\boldsymbol{v}}\rangle_{\mathbb{L}^2(D)}\Big)\, dt \\ &\quad + \sum_{i=1}^{q}\Big(\langle G_i\xi_t^{\boldsymbol{u}} + I_iZ_t^{\boldsymbol{u}}, \xi_t^{\boldsymbol{v}}\rangle_{\mathbb{L}^2(D)} + \langle \xi_t^{\boldsymbol{u}}, G_i\xi_t^{\boldsymbol{v}} + I_iZ_t^{\boldsymbol{v}}\rangle_{\mathbb{L}^2(D)}\Big)\, dW_i(t). \end{aligned} \tag{3.23}$$

Using (3.1) and (3.2), we deduce from (3.23) that

$$\begin{aligned}
d\left\langle \xi_t^{\boldsymbol{u}}, \xi_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} &= \sum_{i=1}^{q} \Big(\tfrac{1}{2} \left\langle H_i Z_t^{\boldsymbol{u}}, \xi_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} + \tfrac{1}{2} \left\langle \xi_t^{\boldsymbol{u}}, H_i Z_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} + \left\langle G_i \xi_t^{\boldsymbol{u}}, I_i Z_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} \\
&\qquad + \left\langle I_i Z_t^{\boldsymbol{u}}, G_i \xi_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} + \left\langle I_i Z_t^{\boldsymbol{u}}, I_i Z_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} \Big) dt \\
&\quad + \sum_{i=1}^{q} \Big(\left\langle I_i Z_t^{\boldsymbol{u}}, \xi_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} + \left\langle \xi_t^{\boldsymbol{u}}, I_i Z_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} \Big) dW_i(t) \\
&= \sum_{i=1}^{q} \Big(\left\langle (\tfrac{1}{2} H_i - G_i I_i) Z_t^{\boldsymbol{u}}, \xi_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} + \left\langle \xi_t^{\boldsymbol{u}}, (\tfrac{1}{2} H_i - G_i I_i) Z_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} \\
&\qquad + \left\langle I_i Z_t^{\boldsymbol{u}}, I_i Z_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} \Big) dt \\
&\quad + \sum_{i=1}^{q} \Big(\left\langle \xi_t^{\boldsymbol{u}}, I_i Z_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} + \left\langle I_i Z_t^{\boldsymbol{u}}, \xi_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} \Big) dW_i(t).
\end{aligned}$$

Integrating by parts for the first and the last term in the right hand side of the above equation and noting the homogeneous Neumann boundary condition of $\boldsymbol{g}_i$, we obtain

$$\begin{aligned}
d\left\langle \xi_t^{\boldsymbol{u}}, \xi_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} &= \sum_{i=1}^{q} \Big(- \left\langle \nabla (\tfrac{1}{2} H_i - G_i I_i) Z_t^{\boldsymbol{u}}, Z_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} + \left\langle \xi_t^{\boldsymbol{u}}, (\tfrac{1}{2} H_i - G_i I_i) Z_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} \\
&\qquad + \left\langle I_i Z_t^{\boldsymbol{u}}, I_i Z_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} \Big) dt \\
&\quad + \sum_{i=1}^{q} \Big(\left\langle \xi_t^{\boldsymbol{u}}, I_i Z_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} - \left\langle \nabla (I_i Z_t^{\boldsymbol{u}}), Z_t^{\boldsymbol{v}} \right\rangle_{\mathbb{L}^2(D)} \Big) dW_i(t). \qquad (3.24)
\end{aligned}$$

Hence, the resutl follows from replacing t by s and intergrating (3.24) over $[0,t]$. □

Remark 3.7. *By using integration by parts and the homogeneous Neumann boundary conditions of $\boldsymbol{g}_i$ for $i = 1, \cdots, q$ we obtain some symmetry properties of functions $F_{1,i}$, $F_{2,i}$ and F: for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{H}^1(D)$,*

$$F_{1,i}(t, \boldsymbol{u}, \boldsymbol{v}) = F_{1,i}(t, \boldsymbol{v}, \boldsymbol{u}); \quad F_{2,i}(t, \boldsymbol{u}, \boldsymbol{v}) = F_{2,i}(t, \boldsymbol{v}, \boldsymbol{u});$$

and hence, $F(t, \boldsymbol{u}, \boldsymbol{v}) = F(t, \boldsymbol{v}, \boldsymbol{u})$. Furthermore, it follows from (3.13) that

$$F(t, \boldsymbol{u}, \boldsymbol{v}_1 + \boldsymbol{v}_2) = F(t, \boldsymbol{u}, \boldsymbol{v}_1) + F(t, \boldsymbol{u}, \boldsymbol{v}_2).$$

The following lemmas state some important properties of F used throughout this paper.

Lemma 3.8. *Assume that $\boldsymbol{g}_i \in \mathbb{W}^{2,\infty}(D)$ for $i = 1, \cdots, q$. Then for any $\boldsymbol{u}, \boldsymbol{v} \in L^2(\Omega; \mathbb{H}^1(D))$ there exists a constant c depending on T and $\{\boldsymbol{g}_i\}_{i=1,\cdots,q}$ such that*

$$\mathbb{E} \sup_{t \in [0,T]} \|\nabla Z_t(\boldsymbol{u})\|^2_{\mathbb{L}^2(D)} \leq c \mathbb{E} \|\boldsymbol{u}\|^2_{\mathbb{H}^1(D)}, \qquad (3.25)$$

and for any $\epsilon > 0$,

$$\mathbb{E} \sup_{s \in [0,t]} \left| F(s, \boldsymbol{u}, \boldsymbol{v}) \right| \leq c\epsilon \mathbb{E} \|\boldsymbol{u}\|^2_{\mathbb{H}^1(D)} + c\epsilon^{-1} \mathbb{E} \|\boldsymbol{v}\|^2_{\mathbb{L}^2(D)}. \qquad (3.26)$$

Proof. It follows from (3.22) that

$$\mathbb{E}\|\nabla Z_t(\boldsymbol{u})\|^2_{\mathbb{L}^2(D)} = \mathbb{E}\|\nabla \boldsymbol{u}\|^2_{\mathbb{L}^2(D)} + \mathbb{E}[F(t,\boldsymbol{u},\boldsymbol{u})] \tag{3.27}$$
$$\leq \mathbb{E}\|\nabla \boldsymbol{u}\|^2_{\mathbb{L}^2(D)} + \sum_{i=1}^{q}\mathbb{E}\int_0^t \big|F_{1,i}(\tau,\boldsymbol{u},\boldsymbol{u})\big|\,d\tau$$
$$+ \mathbb{E}\Big|\sum_{i=1}^{q}\int_0^t F_{2,i}(\tau,\boldsymbol{u},\boldsymbol{u})\,dW_i(\tau)\Big|. \tag{3.28}$$

For convenience, we next estimate $|F(\tau,\boldsymbol{u},\boldsymbol{v})|$, which is a slightly more general version of $|F(\tau,\boldsymbol{u},\boldsymbol{u})|$. By using the elementary inequality

$$ab \leq \tfrac{1}{2}\epsilon a^2 + \tfrac{1}{2}\epsilon^{-1}b^2, \tag{3.29}$$

the assumption $\boldsymbol{g}_i \in \mathbb{W}^{2,\infty}(D)$ and (3.3), there holds

$$\begin{aligned}\big|F_{1,i}(\tau,\boldsymbol{u},\boldsymbol{v})\big| &\leq \big|\big\langle \nabla Z_\tau(\boldsymbol{u}), (\tfrac{1}{2}H_i - G_iI_i)Z_\tau(\boldsymbol{v})\big\rangle_{\mathbb{L}^2(D)}\big| \\ &\quad + \big|\big\langle \nabla(\tfrac{1}{2}H_i - G_iI_i)Z_\tau(\boldsymbol{u}), Z_\tau(\boldsymbol{v})\big\rangle_{\mathbb{L}^2(D)}\big| + \big|\big\langle I_iZ_\tau(\boldsymbol{u}), I_iZ_\tau(\boldsymbol{v})\big\rangle_{\mathbb{L}^2(D)}\big| \\ &\leq c\big(\epsilon\|\nabla Z_\tau(\boldsymbol{u})\|^2_{\mathbb{L}^2(D)} + \epsilon\|Z_\tau(\boldsymbol{u})\|^2_{\mathbb{L}^2(D)} + \epsilon^{-1}\|Z_\tau(\boldsymbol{v})\|^2_{\mathbb{L}^2(D)}\big) \\ &\leq c\big(\epsilon\|\nabla Z_\tau(\boldsymbol{u})\|^2_{\mathbb{L}^2(D)} + \epsilon\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D)} + \epsilon^{-1}\|\boldsymbol{v}\|^2_{\mathbb{L}^2(D)}\big).\end{aligned}$$

This implies that

$$\mathbb{E}\int_0^t \big|F_{1,i}(\tau,\boldsymbol{u},\boldsymbol{v})\big|\,d\tau \leq c\epsilon t\mathbb{E}\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D)} + c\epsilon^{-1}t\mathbb{E}\|\boldsymbol{v}\|^2_{\mathbb{L}^2(D)} + c\epsilon\mathbb{E}\int_0^t \|\nabla Z_\tau(\boldsymbol{u})\|^2_{\mathbb{L}^2(D)}\,d\tau. \tag{3.30}$$

Then, by using the Burkholder-Davis-Gundy inequality, Hölder inequality, (3.3) and (3.29), we estimate

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}\Big|\sum_{i=1}^{q}\int_0^s F_{2,i}(\tau,\boldsymbol{u},\boldsymbol{v})\,dW_i(\tau)\Big| &\leq c\mathbb{E}\Big|\sum_{i=1}^{q}\int_0^t \big(F_{2,i}(\tau,\boldsymbol{u},\boldsymbol{v})\big)^2\,d\tau\Big|^{1/2} \\ &\leq c\mathbb{E}\Big|\int_0^t \big(\|\nabla Z_\tau(\boldsymbol{u})\|_{\mathbb{L}^2(D)}\|Z_\tau(\boldsymbol{v})\|_{\mathbb{L}^2(D)} + \|Z_\tau(\boldsymbol{u})\|_{\mathbb{L}^2(D)}\|Z_\tau(\boldsymbol{v})\|_{\mathbb{L}^2(D)}\big)^2\,d\tau\Big|^{1/2} \\ &\leq c\mathbb{E}\Big|\int_0^t \big(\|\nabla Z_\tau(\boldsymbol{u})\|^2_{\mathbb{L}^2(D)}\|\boldsymbol{v}\|^2_{\mathbb{L}^2(D)} + \|\boldsymbol{u}\|^2_{\mathbb{L}^2(D)}\|\boldsymbol{v}\|^2_{\mathbb{L}^2(D)}\big)\,d\tau\Big|^{1/2} \\ &\leq c\mathbb{E}\Big[\|\boldsymbol{v}\|_{\mathbb{L}^2(D)}\Big(\int_0^t \|\nabla Z_\tau(\boldsymbol{u})\|^2_{\mathbb{L}^2(D)}\,d\tau\Big)^{1/2}\Big] + ct^{1/2}\mathbb{E}\big[\|\boldsymbol{u}\|_{\mathbb{L}^2(D)}\|\boldsymbol{v}\|_{\mathbb{L}^2(D)}\big] \qquad (3.31) \\ &\leq c\epsilon t^{1/2}\mathbb{E}\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D)} + c\epsilon^{-1}(t^{1/2}+1)\mathbb{E}\|\boldsymbol{v}\|^2_{\mathbb{L}^2(D)} + c\epsilon\mathbb{E}\int_0^t \|\nabla Z_\tau(\boldsymbol{u})\|^2_{\mathbb{L}^2(D)}\,d\tau. \qquad (3.32)\end{aligned}$$

We use (3.30) and (3.32) with $\boldsymbol{v} = \boldsymbol{u}$ and $\epsilon = 1$ together with (3.27) to deduce

$$\mathbb{E}\|\nabla Z_t(\boldsymbol{u})\|^2_{\mathbb{L}^2(D)} \leq c\mathbb{E}\|\boldsymbol{u}\|^2_{\mathbb{H}^1(D)} + c\mathbb{E}\int_0^t \|\nabla Z_\tau(\boldsymbol{u})\|^2_{\mathbb{L}^2(D)}\,d\tau.$$

Hence, the result (3.25) follows immediately by using Gronwall's inequality.

To prove (3.26) we note that

$$\mathbb{E}\sup_{s\in[0,t]}\big|F(s,\boldsymbol{u},\boldsymbol{v})\big| \le \sum_{i=1}^{q}\mathbb{E}\int_0^t \big|F_{1,i}(\tau,\boldsymbol{u},\boldsymbol{v})\big|\,d\tau + \mathbb{E}\sup_{s\in[0,t]}\Big|\sum_{i=1}^{q}\int_0^s F_{2,i}(\tau,\boldsymbol{u},\boldsymbol{v})\,dW_i(\tau)\Big|. \tag{3.33}$$

Hence, it follows from (3.30), (3.32) and (3.25) that

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}\Big|\sum_{i=1}^{q}\int_0^s F_{2,i}(\tau,\boldsymbol{u},\boldsymbol{v})\,dW_i(\tau)\Big| &\le c\epsilon\mathbb{E}\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D)} + c\epsilon^{-1}\mathbb{E}\|\boldsymbol{v}\|^2_{\mathbb{L}^2(D)}\\ &\quad + c\epsilon\mathbb{E}\int_0^t \|\nabla Z_\tau(\boldsymbol{u})\|^2_{\mathbb{L}^2(D)}\,d\tau\\ &\le c\big(\epsilon\mathbb{E}\|\boldsymbol{u}\|^2_{\mathbb{H}^1(D)} + \epsilon^{-1}\mathbb{E}\|\boldsymbol{v}\|^2_{\mathbb{L}^2(D)}\big),\end{aligned}$$

which completed the proof of the lemma. □

Lemma 3.9. *For any $\boldsymbol{u}\in L^2\big(\Omega;\mathbb{L}^2(D)\big)$, $\boldsymbol{v}\in L^2\big(\Omega;\mathbb{H}^1(D)\big)$ and $0\le s\le T$, there exists a constant c depending on $\{\boldsymbol{g}_i\}_{i=1,\cdots,q}$ such that*

$$\begin{aligned}\mathbb{E}|F(s,\boldsymbol{u},\boldsymbol{v})| &\le cs\big(\mathbb{E}[\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}[\|\nabla\boldsymbol{v}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\\ &\quad + c(s^{1/2}+s)\big(\mathbb{E}[\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}[\|\boldsymbol{v}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}.\end{aligned}$$

Proof. From the definition of function F in Lemma 3.6 and the triangle inequality, there holds

$$\mathbb{E}|F(s,\boldsymbol{u},\boldsymbol{v})| \le \sum_{i=1}^{q}\mathbb{E}\int_0^s \big|F_{1,i}(\tau,\boldsymbol{u},\boldsymbol{v})\big|\,d\tau + \mathbb{E}\Big|\sum_{i=1}^{q}\int_0^s F_{2,i}(\tau,\boldsymbol{u},\boldsymbol{v})\,dW_i(\tau)\Big|. \tag{3.34}$$

From Remark 3.7, we note that

$$F_{2,i}(\tau,\boldsymbol{u},\boldsymbol{v}) = F_{2,i}(\tau,\boldsymbol{v},\boldsymbol{u}),$$

and therefore, by using (3.31), the last term of (3.34) can be estimated as follows:

$$\begin{aligned}\mathbb{E}\Big|\sum_{i=1}^{q}\int_0^s F_{2,i}(\tau,\boldsymbol{u},\boldsymbol{v})\,dW_i(\tau)\Big| &= \mathbb{E}\Big|\sum_{i=1}^{q}\int_0^s F_{2,i}(\tau,\boldsymbol{v},\boldsymbol{u})\,dW_i(\tau)\Big|\\ &\le c\mathbb{E}\left[\|\boldsymbol{u}\|_{\mathbb{L}^2(D)}\Big(\int_0^s \|\nabla Z_\tau(\boldsymbol{v})\|^2_{\mathbb{L}^2(D)}\,d\tau\Big)^{1/2}\right]\\ &\quad + cs^{1/2}\mathbb{E}\left[\|\boldsymbol{u}\|_{\mathbb{L}^2(D)}\|\boldsymbol{v}\|_{\mathbb{L}^2(D)}\right].\end{aligned} \tag{3.35}$$

We now estimate $\big|F_{1,i}(\tau,\boldsymbol{u},\boldsymbol{v})\big|$ by integrating by parts and then using Hölder's inequality, the assumption $\boldsymbol{g}_i\in\mathbb{W}^{2,\infty}(D)$ and (3.3) as follows:

$$\begin{aligned}\big|F_{1,i}(\tau,\boldsymbol{u},\boldsymbol{v})\big| &= \Big|-\big\langle Z_\tau(\boldsymbol{u}),\nabla\big((\tfrac12 H_i - G_iI_i)Z_\tau(\boldsymbol{v})\big)\big\rangle_{\mathbb{L}^2(D)}\\ &\quad + \big\langle(\tfrac12 H_i - G_iI_i)Z_\tau(\boldsymbol{u}),\nabla Z_\tau(\boldsymbol{v})\big\rangle_{\mathbb{L}^2(D)}\\ &\quad + \langle I_iZ_\tau(\boldsymbol{u}),I_iZ_\tau(\boldsymbol{v})\rangle_{\mathbb{L}^2(D)}\Big|\\ &\le c\|\boldsymbol{u}\|_{\mathbb{L}^2(D)}\big(\|\nabla Z_\tau\boldsymbol{v}\|_{\mathbb{L}^2(D)} + \|\boldsymbol{v}\|_{\mathbb{L}^2(D)}\big),\end{aligned}$$

and therefore,

$$\mathbb{E}\int_0^s \big|F_{1,i}(\tau,\boldsymbol{u},\boldsymbol{v})\big|\,d\tau \le c\mathbb{E}\left[\|\boldsymbol{u}\|_{\mathbb{L}^2(D)}\big(\int_0^s \|\nabla Z_\tau \boldsymbol{v}\|_{\mathbb{L}^2(D)}\,d\tau\big)\right] + cs\mathbb{E}\left[\|\boldsymbol{u}\|_{\mathbb{L}^2(D)}\|\boldsymbol{v}\|_{\mathbb{L}^2(D)}\right]. \tag{3.36}$$

Hence, by using Hölder inequality we obtain from (3.34)–(3.36) that there holds:

$$\begin{aligned}\mathbb{E}|F(s,\boldsymbol{u},\boldsymbol{v})| &\le c\mathbb{E}\left[\|\boldsymbol{u}\|_{\mathbb{L}^2(D)}\big(\int_0^s \|\nabla Z_\tau \boldsymbol{v}\|_{\mathbb{L}^2(D)}\,d\tau\big)\right] \\ &\quad + c(s^{1/2}+s)\mathbb{E}\left[\|\boldsymbol{u}\|_{\mathbb{L}^2(D)}\|\boldsymbol{v}\|_{\mathbb{L}^2(D)}\right] \\ &\le c\big(\mathbb{E}[\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}\big[(\int_0^s \|\nabla Z_\tau \boldsymbol{v}\|_{\mathbb{L}^2(D)}\,d\tau)^2\big]\big)^{1/2} \\ &\quad + c(s^{1/2}+s)\big(\mathbb{E}[\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}[\|\boldsymbol{v}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}.\end{aligned} \tag{3.37}$$

Via the Minkowski inequality and (3.25), we observe that

$$\big(\mathbb{E}\big[(\int_0^s \|\nabla Z_\tau \boldsymbol{v}\|_{\mathbb{L}^2(D)}\,d\tau)^2\big]\big)^{1/2} \le \int_0^s \big(\mathbb{E}[\|\nabla Z_\tau \boldsymbol{v}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\,d\tau \le cs\big(\mathbb{E}[\|\nabla \boldsymbol{v}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}. \tag{3.38}$$

The required result follows from (3.37) and (3.38), which completes the proof of this lemma. □

4. Equivalence of weak solutions

In this section we use the process $(Z_t)_{t\ge 0}$ defined in the preceding section to define a new process $\boldsymbol{m}$ from $\boldsymbol{M}$. Let

$$\boldsymbol{m}(t,\boldsymbol{x}) = Z_t^{-1}\boldsymbol{M}(t,\boldsymbol{x}) \quad \forall t\ge 0,\ a.e.\boldsymbol{x}\in D. \tag{4.1}$$

We will show that this new variable $\boldsymbol{m}$ is differentiable with respect to t.

In the next lemma, we introduce the equation satisfied by $\boldsymbol{m}$ so that $\boldsymbol{M}$ is a solution to (1.3) in the sense of (2.1).

Lemma 4.1. *If $\boldsymbol{m}(\cdot,\omega)\in H^1(0,T;\mathbb{L}^2(D))\cap L^2(0,T;\mathbb{H}^1(D))$, for $\mathbb{P}$-a.s. $\omega\in\Omega$, satisfies*

$$|\boldsymbol{m}(t,\boldsymbol{x})| = 1 \quad \forall t\ge 0,\ a.e.\ \boldsymbol{x}\in D,\ \mathbb{P}-a.s.,$$

and for any $\boldsymbol{\psi}\in L^2(0,T;\mathbb{W}^{1,\infty}(D))$

$$\begin{aligned}&\langle \partial_t \boldsymbol{m},\boldsymbol{\psi}\rangle_{\mathbb{L}^2(D_T)} + \lambda_1\int_0^T \langle Z_s\boldsymbol{m}\times\nabla Z_s\boldsymbol{m},\nabla Z_s\boldsymbol{\psi}\rangle_{\mathbb{L}^2(D)}\,ds \\ &\qquad + \lambda_2\int_0^T \langle Z_s\boldsymbol{m}\times\nabla Z_s\boldsymbol{m},\nabla Z_s(\boldsymbol{m}\times\boldsymbol{\psi})\rangle_{\mathbb{L}^2(D)}\,ds = 0, \quad \mathbb{P}\text{-a.s.}.\end{aligned} \tag{4.2}$$

Then $\boldsymbol{M} = Z_t\boldsymbol{m}$ satisfies (2.1) $\mathbb{P}$-a.s..

Proof. Using Itô's formula for $\boldsymbol{M} = Z_t\boldsymbol{m}$, we deduce

$$\begin{aligned}\boldsymbol{M}(t) &= \boldsymbol{M}(0) + \sum_{i=1}^q \int_0^t Z\boldsymbol{m}\times\boldsymbol{g}_i\circ dW_i(s) + \int_0^t Z(\partial_t\boldsymbol{m})\,ds \\ &= \boldsymbol{M}(0) + \sum_{i=1}^q \int_0^t \boldsymbol{M}\times\boldsymbol{g}_i\circ dW_i(s) + \int_0^t Z_s(\partial_t\boldsymbol{m})\,ds.\end{aligned}$$

Multiplying both sides by a test function $\boldsymbol{\psi} \in \mathbb{C}_0^\infty(D)$ and integrating over D we obtain

$$\begin{aligned}\langle \boldsymbol{M}(t), \boldsymbol{\psi}\rangle_{\mathbb{L}^2(D)} &= \langle \boldsymbol{M}(0), \boldsymbol{\psi}\rangle_{\mathbb{L}^2(D)} + \sum_{i=1}^q \int_0^t \langle \boldsymbol{M}\times \boldsymbol{g}_i, \boldsymbol{\psi}\rangle_{\mathbb{L}^2(D)} \circ dW_i(s)\\ &\quad + \int_0^t \langle Z_s(\partial_t \boldsymbol{m}), \boldsymbol{\psi}\rangle_{\mathbb{L}^2(D)}\, ds\\ &= \langle \boldsymbol{M}(0), \boldsymbol{\psi}\rangle_{\mathbb{L}^2(D)} + \sum_{i=1}^q \int_0^t \langle \boldsymbol{M}\times \boldsymbol{g}_i, \boldsymbol{\psi}\rangle_{\mathbb{L}^2(D)} \circ dW_i(s)\\ &\quad + \int_0^t \left\langle \partial_t \boldsymbol{m}, Z_s^{-1}\boldsymbol{\psi}\right\rangle_{\mathbb{L}^2(D)}\, ds, \end{aligned} \tag{4.3}$$

where in the last step we used (3.16). On the other hand, it follows from (4.2) that, for all $\boldsymbol{\xi} \in L^2(0,t;\mathbb{W}^{1,\infty}(D))$, there holds:

$$\begin{aligned}\int_0^t \langle \partial_t \boldsymbol{m}, \boldsymbol{\xi}\rangle_{\mathbb{L}^2(D)}\, ds &= -\lambda_1 \int_0^t \langle Z_s\boldsymbol{m}\times \nabla Z_s\boldsymbol{m}, \nabla Z_s\boldsymbol{\xi}\rangle_{\mathbb{L}^2(D)}\, ds\\ &\quad - \lambda_2 \int_0^t \langle Z_s\boldsymbol{m}\times \nabla Z_s\boldsymbol{m}, \nabla Z_s(\boldsymbol{m}\times\boldsymbol{\xi})\rangle_{\mathbb{L}^2(D)}\, ds. \end{aligned} \tag{4.4}$$

Using (4.4) with $\boldsymbol{\xi} = Z_s^{-1}\boldsymbol{\psi}$ for the last term on the right hand side of (4.3) we deduce

$$\begin{aligned}\langle \boldsymbol{M}(t), \boldsymbol{\psi}\rangle_{\mathbb{L}^2(D)} &= \langle \boldsymbol{M}(0), \boldsymbol{\psi}\rangle_{\mathbb{L}^2(D)} + \sum_{i=1}^q \int_0^t \langle \boldsymbol{M}\times \boldsymbol{g}_i, \boldsymbol{\psi}\rangle_{\mathbb{L}^2(D)} \circ dW_i(s)\\ &\quad - \lambda_1 \int_0^t \left\langle \boldsymbol{M}\times\nabla\boldsymbol{M}, \nabla\boldsymbol{\psi})\right\rangle_{\mathbb{L}^2(D)}\, ds\\ &\quad - \lambda_2 \int_0^t \left\langle \boldsymbol{M}\times\nabla\boldsymbol{M}, \nabla Z_s(\boldsymbol{m}\times Z_s^{-1}\boldsymbol{\psi})\right\rangle_{\mathbb{L}^2(D)}\, ds. \end{aligned}$$

It follows from (3.17) that

$$\begin{aligned}\langle \boldsymbol{M}(t), \boldsymbol{\psi}\rangle_{\mathbb{L}^2(D)} &= \langle \boldsymbol{M}(0), \boldsymbol{\psi}\rangle_{\mathbb{L}^2(D)} + \sum_{i=1}^q \int_0^t \langle \boldsymbol{M}\times \boldsymbol{g}_i, \boldsymbol{\psi}\rangle_{\mathbb{L}^2(D)} \circ dW_i(s)\\ &\quad - \lambda_1 \int_0^t \left\langle \boldsymbol{M}\times\nabla\boldsymbol{M}, \nabla\boldsymbol{\psi})\right\rangle_{\mathbb{L}^2(D)}\, ds\\ &\quad - \lambda_2 \int_0^t \langle \boldsymbol{M}\times\nabla\boldsymbol{M}, \nabla(\boldsymbol{M}\times\boldsymbol{\psi})\rangle_{\mathbb{L}^2(D)}\, ds, \end{aligned}$$

which complete the proof. □

The following lemma shows that the constraint on $|\boldsymbol{m}|$ is inherited by $|\boldsymbol{M}|$.

Lemma 4.2. *The process $\boldsymbol{M}$ satisfies*

$$|\boldsymbol{M}(t,\boldsymbol{x})| = 1 \quad \forall t \geq 0,\ a.e.\, \boldsymbol{x} \in D,\ \mathbb{P}-a.s.$$

if and only if $\boldsymbol{m}$ defined in (4.1) satisfies

$$|\boldsymbol{m}(t,\boldsymbol{x})| = 1 \quad \forall t \geq 0,\ a.e. \boldsymbol{x} \in D,\ \mathbb{P}-a.s..$$

Proof. The proof follows by using (3.16):

$$|\boldsymbol{m}|^2 = \langle \boldsymbol{m}, \boldsymbol{m}\rangle = \left\langle Z_t^{-1}\boldsymbol{M}, Z_t^{-1}\boldsymbol{M}\right\rangle = \left\langle \boldsymbol{M}, Z_t Z_t^{-1}\boldsymbol{M}\right\rangle = \langle \boldsymbol{M}, \boldsymbol{M}\rangle = |\boldsymbol{M}|^2.$$

□

In the next lemma we provide a relationship between equation (4.2) and its Gilbert form.

Lemma 4.3. *Let $\boldsymbol{m}(\cdot,\omega) \in H^1(0,T;\mathbb{L}^2(D)) \cap L^2(0,T;\mathbb{H}^1(D))$ for $\mathbb{P}$-a.s. $\omega \in \Omega$ satisfy*

$$|\boldsymbol{m}(t,\boldsymbol{x})| = 1, \quad t \in (0,T),\ \boldsymbol{x} \in D, \tag{4.5}$$

and

$$\lambda_1 \langle \partial_t \boldsymbol{m}, \boldsymbol{\varphi}\rangle_{\mathbb{L}^2(D_T)} + \lambda_2 \langle \boldsymbol{m}\times\partial_t\boldsymbol{m}, \boldsymbol{\varphi}\rangle_{\mathbb{L}^2(D_T)} = \mu \int_0^T \langle \nabla Z_s\boldsymbol{m}, \nabla Z_s(\boldsymbol{m}\times\boldsymbol{\varphi})\rangle_{\mathbb{L}^2(D)}\, ds, \tag{4.6}$$

where $\mu = \lambda_1^2 + \lambda_2^2$. Then $\boldsymbol{m}$ satisfies (4.2).

Proof. For each $\boldsymbol{\psi} \in L^2(0,T;\mathbb{W}^{1,\infty}(D))$, using Lemma 7.1 in the Appendix, there exists $\boldsymbol{\varphi} \in L^2(0,T;\mathbb{H}^1(D))$ such that

$$\lambda_1\boldsymbol{\varphi} + \lambda_2\boldsymbol{\varphi}\times\boldsymbol{m} = \boldsymbol{\psi}. \tag{4.7}$$

We can write (4.6) as

$$\begin{aligned}\langle \partial_t\boldsymbol{m}, \lambda_1\boldsymbol{\varphi} + \lambda_2\boldsymbol{\varphi}\times\boldsymbol{m}\rangle_{\mathbb{L}^2(D_T)} &+ \lambda_1\int_0^T \langle Z_s\boldsymbol{m}\times\nabla Z_s\boldsymbol{m}, \nabla Z_s(\lambda_1\boldsymbol{\varphi})\rangle_{\mathbb{L}^2(D)}\, ds \\ &+ \lambda_2\int_0^T \langle \nabla Z_s\boldsymbol{m}, \nabla Z_s(\lambda_2\boldsymbol{\varphi}\times\boldsymbol{m})\rangle_{\mathbb{L}^2(D)}\, ds = 0.\end{aligned} \tag{4.8}$$

From (4.5) and (3.3), we obtain that

$$|Z_t\boldsymbol{m}(t,\boldsymbol{x})| = 1, \quad \forall t \in (0,T) \text{ and } \boldsymbol{x} \in D. \tag{4.9}$$

On the other hand, by using (4.9), (3.19) and a standard identity

$$\langle \boldsymbol{a}, \boldsymbol{b}\times\boldsymbol{c}\rangle = \langle \boldsymbol{b}, \boldsymbol{c}\times\boldsymbol{a}\rangle = \langle \boldsymbol{c}, \boldsymbol{a}\times\boldsymbol{b}\rangle, \quad \text{for all } \boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in \mathbb{R}^3, \tag{4.10}$$

we obtain

$$\begin{aligned}\lambda_1\int_0^T \langle Z_s\boldsymbol{m}\times\nabla Z_s\boldsymbol{m}, \nabla Z_s(\lambda_2\boldsymbol{\varphi}\times\boldsymbol{m})\rangle_{\mathbb{L}^2(D)}\, ds &+ \lambda_2\int_0^T \langle \nabla Z_s\boldsymbol{m}, \nabla Z_s(\lambda_1\boldsymbol{\varphi})\rangle_{\mathbb{L}^2(D)}\, ds \\ &- \lambda_2\int_0^T \left\langle |\nabla Z_s\boldsymbol{m}|^2 Z_s\boldsymbol{m}, Z_s(\lambda_1\boldsymbol{\varphi})\right\rangle_{\mathbb{L}^2(D)}\, ds = 0.\end{aligned} \tag{4.11}$$

Moreover, we have

$$-\lambda_2\int_0^T \left\langle |\nabla Z_s\boldsymbol{m}|^2 Z_s\boldsymbol{m}, \lambda_2 Z_s\boldsymbol{\varphi}\times Z_s\boldsymbol{m}\right\rangle_{\mathbb{L}^2(D)}\, ds = 0. \tag{4.12}$$

Summing (4.8), (4.11) and (4.12) gives

$$\begin{aligned}\langle \partial_t\boldsymbol{m}, \lambda_1\boldsymbol{\varphi} + \lambda_2\boldsymbol{\varphi}\times\boldsymbol{m}\rangle_{\mathbb{L}^2(D_T)} &+ \lambda_1\int_0^T \langle Z_s\boldsymbol{m}\times\nabla Z_s\boldsymbol{m}, \nabla Z_s(\lambda_1\boldsymbol{\varphi} + \lambda_2\boldsymbol{\varphi}\times\boldsymbol{m})\rangle_{\mathbb{L}^2(D)}\, ds \\ &+ \lambda_2\int_0^T \langle \nabla Z_s\boldsymbol{m}, \nabla Z_s(\lambda_1\boldsymbol{\varphi} + \lambda_2\boldsymbol{\varphi}\times\boldsymbol{m})\rangle_{\mathbb{L}^2(D)}\, ds \\ &- \lambda_2\int_0^T \left\langle |\nabla Z_s\boldsymbol{m}|^2 Z_s\boldsymbol{m}, Z_s(\lambda_1\boldsymbol{\varphi} + \lambda_2\boldsymbol{\varphi}\times\boldsymbol{m})\right\rangle_{\mathbb{L}^2(D)}\, ds = 0\end{aligned}$$

The desired equation (4.2) follows by noting (4.7) and using (3.19), (4.10) and (4.9). □

Remark 4.4. *By using* (4.10) *and* (4.5) *we can rewrite* (4.6) *as*

$$\lambda_1 \langle \boldsymbol{m} \times \partial_t \boldsymbol{m}, \boldsymbol{w} \rangle_{\mathbb{L}^2(D_T)} - \lambda_2 \langle \partial_t \boldsymbol{m}, \boldsymbol{w} \rangle_{\mathbb{L}^2(D_T)} = \mu \int_0^T \langle \nabla Z_s \boldsymbol{m}, \nabla Z_s \boldsymbol{w} \rangle_{\mathbb{L}^2(D)} \, ds, \tag{4.13}$$

or equivalently, thanks to Lemma 3.6,

$$\begin{aligned} \lambda_1 \langle \boldsymbol{m} \times \partial_t \boldsymbol{m}, \boldsymbol{w} \rangle_{\mathbb{L}^2(D_T)} - \lambda_2 \langle \partial_t \boldsymbol{m}, \boldsymbol{w} \rangle_{\mathbb{L}^2(D_T)} =& \mu \langle \nabla \boldsymbol{m}, \nabla \boldsymbol{w} \rangle_{\mathbb{L}^2(D_T)} \\ &+ \mu \int_0^T F(t, \boldsymbol{m}(t,\cdot), \boldsymbol{w}(t,\cdot)) \, dt, \end{aligned} \tag{4.14}$$

where $\boldsymbol{w} = \boldsymbol{m} \times \boldsymbol{\varphi}$ *for* $\boldsymbol{\varphi} \in L^2(0,T;\mathbb{H}^1(D))$. *We note in particular that* $\boldsymbol{w} \cdot \boldsymbol{m} = 0$. *This property will be exploited later in the design of the finite element scheme.*

We state the following lemma as a consequence of Lemmas 4.3, 4.2 and 4.1.

Lemma 4.5. *Let* $\boldsymbol{m}(\cdot,\omega) \in H^1(0,T;\mathbb{L}^2(D)) \cap L^2(0,T;\mathbb{H}^1(D))$ *for* $\mathbb{P}$*-a.s.* $\omega \in \Omega$. *If* $\boldsymbol{m}$ *is a solution of* (4.5)– (4.6)*, then* $\boldsymbol{M} = Z_t \boldsymbol{m}$ *is a weak martingale solution of* (1.3) *in the sense of Definition 2.2.*

Proof. By using Lemmas 4.1, 4.2 and 4.3 together with the imbedding $H^1(0,T;\mathbb{L}^2(D)) \hookrightarrow C(0,T;\mathbb{H}^{-1}(D)$, we deduce that $\boldsymbol{M}$ satisfies (1), (2), (3), (4) in Definition 2.2, which completes the proof. □

Thanks to the above lemma, we now can now restrict our attention to solving equation (4.6) rather than (2.1).

5. The finite element scheme

In this section we design a finite element scheme to find approximate solutions to (4.6). In the next section, we prove that the finite element solutions converge to a solution of (4.6). Then, thanks to Lemma 4.5, we obtain a weak solution of (2.1).

Let $\mathbb{T}_h$ be a regular tetrahedrization of the domain D into tetrahedra of maximal mesh-size h. We denote by $\mathcal{N}_h := \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ the set of vertices and introduce the finite-element space $\mathbb{V}_h \subset \mathbb{H}^1(D)$, which is the space of all continuous piecewise linear functions on $\mathbb{T}_h$. A basis for $\mathbb{V}_h$ can be chosen to be $\{\phi_n \boldsymbol{\xi}_1, \phi_n \boldsymbol{\xi}_2, \phi_n \boldsymbol{\xi}_3\}_{1 \le n \le N}$, where $\{\boldsymbol{\xi}_i\}_{i=1,\cdots,3}$ is the canonical basis for $\mathbb{R}^3$ and $\phi_n(\boldsymbol{x}_m) = \delta_{n,m}$. Here $\delta_{n,m}$ denotes the Kronecker delta symbol. The interpolation operator from $\mathbb{C}^0(D)$ onto $\mathbb{V}_h$, denoted by $I_{\mathbb{V}_h}$, is defined by

$$I_{\mathbb{V}_h}(\boldsymbol{v}) = \sum_{n=1}^{N} \boldsymbol{v}(\boldsymbol{x}_n) \phi_n(\boldsymbol{x}) \quad \forall \boldsymbol{v} \in \mathbb{C}^0(D, \mathbb{R}^3).$$

Before introducing the finite element scheme, we state the following result proved by Bartels [4], which will be used in the subsequent analysis.

Lemma 5.1. *Assume that*

$$\int_D \nabla \phi_i \cdot \nabla \phi_j \, d\boldsymbol{x} \le 0 \quad \text{for all} \quad i,j \in \{1,2,\cdots,J\} \text{ and } i \ne j. \tag{5.1}$$

Then for all $\boldsymbol{u} \in \mathbb{V}_h$ *satisfying* $|\boldsymbol{u}(\boldsymbol{x}_l)| \ge 1$, $l = 1,2,\cdots,J$, *there holds*

$$\int_D \left| \nabla I_{\mathbb{V}_h} \left(\frac{\boldsymbol{u}}{|\boldsymbol{u}|} \right) \right|^2 d\boldsymbol{x} \le \int_D |\nabla \boldsymbol{u}|^2 \, d\boldsymbol{x}. \tag{5.2}$$

When $d = 2$, we note that condition (5.1) holds for Delaunay triangulation. Roughly speaking, a Delaunay triangulation is one in which no vertex is contained inside the perimeter of any triangle. When $d = 3$, condition (5.1) holds if all dihedral angles of the tetrahedra in $\mathbb{T}_h|_{\mathbb{L}^2(D)}$ are less than or equal to $\pi/2$; see [4]. In what follows we assume that (5.1) holds.

To discretize the equation (4.6), we fix a positive integer J, choose the time step k to be $k = T/J$ and define $t_j = jk$, $j = 0, \cdots, J$. For $j = 1, 2, \ldots, J$, the solution $\boldsymbol{m}(t_j, \cdot)$ is approximated by $\boldsymbol{m}_h^{(j)} \in \mathbb{V}_h$, which is computed as follows.

Since

$$\boldsymbol{m}_t(t_j, \cdot) \approx \frac{\boldsymbol{m}(t_{j+1}, \cdot) - \boldsymbol{m}(t_j, \cdot)}{k} \approx \frac{\boldsymbol{m}_h^{(j+1)} - \boldsymbol{m}_h^{(j)}}{k},$$

we can define $\boldsymbol{m}_h^{(j+1)}$ from $\boldsymbol{m}_h^{(j)}$ by

$$\boldsymbol{m}_h^{(j+1)} = \boldsymbol{m}_h^{(j)} + k\boldsymbol{v}_h^{(j)}, \tag{5.3}$$

where $\boldsymbol{v}_h^{(j)}$ is an approximation of $\boldsymbol{m}_t(t_j, \cdot)$. Hence, it suffices to propose a scheme to compute $\boldsymbol{v}_h^{(j)}$.

Motivated by the property $\partial_t \boldsymbol{m} \cdot \boldsymbol{m} = 0$, we will find $\boldsymbol{v}_h^{(j)}$ in the space $\mathbb{W}_h^{(j)}$ defined by

$$\mathbb{W}_h^{(j)} := \left\{ \boldsymbol{w} \in \mathbb{V}_h \mid \boldsymbol{w}(\boldsymbol{x}_n) \cdot \boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n) = 0,\ n = 1, \ldots, N \right\}. \tag{5.4}$$

Given $\boldsymbol{m}_h^{(j)} \in \mathbb{V}_h$, we use (4.14) to define $\boldsymbol{v}_h^{(j)}$ instead of using (4.6) so that the same test and trial functions can be used (see Remark 4.4). Hence, we define by $\boldsymbol{v}_h^{(j)} \in \mathbb{W}_h^{(j)}$ satisfying the following equation

$$\begin{aligned} \lambda_1 \left\langle \boldsymbol{m}_h^{(j)} \times \boldsymbol{v}_h^{(j)}, \boldsymbol{w}_h^{(j)} \right\rangle_{\mathbb{L}^2(D)} - \lambda_2 \left\langle \boldsymbol{v}_h^{(j)}, \boldsymbol{w}_h^{(j)} \right\rangle_{\mathbb{L}^2(D)} &= \mu \left\langle \nabla(\boldsymbol{m}_h^{(j)} + k\theta \boldsymbol{v}_h^{(j)}), \nabla \boldsymbol{w}_h^{(j)} \right\rangle_{\mathbb{L}^2(D)} \\ &\quad + \mu F(t_j, \boldsymbol{m}_h^{(j)}, \boldsymbol{w}_h^{(j)}) \quad \mathbb{P}\text{-a.s.}. \end{aligned} \tag{5.5}$$

We summarise the algorithm as follows.

Algorithm 5.1.

Step 1: *Set* $j = 0$. *Choose* $\boldsymbol{m}_h^{(0)} = I_{\mathbb{V}_h} \boldsymbol{m}_0$.

Step 2: *Find* $\boldsymbol{v}_h^{(j)} \in \mathbb{W}_h^{(j)}$ *satisfying* (5.5).

Step 3: *Define*

$$\boldsymbol{m}_h^{(j+1)}(\boldsymbol{x}) := \sum_{n=1}^{N} \frac{\boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n) + k\boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n)}{\left| \boldsymbol{m}_h^{(j)}(\boldsymbol{x}_n) + k\boldsymbol{v}_h^{(j)}(\boldsymbol{x}_n) \right|} \phi_n(\boldsymbol{x}).$$

Step 4: *Set* $j = j + 1$, *and return to Step 2 if* $j < J$. *Stop if* $j = J$.

Since $\left| \boldsymbol{m}_h^{(0)}(x_n) \right| = 1$ and $\boldsymbol{v}_h^{(j)}(x_n) \cdot \boldsymbol{m}_h^{(j)}(x_n) = 0$ for all $n = 1, \ldots, N$ and $j = 0, \ldots, J$, we obtain (by induction)

$$\left| \boldsymbol{m}_h^{(j)}(x_n) + k\boldsymbol{v}_h^{(j)}(x_n) \right| \geq 1 \quad \text{and} \quad \left| \boldsymbol{m}_h^{(j)}(x_n) \right| = 1, \quad j = 0, \ldots, J. \tag{5.6}$$

In particular, (5.6) shows that the algorithm is well defined.

We finish this section by proving the following three lemmas concerning some properties of $\boldsymbol{m}_h^{(j)}$ and $R_{h,k}$.

Lemma 5.2. *For any $j = 0, \dots, J$,*

$$\|\boldsymbol{m}_h^{(j)}\|_{\mathbb{L}^\infty(D)} \leq 1 \quad \textit{and} \quad \|\boldsymbol{m}_h^{(j)}\|_{\mathbb{L}^2(D)} \leq |D|,$$

where $|D|$ denotes the measure of D.

Proof. The first inequality follows from (5.6) and the second can be obtained by integrating $\boldsymbol{m}_h^{(j)}(\boldsymbol{x})$ over D. □

Lemma 5.3. *There exist a deterministic constant c depending on $\boldsymbol{m}_0$, $\{\boldsymbol{g}_i\}_{i=1}^q$, λ_1 and λ_2 such that for any $\theta \in [1/2, 1]$ and for $j = 1, \dots, J$ there holds*

$$\mathbb{E}\left\|\nabla \boldsymbol{m}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 + k\sum_{i=1}^{j-1}\mu^{-1}\lambda_2\mathbb{E}\left\|\boldsymbol{v}_h^{(i)}\right\|_{\mathbb{L}^2(D)}^2 + k^2(2\theta-1)\sum_{i=1}^{j-1}\mathbb{E}\left\|\nabla \boldsymbol{v}_h^{(i)}\right\|_{\mathbb{L}^2(D)}^2 \leq c.$$

Proof. Taking $\boldsymbol{w}_h^{(j)} = \boldsymbol{v}_h^{(j)}$ in equation (5.5) yields to the following identity:

$$-\lambda_2\left\|\boldsymbol{v}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 = \mu\left\langle\nabla \boldsymbol{m}_h^{(j)}, \nabla \boldsymbol{v}_h^{(j)}\right\rangle_{\mathbb{L}^2(D)} + \mu k\theta\left\|\nabla \boldsymbol{v}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 + \mu F\big(t_j, \boldsymbol{m}_h^{(j)}, \boldsymbol{v}_h^{(j)}\big),$$

or equivalently

$$\mu\left\langle\nabla \boldsymbol{m}_h^{(j)}, \nabla \boldsymbol{v}_h^{(j)}\right\rangle_{\mathbb{L}^2(D)} = -\lambda_2\left\|\boldsymbol{v}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 - \mu k\theta\left\|\nabla \boldsymbol{v}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 - \mu F\big(t_j, \boldsymbol{m}_h^{(j)}, \boldsymbol{v}_h^{(j)}\big). \tag{5.7}$$

From Lemma 5.1 it follows that

$$\left\|\nabla \boldsymbol{m}_h^{(j+1)}\right\|_{\mathbb{L}^2(D)}^2 \leq \left\|\nabla(\boldsymbol{m}_h^{(j)} + k\boldsymbol{v}_h^{(j)})\right\|_{\mathbb{L}^2(D)}^2,$$

or equivalently,

$$\left\|\nabla \boldsymbol{m}_h^{(j+1)}\right\|_{\mathbb{L}^2(D)}^2 \leq \left\|\nabla \boldsymbol{m}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 + k^2\left\|\nabla \boldsymbol{v}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 + 2k\left\langle\nabla \boldsymbol{m}_h^{(j)}, \nabla \boldsymbol{v}_h^{(j)}\right\rangle_{\mathbb{L}^2(D)}.$$

Therefore, together with (5.7), we deduce

$$\begin{aligned}\left\|\nabla \boldsymbol{m}_h^{(j+1)}\right\|_{\mathbb{L}^2(D)}^2 + 2k\mu^{-1}\lambda_2\left\|\boldsymbol{v}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 + k^2(2\theta-1)\left\|\nabla \boldsymbol{v}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 &\leq \left\|\nabla \boldsymbol{m}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 \\ &\quad - 2kF\big(t_j, \boldsymbol{m}_h^{(j)}, \boldsymbol{v}_h^{(j)}\big).\end{aligned}$$

Thus, it follows from (3.26) that

$$\begin{aligned}\mathbb{E}\left\|\nabla \boldsymbol{m}_h^{(j+1)}\right\|_{\mathbb{L}^2(D)}^2 &+ 2k\mu^{-1}\lambda_2\mathbb{E}\left\|\boldsymbol{v}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 + k^2(2\theta-1)\mathbb{E}\left\|\nabla \boldsymbol{v}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 \\ &\leq \mathbb{E}\left\|\nabla \boldsymbol{m}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 + 2k\mathbb{E}\left[\left|F\big(t_j, \boldsymbol{m}_h^{(j)}, \boldsymbol{v}_h^{(j)}\big)\right|\right] \\ &\leq (1 + kc\epsilon T)\mathbb{E}\left\|\nabla \boldsymbol{m}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 + ck\epsilon(T + T^{1/2})\mathbb{E}\left\|\boldsymbol{m}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 \\ &\quad + ck\epsilon^{-1}(T + T^{1/2} + 1)\mathbb{E}\left\|\boldsymbol{v}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2.\end{aligned}$$

By choosing $\epsilon = \frac{\mu^{-1}\lambda_2}{c(T+T^{1/2}+1)}$ in the right hand side of this inequality and using Lemma 5.2 we deduce

$$\mathbb{E}\left\|\nabla \boldsymbol{m}_h^{(j+1)}\right\|_{\mathbb{L}^2(D)}^2 + k\mu^{-1}\lambda_2\mathbb{E}\left\|\boldsymbol{v}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 + k^2(2\theta-1)\mathbb{E}\left\|\nabla \boldsymbol{v}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 \leq ck + (1+kc)\mathbb{E}\left\|\nabla \boldsymbol{m}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2.$$

Replacing j by i in the above inequality and summing for i from 0 to $j-1$ yeilds

$$\mathbb{E}\left\|\nabla \boldsymbol{m}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 + k\sum_{i=1}^{j-1}\mu^{-1}\lambda_2\mathbb{E}\left\|\boldsymbol{v}_h^{(i)}\right\|_{\mathbb{L}^2(D)}^2 + k^2(2\theta-1)\sum_{i=1}^{j-1}\mathbb{E}\left\|\nabla \boldsymbol{v}_h^{(i)}\right\|_{\mathbb{L}^2(D)}^2 \leq ckj + c\|\nabla \boldsymbol{m}_h^{(0)}\|_{\mathbb{L}^2(D)}^2 + kc\sum_{i=1}^{j-1}\mathbb{E}\left\|\nabla \boldsymbol{m}_h^{(i)}\right\|_{\mathbb{L}^2(D)}^2. \tag{5.8}$$

Since $\boldsymbol{m}_0 \in \mathbb{H}^2(D)$ it can be shown that there exists a deterministic constant c depending only on $\boldsymbol{m}_0$ such that

$$\|\nabla \boldsymbol{m}_h^{(0)}\|_{\mathbb{L}^2(D)} \leq c. \tag{5.9}$$

Hence, inequality (5.8) implies

$$\mathbb{E}\left\|\nabla \boldsymbol{m}_h^{(j)}\right\|_{\mathbb{L}^2(D)}^2 + k\sum_{i=1}^{j-1}\mu^{-1}\lambda_2\mathbb{E}\left\|\boldsymbol{v}_h^{(i)}\right\|_{\mathbb{L}^2(D)}^2 + k^2(2\theta-1)\sum_{i=1}^{j-1}\mathbb{E}\left\|\nabla \boldsymbol{v}_h^{(i)}\right\|_{\mathbb{L}^2(D)}^2 \leq c + kc\sum_{i=1}^{j-1}\mathbb{E}\left\|\nabla \boldsymbol{m}_h^{(i)}\right\|_{\mathbb{L}^2(D)}^2. \tag{5.10}$$

By using induction and (5.9) we can show that

$$\mathbb{E}\|\nabla \boldsymbol{m}_h^i\|_{\mathbb{L}^2(D)}^2 \leq c(1+ck)^i.$$

Summing over i from 0 to $j-1$ and using $1+x \leq e^x$ we obtain

$$k\sum_{i=0}^{j-1}\mathbb{E}\left\|\nabla \boldsymbol{m}_h^i\right\|_{\mathbb{L}^2(D)}^2 \leq ck\frac{(1+ck)^j - 1}{ck} \leq e^{ckJ} = c.$$

This together with (5.10) gives the desired result. □

6. The main result

In this section, we will use the finite element function $\boldsymbol{m}_h^{(j)}$ to construct a sequence of functions that converges in an appropriate sense to a function that is a weak martingale solution of (1.3) in the sense of Definition 2.2.

The discrete solutions $\boldsymbol{m}_h^{(j)}$ and $\boldsymbol{v}_h^{(j)}$ constructed via Algorithm 5.1 are interpolated in time in the following definition.

Definition 6.1. *For all $x \in D$, $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}_h$ and all $t \in [0,T]$, let $j \in \{0, ..., J\}$ be such that $t \in [t_j, t_{j+1})$. We then define*

$$\begin{aligned}
\boldsymbol{m}_{h,k}(t,x) &:= \frac{t-t_j}{k}\boldsymbol{m}_h^{(j+1)}(x) + \frac{t_{j+1}-t}{k}\boldsymbol{m}_h^{(j)}(x),\\
\boldsymbol{m}_{h,k}^-(t,x) &:= \boldsymbol{m}_h^{(j)}(x),\\
\boldsymbol{v}_{h,k}(t,x) &:= \boldsymbol{v}_h^{(j)}(x),\\
F_k(t,\boldsymbol{u},\boldsymbol{v}) &:= F(t_j,\boldsymbol{u},\boldsymbol{v}) \quad \mathbb{P}-a.s..
\end{aligned}$$

We note that $\boldsymbol{m}_{h,k}(t)$ is an $\mathcal{F}_{t_j}$ adapted process for $t \in [t_j, t_{j+1})$. The above sequences have the following obvious bounds.

Lemma 6.2. *There exist a deterministic constant c depending on $\boldsymbol{m}_0$, $\boldsymbol{g}$, μ_1, μ_2 and T such that for all $\theta \in [1/2, 1]$,*

$$\mathbb{E}\|\boldsymbol{m}_{h,k}^*\|^2_{\mathbb{L}^2(D_T)} + \mathbb{E}\left\|\nabla \boldsymbol{m}_{h,k}^*\right\|^2_{\mathbb{L}^2(D_T)} + \mathbb{E}\left\|\boldsymbol{v}_{h,k}\right\|^2_{\mathbb{L}^2(D_T)} + k(2\theta-1)\mathbb{E}\left\|\nabla \boldsymbol{v}_{h,k}\right\|^2_{\mathbb{L}^2(D_T)} \le c,$$

where $\boldsymbol{m}_{h,k}^* = \boldsymbol{m}_{h,k}$ or $\boldsymbol{m}_{h,k}^-$. In particular, when $\theta \in [0, \frac{1}{2})$,

$$\mathbb{E}\|\boldsymbol{m}_{h,k}^*\|^2_{\mathbb{L}^2(D_T)} + \mathbb{E}\left\|\nabla \boldsymbol{m}_{h,k}^*\right\|^2_{\mathbb{L}^2(D_T)} + \left(1 + (2\theta-1)kh^{-2}\right)\mathbb{E}\left\|\boldsymbol{v}_{h,k}\right\|^2_{\mathbb{L}^2(D_T)} \le c.$$

Proof. It is easy to see that

$$\|\boldsymbol{m}_{h,k}^-\|^2_{\mathbb{L}^2(D_T)} = k\sum_{i=0}^{J-1}\|\boldsymbol{m}_h^{(i)}\|^2_{\mathbb{L}^2(D)} \quad \text{and} \quad \|\boldsymbol{v}_{h,k}\|^2_{\mathbb{L}^2(D_T)} = k\sum_{i=0}^{J-1}\|\boldsymbol{v}_h^{(i)}\|^2_{\mathbb{L}^2(D)}.$$

Both inequalities are direct consequences of Definition 6.1, Lemmas 5.2, and 5.3, noting that the second inequality requires the use of the inverse estimate (see e.g. [14])

$$\|\nabla \boldsymbol{v}_h^{(i)}\|^2_{\mathbb{L}^2(D)} \le ch^{-2}\|\boldsymbol{v}_h^{(i)}\|^2_{\mathbb{L}^2(D)}.$$

□

The next lemma provides a bound of $\boldsymbol{m}_{h,k}$ in the $\mathbb{H}^1$-norm and establishes relationships between $\boldsymbol{m}_{h,k}^-$, $\boldsymbol{m}_{h,k}$ and $\boldsymbol{v}_{h,k}$.

Lemma 6.3. *Assume that h and k approach 0, with the further condition $k = o(h^2)$ when $\theta \in [0, \frac{1}{2})$. The sequences $\{\boldsymbol{m}_{h,k}\}$, $\{\boldsymbol{m}_{h,k}^-\}$, and $\{\boldsymbol{v}_{h,k}\}$ defined in Definition 6.1 satisfy the following properties:*

$$\mathbb{E}\|\boldsymbol{m}_{h,k}\|^2_{\mathbb{H}^1(D_T)} \le c, \tag{6.1}$$

$$\mathbb{E}\|\boldsymbol{m}_{h,k} - \boldsymbol{m}_{h,k}^-\|^2_{\mathbb{L}^2(D_T)} \le ck^2, \tag{6.2}$$

$$\mathbb{E}\|\boldsymbol{v}_{h,k} - \partial_t \boldsymbol{m}_{h,k}\|_{\mathbb{L}^1(D_T)} \le ck, \tag{6.3}$$

$$\mathbb{E}\||\boldsymbol{m}_{h,k}| - 1\|^2_{\mathbb{L}^2(D_T)} \le c(h^2 + k^2). \tag{6.4}$$

Proof. The results can be obtained by using Lemma 6.2 and the arguments in the proof of [13, Lemma 6.3]. □

We now prove some properties of F and F_k, which will be used in the next two lemmas.

Lemma 6.4. *For any $\boldsymbol{u}, \boldsymbol{v} \in L^2\big(\Omega; L^2(0,T;\mathbb{H}^1(D))\big)$, there exists a constant c depending on T and $\{\boldsymbol{g}_i\}_{i=1,\cdots,q}$ such that*

(6.5)
$$\mathbb{E}[\int_0^T |F^*(t,\boldsymbol{u}(t,\cdot),\boldsymbol{v}(t,\cdot))|\,dt] \le c\big(\mathbb{E}[\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D_T)}]\big)^{1/2}\bigg(\big(\mathbb{E}[\|\nabla\boldsymbol{v}\|^2_{\mathbb{L}^2(D_T)}]\big)^{1/2}+\big(\mathbb{E}[\|\boldsymbol{v}\|^2_{\mathbb{L}^2(D_T)}]\big)^{1/2}\bigg),$$

here, $F^* = F$ or F_k. Furthermore,

$$\begin{aligned}&\mathbb{E}[\int_0^T |F(t,\boldsymbol{u}(t,\cdot),\boldsymbol{v}(t,\cdot)) - F_k(t,\boldsymbol{u}(t,\cdot),\boldsymbol{v}(t,\cdot))|\,dt]\\ &\qquad\le ck^{1/2}\big(\mathbb{E}[\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D_T)}]\big)^{1/2}\bigg(\big(\mathbb{E}[\|\nabla\boldsymbol{v}\|^2_{\mathbb{L}^2(D_T)}]\big)^{1/2} + \big(\mathbb{E}[\|\boldsymbol{v}\|^2_{\mathbb{L}^2(D_T)}]\big)^{1/2}\bigg).\end{aligned} \tag{6.6}$$

Proof. Proof of (6.5): The first result of the lemma for $F^* = F$ can be deduced from Lemma 3.9 by replacing $s \equiv t$, $\boldsymbol{u} \equiv \boldsymbol{u}(t,\cdot)$, $\boldsymbol{v} \equiv \boldsymbol{v}(t,\cdot)$ and using Hölder's inequality as follows:

$$\begin{aligned}\mathbb{E}[\int_0^T |F(t,\boldsymbol{u}(t,\cdot),\boldsymbol{v}(t,\cdot))|\,dt] &= \int_0^T \mathbb{E}[|F(t,\boldsymbol{u}(t,\cdot),\boldsymbol{v}(t,\cdot))|]\,dt\\ &\le c\int_0^T \big(\mathbb{E}[\|\boldsymbol{u}(t,\cdot)\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}[\|\nabla\boldsymbol{v}(t,\cdot)\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\,dt\\ &\quad + c\int_0^T \big(\mathbb{E}[\|\boldsymbol{u}(t,\cdot)\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}[\|\boldsymbol{v}(t,\cdot)\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\,dt\\ &\le c\big(\mathbb{E}[\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D_T)}]\big)^{1/2}\bigg(\big(\mathbb{E}[\|\nabla\boldsymbol{v}\|^2_{\mathbb{L}^2(D_T)}]\big)^{1/2} + \big(\mathbb{E}[\|\boldsymbol{v}\|^2_{\mathbb{L}^2(D_T)}]\big)^{1/2}\bigg).\end{aligned} \tag{6.7}$$

We first note that

$$\begin{aligned}\mathbb{E}[\int_0^T |F_k(t,\boldsymbol{u}(t,\cdot),\boldsymbol{v}(t,\cdot))|\,dt] &= \int_0^T \mathbb{E}[|F_k(t,\boldsymbol{u}(t,\cdot),\boldsymbol{v}(t,\cdot))|]\,dt\\ &= \sum_{j=0}^{J-1}\int_{t_j}^{t_{j+1}} \mathbb{E}[|F(t_j,\boldsymbol{u}(t,\cdot),\boldsymbol{v}(t,\cdot))|]\,dt,\end{aligned}$$

then apply Lemma 3.9 for $s \equiv t_j$, $\boldsymbol{u} \equiv \boldsymbol{u}(t,\cdot)$ and $\boldsymbol{v} \equiv \boldsymbol{v}(t,\cdot)$ to deduce

$$\begin{aligned}\mathbb{E}[\int_0^T |F_k(t,\boldsymbol{u}(t,\cdot),\boldsymbol{v}(t,\cdot))|\,dt] &\le c\sum_{j=0}^{J-1} t_j \int_{t_j}^{t_{j+1}} \big(\mathbb{E}[\|\boldsymbol{u}(t,\cdot)\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}[\|\nabla\boldsymbol{v}(t,\cdot)\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\,dt\\ &\quad + c\sum_{j=0}^{J-1}(t_j^{1/2}+t_j)\int_{t_j}^{t_{j+1}} \big(\mathbb{E}[\|\boldsymbol{u}(t,\cdot)\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}[\|\boldsymbol{v}(t,\cdot)\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\,dt\\ &\le cT\int_0^T \big(\mathbb{E}[\|\boldsymbol{u}(t,\cdot)\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}[\|\nabla\boldsymbol{v}(t,\cdot)\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\,dt\\ &\quad + c(T+T^{1/2})\int_0^T \big(\mathbb{E}[\|\boldsymbol{u}(t,\cdot)\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}[\|\boldsymbol{v}(t,\cdot)\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\,dt.\end{aligned}$$

Hence, (6.5) with function $F^* = F_k$ follows by using Hölder's inequality.

Proof of (6.6): Noting that

$$\begin{aligned}
&\mathbb{E}[\int_0^T |F(t,\boldsymbol{u}(t,\cdot),\boldsymbol{v}(t,\cdot)) - F_k(t,\boldsymbol{u}(t,\cdot),\boldsymbol{v}(t,\cdot))|\,dt\,] \\
&\quad= \sum_{j=0}^{J}\int_{t_j}^{t_{j+1}} \mathbb{E}[|F(t,\boldsymbol{u}(t,\cdot),\boldsymbol{v}(t,\cdot)) - F(t_j,\boldsymbol{u}(t,\cdot),\boldsymbol{v}(t,\cdot))|]\,dt \\
&\quad:= \sum_{j=0}^{J}\int_{t_j}^{t_{j+1}} \mathbb{E}[|\tilde{F}^j(t-t_j,\boldsymbol{u}(t,\cdot),\boldsymbol{v}(t,\cdot))|]\,dt. \qquad (6.8)
\end{aligned}$$

Here

$$\tilde{F}^j(t,\boldsymbol{x},\boldsymbol{y}) := \sum_{i=1}^{q}\int_0^t \tilde{F}^j_{1,i}(s,\boldsymbol{x},\boldsymbol{y})\,ds + \sum_{i=1}^{q}\int_0^t \tilde{F}^j_{2,i}(s,\boldsymbol{x},\boldsymbol{y})\,d\tilde{W}_i(s),$$

in which $\tilde{F}^j_{1,i}(s,\boldsymbol{x},\boldsymbol{y}) = F_{1,i}(s+t_j,\boldsymbol{x},\boldsymbol{y})$, $\tilde{F}^j_{2,i}(s,\boldsymbol{x},\boldsymbol{y}) = F_{2,i}(s+t_j,\boldsymbol{x},\boldsymbol{y})$ and $\tilde{W}_i(s) = W_i(s+t_j) - W_i(t_j)$. By using the same arguments as in the proof of Lemma 3.9 we obtain the same result for the upper bound of $\tilde{F}^j$, namely

$$\begin{aligned}
\mathbb{E}|\tilde{F}^j(s,\boldsymbol{u},\boldsymbol{v})| &\le cs\big(\mathbb{E}[\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}[\|\nabla\boldsymbol{v}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2} \\
&\quad + c(s^{1/2}+s)\big(\mathbb{E}[\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}[\|\boldsymbol{v}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}.
\end{aligned}$$

Hence, there holds

$$\begin{aligned}
\int_{t_j}^{t_{j+1}} \mathbb{E}|\tilde{F}^j(s,\boldsymbol{u},\boldsymbol{v})|\,ds &\le c\int_{t_j}^{t_{j+1}} s\big(\mathbb{E}[\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}[\|\nabla\boldsymbol{v}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\,ds \\
&\quad + c\int_{t_j}^{t_{j+1}} (s^{1/2}+s)\big(\mathbb{E}[\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}[\|\boldsymbol{v}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\,ds, \\
&\le ck\int_{t_j}^{t_{j+1}} \big(\mathbb{E}[\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}[\|\nabla\boldsymbol{v}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\,ds \\
&\quad + c(k^{1/2}+k)\int_{t_j}^{t_{j+1}} \big(\mathbb{E}[\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}[\|\boldsymbol{v}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\,ds. \qquad (6.9)
\end{aligned}$$

Therefore, it follows from (6.8) and (6.9) that

$$\begin{aligned}
&\mathbb{E}[\int_0^T |F(t,\boldsymbol{u}(t,\cdot),\boldsymbol{v}(t,\cdot)) - F_k(t,\boldsymbol{u}(t,\cdot),\boldsymbol{v}(t,\cdot))|\,dt\,] \\
&\quad\le ck\int_0^T \big(\mathbb{E}[\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}[\|\nabla\boldsymbol{v}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\,ds \\
&\qquad + c(k^{1/2}+k)\int_0^T \big(\mathbb{E}[\|\boldsymbol{u}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\big(\mathbb{E}[\|\boldsymbol{v}\|^2_{\mathbb{L}^2(D)}]\big)^{1/2}\,ds.
\end{aligned}$$

The result follows immediately by using Hölder's inequality, which completes the proof of the lemma. □

The following two Lemmas 6.5 and 6.6 show that $\boldsymbol{m}^-_{h,k}$ and $\boldsymbol{m}_{h,k}$, respectively, satisfy a discrete form of (4.6).

Lemma 6.5. *Assume that h and k approach 0 with the following conditions*

$$\begin{cases} k = o(h^2) & \text{when } 0 \le \theta < 1/2, \\ k = o(h) & \text{when } \theta = 1/2, \\ \text{no condition} & \text{when } 1/2 < \theta \le 1. \end{cases} \tag{6.10}$$

Then for any $\boldsymbol{\psi} \in C_0^\infty\big((0,T);\mathbb{C}^\infty(D)\big)$, there holds $\mathbb{P}$-a.s.

$$\begin{aligned} &-\lambda_1 \left\langle \boldsymbol{m}_{h,k}^- \times \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} \right\rangle_{\mathbb{L}^2(D_T)} + \lambda_2 \left\langle \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} \right\rangle_{\mathbb{L}^2(D_T)} \\ &+ \mu \left\langle \nabla(\boldsymbol{m}_{h,k}^- + k\theta \boldsymbol{v}_{h,k}), \nabla(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}) \right\rangle_{\mathbb{L}^2(D_T)} + \mu \int_0^T F_k(t, \boldsymbol{m}_{h,k}^-, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})\, dt = \sum_{j=1}^3 I_j, \end{aligned}$$

where

$$\begin{aligned} I_1 &:= \left\langle -\lambda_1 \boldsymbol{m}_{h,k}^- \times \boldsymbol{v}_{h,k} + \lambda_2 \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}) \right\rangle_{\mathbb{L}^2(D_T)}, \\ I_2 &:= \mu \left\langle \nabla(\boldsymbol{m}_{h,k}^- + k\theta \boldsymbol{v}_{h,k}), \nabla(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})) \right\rangle_{\mathbb{L}^2(D_T)}, \\ I_3 &:= \mu \int_0^T F_k(t, \boldsymbol{m}_{h,k}^-, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}) - F_k(t, \boldsymbol{m}_{h,k}^-, I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}))\, dt. \end{aligned}$$

Furthermore, $\mathbb{E}|I_i| = O(h)$ for $i = 1, 2, 3$.

Proof. For $t \in [t_j, t_{j+1})$, we use equation (5.5) with $\boldsymbol{w}_h^{(j)} = I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{\psi}(t,\cdot)\big)$ to see

$$\begin{aligned} &-\lambda_1 \left\langle \boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{v}_{h,k}(t,\cdot), I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{\psi}(t,\cdot)\big) \right\rangle_{\mathbb{L}^2(D)} \\ &\quad + \lambda_2 \left\langle \boldsymbol{v}_{h,k}(t,\cdot), I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{\psi}(t,\cdot)\big) \right\rangle_{\mathbb{L}^2(D)} \\ &\quad + \mu \left\langle \nabla(\boldsymbol{m}_{h,k}^-(t,\cdot) + k\theta \boldsymbol{v}_{h,k}(t,\cdot)), \nabla I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{\psi}(t,\cdot)\big) \right\rangle_{\mathbb{L}^2(D)} \\ &\quad + \mu F_k(t, \boldsymbol{m}_{h,k}^-(t,\cdot), I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^-(t,\cdot) \times \boldsymbol{\psi}(t,\cdot)\big)) = 0. \end{aligned}$$

Integrating both sides of the above equation over (t_j, t_{j+1}) and summing over $j = 0, \dots, J-1$ we deduce

$$\begin{aligned} &-\lambda_1 \left\langle \boldsymbol{m}_{h,k}^- \times \boldsymbol{v}_{h,k}, I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}\big) \right\rangle_{\mathbb{L}^2(D_T)} + \lambda_2 \left\langle \boldsymbol{v}_{h,k}, I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}\big) \right\rangle_{\mathbb{L}^2(D_T)} \\ &+ \mu \left\langle \nabla(\boldsymbol{m}_{h,k}^- + k\theta \boldsymbol{v}_{h,k}), \nabla I_{\mathbb{V}_h}\big(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}\big) \right\rangle_{\mathbb{L}^2(D_T)} + \mu \int_0^T F_k(t, \boldsymbol{m}_{h,k}^-, I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}))\, dt = 0. \end{aligned}$$

This implies

$$\begin{aligned} &-\lambda_1 \left\langle \boldsymbol{m}_{h,k}^- \times \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} \right\rangle_{\mathbb{L}^2(D_T)} + \lambda_2 \left\langle \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} \right\rangle_{\mathbb{L}^2(D_T)} \\ &\quad + \mu \left\langle \nabla(\boldsymbol{m}_{h,k}^- + k\theta \boldsymbol{v}_{h,k}), \nabla(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}) \right\rangle_{\mathbb{L}^2(D_T)} + \mu \int_0^T F_k(t, \boldsymbol{m}_{h,k}^-, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})\, dt \\ &\quad = I_1 + I_2 + I_3. \end{aligned}$$

Hence it suffices to prove that $\mathbb{E}|I_i| = O(h)$ for $i = 1, 2, 3$. First, by using Lemma 5.2 we obtain

$$\|\boldsymbol{m}_{h,k}^-\|_{\mathbb{L}^\infty(D_T)} \leq \sup_{0\leq j\leq J} \|\boldsymbol{m}_h^{(j)}\|_{\mathbb{L}^\infty(D)} \leq 1, \tag{6.11}$$

and

$$\|\boldsymbol{m}_{h,k}\|_{\mathbb{L}^\infty(D_T)} \leq 2 \sup_{0\leq j\leq J} \|\boldsymbol{m}_h^{(j)}\|_{\mathbb{L}^\infty(D)} \leq 2. \tag{6.12}$$

Lemma 6.2 and (6.11) together with Hölder's inequality and Lemma 7.2 yield

$$\begin{aligned}
\mathbb{E}|I_1| &\leq c\mathbb{E}\left[\left(\|\boldsymbol{m}_{h,k}^-\|_{\mathbb{L}^\infty(D_T)} + 1\right)\|\boldsymbol{v}_{h,k}\|_{\mathbb{L}^2(D_T)}\|\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})\|_{\mathbb{L}^2(D_T)}\right] \\
&\leq c\mathbb{E}\left[\|\boldsymbol{v}_{h,k}\|_{\mathbb{L}^2(D_T)}\|\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})\|_{\mathbb{L}^2(D_T)}\right] \\
&\leq c\big(\mathbb{E}[\|\boldsymbol{v}_{h,k}\|^2_{\mathbb{L}^2(D_T)}]\big)^{1/2}\big(\mathbb{E}[\|\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})\|^2_{\mathbb{L}^2(D_T)}]\big)^{1/2} \leq ch.
\end{aligned}$$

The bound for $\mathbb{E}|I_2|$ can be obtained similarly, using Lemma 6.2 and noting that when $\theta \in [0, \frac{1}{2}]$, a suitable bound on $k\,\|\nabla \boldsymbol{v}_{h,k}\|_{\mathbb{L}^2(D_T)}$ can be deduced from the inverse estimate as follows:

$$k\,\|\nabla \boldsymbol{v}_{h,k}\|_{\mathbb{L}^2(D_T)} \leq ckh^{-1}\,\|\boldsymbol{v}_{h,k}\|_{\mathbb{L}^2(D_T)} \leq ckh^{-1}.$$

The bound for $\mathbb{E}|I_3|$ can be obtained by noting the linearity of F in Remark 3.7 and using Lemmas 6.4 and 7.2. Indeed,

$$\begin{aligned}
\mathbb{E}|I_3| &= \mu\mathbb{E}\Big|\int_0^T F_k(t, \boldsymbol{m}_{h,k}^-, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}))\,dt\Big| \\
&\leq \mu\mathbb{E}\int_0^T \big|F_k(t, \boldsymbol{m}_{h,k}^-, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}))\big|\,dt \\
&\leq c\big(\mathbb{E}[\|\boldsymbol{m}_{h,k}^-\|^2_{\mathbb{L}^2(D_T)}]\big)^{1/2}\Big(\big(\mathbb{E}[\|\nabla\big(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})\big)\|^2_{\mathbb{L}^2(D_T)}]\big)^{1/2} \\
&\qquad + \big(\mathbb{E}[\|\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})\|^2_{\mathbb{L}^2(D_T)}]\big)^{1/2}\Big) \\
&\leq ch.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 6.6. *Assume that h and k approach 0 satisfying (6.10). Then for any $\boldsymbol{\psi} \in C_0^\infty\big((0,T); \mathbb{C}^\infty(D)\big)$, there holds $\mathbb{P}$-a.s.*

$$\begin{aligned}
&-\lambda_1\,\langle \boldsymbol{m}_{h,k} \times \partial_t \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi}\rangle_{\mathbb{L}^2(D_T)} + \lambda_2\,\langle \partial_t \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi}\rangle_{\mathbb{L}^2(D_T)} \\
&+\mu\,\langle \nabla(\boldsymbol{m}_{h,k}), \nabla(\boldsymbol{m}_{h,k} \times \boldsymbol{\psi})\rangle_{\mathbb{L}^2(D_T)} + \mu\int_0^T F(t, \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi})\,dt = \sum_{j=1}^7 I_j,
\end{aligned} \tag{6.13}$$

where

$$I_4 = -\lambda_1 \left\langle \boldsymbol{m}_{h,k}^- \times \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} \right\rangle_{\mathbb{L}^2(D_T)} + \lambda_1 \left\langle \boldsymbol{m}_{h,k} \times \partial_t \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi} \right\rangle_{\mathbb{L}^2(D_T)},$$

$$I_5 = \lambda_2 \left\langle \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} \right\rangle_{\mathbb{L}^2(D_T)} - \lambda_2 \left\langle \partial_t \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi} \right\rangle_{\mathbb{L}^2(D_T)},$$

$$I_6 = \mu \left\langle \nabla(\boldsymbol{m}_{h,k}^- + k\theta \boldsymbol{v}_{h,k}), \nabla(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}) \right\rangle_{\mathbb{L}^2(D_T)} - \mu \left\langle \nabla(\boldsymbol{m}_{h,k}), \nabla(\boldsymbol{m}_{h,k} \times \boldsymbol{\psi}) \right\rangle_{\mathbb{L}^2(D_T)},$$

$$I_7 = \mu \int_0^T \left(F(t, \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi}) - F_k(t, \boldsymbol{m}_{h,k}^-, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi}) \right) dt.$$

Furthermore, $\mathbb{E}|I_i| = O(h)$ *for* $i = 1, \cdots, 6$ *and* $\mathbb{E}|I_7| = O(h + k^{1/2})$.

Proof. From Lemma 6.5 it follows that

$$\begin{aligned} &-\lambda_1 \left\langle \boldsymbol{m}_{h,k} \times \partial_t \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi} \right\rangle_{\mathbb{L}^2(D_T)} + \lambda_2 \left\langle \partial_t \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi} \right\rangle_{\mathbb{L}^2(D_T)} \\ &\quad + \mu \left\langle \nabla(\boldsymbol{m}_{h,k}), \nabla(\boldsymbol{m}_{h,k} \times \boldsymbol{\psi}) \right\rangle_{\mathbb{L}^2(D_T)} + \mu \int_0^T F_k(t, \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi})\, dt \\ &\quad = I_1 + \cdots + I_7. \end{aligned}$$

Hence it suffices to prove that $\mathbb{E}|I_i| = O(h)$ for $i = 4, \cdots, 6$. First, by using the triangle inequality and Hölder's inequality, we obtain

$$\begin{aligned} \lambda_1^{-1}|I_4| &\le \left| \left\langle (\boldsymbol{m}_{h,k}^- - \boldsymbol{m}_{h,k}) \times \boldsymbol{v}_{h,k}, \boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} \right\rangle_{\mathbb{L}^2(D_T)} \right| \\ &\quad + \left| \left\langle \boldsymbol{m}_{h,k} \times \boldsymbol{v}_{h,k}, (\boldsymbol{m}_{h,k}^- - \boldsymbol{m}_{h,k}) \times \boldsymbol{\psi} \right\rangle_{\mathbb{L}^2(D_T)} \right| \\ &\quad + \left| \left\langle \boldsymbol{m}_{h,k} \times (\boldsymbol{v}_{h,k} - \partial_t \boldsymbol{m}_{h,k}), \boldsymbol{m}_{h,k} \times \boldsymbol{\psi} \right\rangle_{\mathbb{L}^2(D_T)} \right|, \\ &\le 2\|\boldsymbol{m}_{h,k}^- - \boldsymbol{m}_{h,k}\|_{\mathbb{L}^2(D_T)} \|\boldsymbol{v}_{h,k}\|_{\mathbb{L}^2(D_T)} \big(\|\boldsymbol{m}_{h,k}^-\|_{\mathbb{L}^\infty(D_T)} + \|\boldsymbol{m}_{h,k}\|_{\mathbb{L}^\infty(D_T)} \big) \|\boldsymbol{\psi}\|_{\mathbb{L}^\infty(D_T)} \\ &\quad + \|\boldsymbol{v}_{h,k} - \partial_t \boldsymbol{m}_{h,k}\|_{\mathbb{L}^1(D_T)} \|\boldsymbol{m}_{h,k}\|^2_{\mathbb{L}^\infty(D_T)} \|\boldsymbol{\psi}\|_{\mathbb{L}^\infty(D_T)}. \end{aligned}$$

Therefore, the required bound on $\mathbb{E}|I_4|$ can be obtained by using (6.11), (6.12) and Lemmas 6.2, 6.3. The bounds on $\mathbb{E}|I_5|$ and $\mathbb{E}|I_6|$ can be obtaineded similarly.

In order to prove the bound for $\mathbb{E}|I_7|$, we first use the triangle inequality then Remark 3.7 and Lemma 6.4 to obtain

$$\begin{aligned} \mathbb{E}|I_7| &\le \mathbb{E} \int_0^T |F(t, \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi}) - F_k(t, \boldsymbol{m}_{h,k}, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi})|\, dt \\ &\quad + \mathbb{E} \int_0^T \left| F_k(t, \boldsymbol{m}_{h,k} - \boldsymbol{m}_{h,k}^-, \boldsymbol{m}_{h,k} \times \boldsymbol{\psi}) \right| dt + \mathbb{E} \int_0^T \left| F_k(t, \boldsymbol{m}_{h,k}^-, (\boldsymbol{m}_{h,k} - \boldsymbol{m}_{h,k}^-) \times \boldsymbol{\psi}) \right| dt \\ &\le c k^{1/2} \big(\mathbb{E}[\|\boldsymbol{m}_{h,k}\|^2_{\mathbb{L}^2(D_T)}] \big)^{1/2} \left(\big(\mathbb{E}[\|\nabla \boldsymbol{m}_{h,k}\|^2_{\mathbb{L}^2(D_T)}] \big)^{1/2} + \big(\mathbb{E}[\|\boldsymbol{m}_{h,k}\|^2_{\mathbb{L}^2(D_T)}] \big)^{1/2} \right) \\ &\quad + c \big(\mathbb{E}[\|\boldsymbol{m}_{h,k} - \boldsymbol{m}_{h,k}^-\|^2_{\mathbb{L}^2(D_T)}] \big)^{1/2} \left(\big(\mathbb{E}[\|\nabla \boldsymbol{m}_{h,k}\|^2_{\mathbb{L}^2(D_T)}] \big)^{1/2} + \big(\mathbb{E}[\|\boldsymbol{m}_{h,k}\|^2_{\mathbb{L}^2(D_T)}] \big)^{1/2} \right) \\ &\le c(h + k^{1/2}), \end{aligned}$$

in which (6.1) and (6.2) are used to obtain the last inequality. This completes the proof of the lemma. □

In order to prove the convergence of random variables $\boldsymbol{m}_{h,k}$, we first state a result of tightness for the family $\mathcal{L}(\boldsymbol{m}_{h,k})$. We then use the Skorohod theorem to define another probability space and an almost surely convergent sequence defined in this space whose limit is a weak martingale solution of equation (4.14). The proof of the following results are omitted since they are relatively simple modification of the proof of the corresponding results from [13].

Lemma 6.7. *Assume that h and k approach 0, and further that (6.10) holds. Then the set of laws $\{\mathcal{L}(\boldsymbol{m}_{h,k})\}$ on the Banach space $C\big([0,T];\mathbb{H}^{-1}(D)\big)\cap L^2(0,T;\mathbb{L}^2(D))$ is tight.*

Proposition 6.8. *Assume that h and k approach 0, and further that (6.10) holds. Then there exist:*

(a) *a probability space $(\Omega',\mathcal{F}',\mathbb{P}')$;*
(b) *a sequence $\{\boldsymbol{m}'_{h,k}\}$ of random variables defined on $(\Omega',\mathcal{F}',\mathbb{P}')$ and taking values in $C\big([0,T];\mathbb{H}^{-1}(D)\big)\cap L^2(0,T;\mathbb{L}^2(D))$; and*
(c) *a random variable $\boldsymbol{m}'$ defined on $(\Omega',\mathcal{F}',\mathbb{P}')$ and taking values in $C\big([0,T];\mathbb{H}^{-1}(D)\big)\cap L^2(0,T;\mathbb{L}^2(D))$,*

satisfying

(1) $\mathcal{L}(\boldsymbol{m}_{h,k})=\mathcal{L}(\boldsymbol{m}'_{h,k})$,
(2) $\boldsymbol{m}'_{h,k}\to\boldsymbol{m}'$ *in* $C\big([0,T];\mathbb{H}^{-1}(D)\big)\cap L^2(0,T;\mathbb{L}^2(D))$ *strongly,* $\mathbb{P}'$*-a.s..*

Moreover, the sequence $\{\boldsymbol{m}'_{h,k}\}$ satisfies

$$\mathbb{E}[\|\boldsymbol{m}'_{h,k}\|^2_{\mathbb{H}(D_T)}]\le c, \tag{6.14}$$

$$\mathbb{E}[\||\boldsymbol{m}'_{h,k}|-1\|^2_{\mathbb{L}^2(D_T)}]\le c(h^2+k^2), \tag{6.15}$$

$$\|\boldsymbol{m}'_{h,k}\|^2_{\mathbb{L}^\infty(D_T)}\le c\quad \mathbb{P}'\text{-a.s.}, \tag{6.16}$$

here, c is a positive constant only depending on $\{\boldsymbol{g}_i\}_{i=1,\cdots,q}$.

We are now ready to state and prove our main theorem.

Theorem 6.9. *Assume that $T>0$, $\boldsymbol{M}_0\in\mathbb{H}^1(D)$ satisfies (??) and $\boldsymbol{g}_i\in\mathbb{W}^{2,\infty}(D)$ for $i=1,\cdots,q$ satisfy the homogeneous Neumann boundary condition. Then $\boldsymbol{m}'$, the sequence $\{\boldsymbol{m}'_{h,k}\}$ and the probability space $(\Omega',\mathcal{F}',\mathbb{P}')$ given by Proposition 6.8 satisfy:*

(1) *the sequence of $\{\boldsymbol{m}'_{h,k}\}$ converges to $\boldsymbol{m}'$ weakly in $L^2(\Omega';\mathbb{H}^1(D_T))$; and*
(2) $\big(\Omega',\mathcal{F}',(\mathcal{F}'_t)_{t\in[0,T]},\mathbb{P}',\boldsymbol{M}'\big)$ *is a weak martingale solution of (1.3), where*

$$\boldsymbol{M}'(t):=Z_t\boldsymbol{m}'(t)\quad\forall t\in[0,T],\ \text{a.e. } \boldsymbol{x}\in D.$$

Proof. From (6.16) and property (2) of Proposition 6.8, there exists a set $V\subset\Omega'$ such that $\mathbb{P}'(V)=1$ and for all $\omega'\in V$ there hold

$$\|\boldsymbol{m}'_{h,k}(\omega')\|^2_{\mathbb{L}^2(D_T)}\le c\quad\text{and}\quad \boldsymbol{m}'_{h,k}(\omega')\to\boldsymbol{m}'(\omega')\text{ in }\mathbb{L}^2(D_T)\text{ strongly}.$$

Hence, by using Lebesgue's dominated convergence theorem, we deduce

$$\boldsymbol{m}'_{h,k}\to\boldsymbol{m}'\text{ in }L^2(\Omega';\mathbb{L}^2(D_T))\text{ strongly}, \tag{6.17}$$

which implies from (6.14) that

$$\boldsymbol{m}'_{h,k}\to\boldsymbol{m}'\text{ in }L^2(\Omega';\mathbb{H}^1(D_T))\text{ weakly}. \tag{6.18}$$

In order to prove Part (2), by noting Lemma 4.5 and Remark 4.4 we only need to prove that $\boldsymbol{m}'$ satisfies (4.5) and (4.14), namely

$$|\boldsymbol{m}'(t,\boldsymbol{x})| = 1, \quad t \in (0,T), \quad \boldsymbol{x} \in D, \quad \mathbb{P}'\text{-a.s.} \tag{6.19}$$

and

$$\mathcal{I}(\boldsymbol{m}', \boldsymbol{\varphi}) = 0 \quad \mathbb{P}'\text{-a.s.} \quad \forall \boldsymbol{\varphi} \in L^2(0,T;\mathbb{H}^1(D)), \tag{6.20}$$

where

$$\begin{aligned}\mathcal{I}(\boldsymbol{m}', \boldsymbol{\varphi}) :=& \lambda_1 \left\langle \boldsymbol{m}' \times \partial_t \boldsymbol{m}', \boldsymbol{m}' \times \boldsymbol{\varphi} \right\rangle_{\mathbb{L}^2(D_T)} - \lambda_2 \left\langle \partial_t \boldsymbol{m}', \boldsymbol{m}' \times \boldsymbol{\varphi} \right\rangle_{\mathbb{L}^2(D_T)} \\ & - \mu \left\langle \nabla \boldsymbol{m}', \nabla(\boldsymbol{m}' \times \boldsymbol{\varphi}) \right\rangle_{\mathbb{L}^2(D_T)} - \mu \int_0^T F(t, \boldsymbol{m}'(t,\cdot), \boldsymbol{m}'(t,\cdot) \times \boldsymbol{\varphi}(t,\cdot))\, dt.\end{aligned}$$

By using (6.15) and (6.17), we obtain (6.19) immediately.

In order to prove (6.20), we first find the equation satisfied by $\boldsymbol{m}'_{h,k}$ and then pass to the limit when h and k approach 0.

By using Lemmas 6.6 and property (1) of Proposition 6.8, it follows that for any $\boldsymbol{\psi} \in C_0^\infty(0,T;\mathbb{C}^\infty(D))$ that there holds

$$\mathbb{E}|\mathcal{I}(\boldsymbol{m}'_{h,k}, \boldsymbol{\psi})| = O(h + k^{1/2}). \tag{6.21}$$

To pass to the limit in (6.21), we first using (6.17)–(6.19) and the same arguments as in [13, Theorem 6.8] to obtain that as h and k tend to 0,

$$\left\langle \boldsymbol{m}'_{h,k} \times \partial_t \boldsymbol{m}'_{h,k}, \boldsymbol{m}'_{h,k} \times \boldsymbol{\varphi} \right\rangle_{L^2(\Omega';\mathbb{L}^2(D_T))} \to \left\langle \boldsymbol{m}' \times \partial_t \boldsymbol{m}', \boldsymbol{m}' \times \boldsymbol{\varphi} \right\rangle_{L^2(\Omega';\mathbb{L}^2(D_T))}, \tag{6.22}$$

$$\left\langle \partial_t \boldsymbol{m}'_{h,k}, \boldsymbol{m}'_{h,k} \times \boldsymbol{\varphi} \right\rangle_{L^2(\Omega';\mathbb{L}^2(D_T))} \to \left\langle \partial_t \boldsymbol{m}', \boldsymbol{m}' \times \boldsymbol{\varphi} \right\rangle_{L^2(\Omega';\mathbb{L}^2(D_T))}, \tag{6.23}$$

$$\left\langle \nabla \boldsymbol{m}'_{h,k}, \nabla(\boldsymbol{m}'_{h,k} \times \boldsymbol{\varphi}) \right\rangle_{L^2(\Omega';\mathbb{L}^2(D_T))} \to \left\langle \nabla \boldsymbol{m}', \nabla(\boldsymbol{m}' \times \boldsymbol{\varphi}) \right\rangle_{L^2(\Omega';\mathbb{L}^2(D_T))}. \tag{6.24}$$

Then, by using Remark 3.7 and (6.5) with $F^* = F$, we estimate

$$\begin{aligned}&\mathbb{E} \int_0^T \left| F(t, \boldsymbol{m}'_{h,k}(t,\cdot), \boldsymbol{m}'_{h,k}(t,\cdot) \times \boldsymbol{\varphi}(t,\cdot)) - F(t, \boldsymbol{m}'(t,\cdot), \boldsymbol{m}'(t,\cdot) \times \boldsymbol{\varphi}(t,\cdot)) \right| dt \\ &\quad \le \mathbb{E} \int_0^T \left| F(t, \boldsymbol{m}'_{h,k}(t,\cdot) - \boldsymbol{m}'(t,\cdot), \boldsymbol{m}'_{h,k}(t,\cdot) \times \boldsymbol{\varphi}(t,\cdot)) \right| dt \\ &\qquad + \mathbb{E} \int_0^T \left| F(t, \big(\boldsymbol{m}'_{h,k}(t,\cdot) - \boldsymbol{m}'(t,\cdot)\big) \times \boldsymbol{\varphi}(t,\cdot), \boldsymbol{m}'(t,\cdot)) \right| dt \\ &\quad \le c \|\boldsymbol{m}'_{h,k} - \boldsymbol{m}'\|_{L^2(\Omega';\mathbb{L}^2(D_T))} \big(\|\nabla \boldsymbol{m}'_{h,k}\|_{L^2(\Omega';\mathbb{L}^2(D_T))} + \|\boldsymbol{m}'_{h,k}\|_{L^2(\Omega';\mathbb{L}^2(D_T))} \\ &\qquad + \|\nabla \boldsymbol{m}'\|_{L^2(\Omega';\mathbb{L}^2(D_T))} + \|\boldsymbol{m}'\|_{L^2(\Omega';\mathbb{L}^2(D_T))} \big).\end{aligned}$$

Since $\boldsymbol{m}' \in L^2(\Omega';\mathbb{H}^1(D_T))$, it follows from (6.14) and (6.17) that

$$\mathbb{E}\Big[\int_0^T F(t, \boldsymbol{m}'_{h,k}(t,\cdot), \boldsymbol{m}'_{h,k}(t,\cdot) \times \boldsymbol{\varphi}(t,\cdot))\, dt \Big] \to \mathbb{E}\Big[\int_0^T F(t, \boldsymbol{m}'(t,\cdot), \boldsymbol{m}'(t,\cdot) \times \boldsymbol{\varphi}(t,\cdot))\, dt \Big], \tag{6.25}$$

as h and k tend to 0. From (6.22)–(6.25) we deduce that

$$\mathbb{E}|\mathcal{I}(\boldsymbol{m}'_{h,k}, \boldsymbol{\psi}) - \mathcal{I}(\boldsymbol{m}', \boldsymbol{\psi})| \to 0,$$

and hence, together with (6.21) $\mathbb{E}|\mathcal{I}(\boldsymbol{m}', \boldsymbol{\psi})| = 0$. This implies (6.20) which completes the proof of our main theorem. □

7. Appendix

For the reader's convenience we will recall the following results, which are proved in [13].

Lemma 7.1. *For any real constants λ_1 and λ_2 with $\lambda_1 \neq 0$, if $\boldsymbol{\psi}, \boldsymbol{\zeta} \in \mathbb{R}^3$ satisfy $|\boldsymbol{\zeta}| = 1$, then there exists $\boldsymbol{\varphi} \in \mathbb{R}^3$ satisfying*

$$\lambda_1 \boldsymbol{\varphi} + \lambda_2 \boldsymbol{\varphi} \times \boldsymbol{\zeta} = \boldsymbol{\psi}. \tag{7.1}$$

As a consequence, if $\boldsymbol{\zeta} \in \mathbb{H}^1(D_T)$ with $|\boldsymbol{\zeta}(t,x)| = 1$ a.e. in D_T and $\boldsymbol{\psi} \in L^2(0,T;\mathbb{W}^{1,\infty}(D))$, then $\boldsymbol{\varphi} \in L^2(0,T;\mathbb{H}^1(D))$.

Lemma 7.2. *For any $\boldsymbol{v} \in \mathbb{C}(D)$, $\boldsymbol{v}_h \in \mathbb{V}_h$ and $\boldsymbol{\psi} \in \mathbb{C}_0^\infty(D_T)$,*

$$\|I_{\mathbb{V}_h} \boldsymbol{v}\|_{\mathbb{L}^\infty(D)} \leq \|\boldsymbol{v}\|_{\mathbb{L}^\infty(D)},$$

$$\|\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi} - I_{\mathbb{V}_h}(\boldsymbol{m}_{h,k}^- \times \boldsymbol{\psi})\|^2_{\mathbb{L}([0,T],\mathbb{H}^1(D))} \leq ch^2 \|\boldsymbol{m}_{h,k}^-\|^2_{\mathbb{L}([0,T],\mathbb{H}^1(D))} \|\boldsymbol{\psi}\|^2_{\mathbb{W}^{2,\infty}(D_T)},$$

where $\boldsymbol{m}_{h,k}^-$ is defined in Defintion 6.1

The next lemma defines a discrete $\mathbb{L}^p$-norm in $\mathbb{V}_h$, equivalent to the usual $\mathbb{L}^p$-norm.

Lemma 7.3. *There exist h-independent positive constants C_1 and C_2 such that for all $p \in [1,\infty]$ and $\boldsymbol{u} \in \mathbb{V}_h$,*

$$C_1 \|\boldsymbol{u}\|^p_{\mathbb{L}^p(\Omega)} \leq h^d \sum_{n=1}^{N} |\boldsymbol{u}(\boldsymbol{x}_n)|^p \leq C_2 \|\boldsymbol{u}\|^p_{\mathbb{L}^p(\Omega)},$$

where $\Omega \subset \mathbb{R}^d$, d=1,2,3.

Acknowledgements

The authors acknowledge financial support through the ARC Discovery projects DP140101193 and DP120101886. They are grateful to Vivien Challis for a number of helpful conversations.

References

[1] F. Alouges. A new finite element scheme for Landau-Lifchitz equations. Discrete Contin. Dyn. Syst. Ser. S, **1** (2008), 187–196.
[2] F. Alouges, A. de Bouard, and A. Hocquet. A semi-discrete scheme for the stochastic Landau–Lifshitz equation. Stochastic Partial Differential Equations: Analysis and Computations, **2** (2014), 281–315.
[3] F. Alouges and P. Jaisson. Convergence of a finite element discretization for the Landau-Lifshitz equations in micromagnetism. Math. Models Methods Appl. Sci., **16** (2006), 299–316.
[4] S. Bartels. Stability and convergence of finite-element approximation schemes for harmonic maps. SIAM J. Numer. Anal., **43** (2005), 220–238 (electronic).
[5] L. Baňas, Z. Brzeźniak, A. Prohl, and M. Neklyudov. A convergent finite-element-based discretization of the stochastic Landau–Lifshitz–Gilbert equation. IMA Journal of Numerical Analysis, (2013).
[6] W. Brown. Thermal fluctuation of fine ferromagnetic particles. IEEE Transactions on Magnetics, **15** (1979), 1196–1208.
[7] W. F. Brown. Thermal fluctuations of a single-domain particle. Phys. Rev., **130** (1963), 1677–1686.
[8] Z. Brzeźniak, B. Goldys, and T. Jegaraj. Weak solutions of a stochastic Landau–Lifshitz–Gilbert equation. Applied Mathematics Research eXpress, (2012), 1–33.
[9] I. Cimrák. A survey on the numerics and computations for the Landau-Lifshitz equation of micromagnetism. Arch. Comput. Methods Eng., **15** (2008), 277–309.
[10] G. Da Prato and J. Zabczyk. Stochastic Equations in Infinite Dimensions. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2014.
[11] H. Doss. Liens entre quations diffrentielles stochastiques et ordinaires. Annales de l'I.H.P. Probabilits et statistiques, **13** (1977), 99–125.

[12] T. Gilbert. A Lagrangian formulation of the gyromagnetic equation of the magnetic field. Phys Rev, **100** (1955), 1243–1255.
[13] B. Goldys, K.-N. Le, and T. Tran. A finite element approximation for the stochastic Landau–Lifshitz–Gilbert equation. Journal of Differential Equations, **260** (2016), 937 – 970.
[14] C. Johnson. Numerical Solution of Partial Differential Equations by the Finite Element Method. Cambridge University Press, Cambridge, 1987.
[15] G. Ju, Y. Peng, E. K. C. Chang, Y. Ding, A. Q. Wu, X. Zhu, Y. Kubota, T. J. Klemmer, H. Amini, L. Gao, Z. Fan, T. Rausch, P. Subedi, M. Ma, S. Kalarickal, C. J. Rea, D. V. Dimitrov, P. W. Huang, K. Wang, X. Chen, C. Peng, W. Chen, J. W. Dykes, M. A. Seigler, E. C. Gage, R. Chantrell, and J. U. Thiele. High density heat-assisted magnetic recording media and advanced characterization –progress and challenges. IEEE Transactions on Magnetics, **51** (2015), 1–9.
[16] O. Kallenberg. Foundations of Modern Probability. Probability and Its Applications. Springer-Verlag New York, Cambridge, 2002.
[17] L. Landau and E. Lifschitz. On the theory of the dispersion of magnetic permeability in ferromagnetic bodies. Phys Z Sowjetunion, **8** (1935), 153–168.
[18] H. J. Sussmann. An interpretation of stochastic differential equations as ordinary differential equations which depend on the sample point. Bulletin of the American Mathematical Society, **83** (1977), 296–298.

School of Mathematics and Statistics, The University of Sydney, Sydney 2006, Australia
E-mail address: beniamin.goldys@sydney.edu.au

School of Mathematics and Physics, The University of Queensland, QLD 4072, Australia
E-mail address: j.grotowski@uq.edu.au

School of Mathematics and Statistics, The University of New South Wales, Sydney 2052, Australia
E-mail address: n.le-kim@unsw.edu.au