FRACTIONAL POWERS OF MONOTONE OPERATORS IN HILBERT SPACES

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Abstract. In this article, we show that if A is a maximal monotone operator on a Hilbert space H with 0 in the range $\mathrm{Rg}(A)$ of A, then for every $0 < s < 1$, the Dirichlet problem associated with the Bessel-type equation

$$A_{1-2s}u := -\frac{1-2s}{t}u_t - u_{tt} + Au \ni 0$$

is well-posed for boundary values $\varphi \in \overline{D(A)}^H$. This allows us to define the Dirichlet-to-Neumann (DtN) operator Λ_s associated with A_{1-2s} as

$$\varphi \mapsto \Lambda_s\varphi := -\lim_{t\to 0+} t^{1-2s}u_t(t) \qquad \text{in H.}$$

The existence of the DtN operator Λ_s associated with A_{1-2s} is the first step to define fractional powers A^α of monotone (possibly, nonlinear and multivalued) operators A on H. We prove that Λ_s is monotone on H and if $\overline{\Lambda}_s$ is the closure of Λ_s in $H \times H_w$ then we provide sufficient conditions implying that $-\overline{\Lambda}_s$ generates a strongly continuous semigroup on $\overline{D(A)}^H$. In addition, we show that if A is completely accretive on $L^2(\Sigma,\mu)$ for a σ-finite measure space (Σ,μ), then Λ_s inherits this property from A.

1. Introduction and main results

In the pioneering work [CS07], Caffarelli and Silvestre constructed three analytical proofs to show that the fractional Laplacian $(-\Delta)^s$, $(0 < s < 1)$, on $\mathbb{R}^d$, $(d \geq 1)$, coincides up to a multiple constant with the Dirichlet-to-Neumann (DtN) operator

$$\varphi \mapsto \Lambda_s\varphi := \lim_{t\to 0+} -t^{1-2s}u'(\cdot,t) \qquad \text{on } \mathbb{R}^d$$

Date: May 8, 2018.

2010 *Mathematics Subject Classification.* 35R11,47H05,47H07,35B65.

Key words and phrases. Monotone operators, Hilbert space, evolution equations, fractional operators.

The results presented in this paper are the outgrows of a summer vacation research project at the University of Sydney under the supervision of Mr Hauer. Ms He and Mr Liu are undergraduate students at the University of Sydney who participated at this research project and actively contributed.

associated with the Bessel type equation

$$A_{1-2s}u := -\frac{1-2s}{t}u_t - u_{tt} + Au \ni 0 \tag{1.1}$$

on the half-space $\mathbb{R}^{d+1}_+ := \mathbb{R}^d \times (0, +\infty)$ with $A = -\Delta$. One crucial *Ansatz* in [CS07] to obtain this identification is to employ the change of variable

$$z = \left(\frac{t}{2s}\right)^{2s}, \tag{1.2}$$

which transforms (1.1) into the equation

$$\tilde{A}_{1-2s}v := -z^{-\frac{1-2s}{s}}v'' + Av \ni 0. \tag{1.3}$$

The identification of the fractional Laplacian $(-\Delta)^s$ with the DtN operator Λ_s has far reaching applications; for example, in the study of nonlocal partial differential equations, it provides new boundary regularity results including boundary Harnack inequalities, a monotonicity formula, and several others (for example, cf [GMS13] or [ATEW18] and the references theirin). The ideas in [CS07] were followed up quickly by many authors and extended to several more abstract *linear* settings. For instance, if for $\alpha \geq 0$, $\mathcal{I}_\alpha$ denotes the family of linear operators A on a Banach space X generating a *tempered α-times integrated semigroup $\{T(t)\}_{t\geq 0}$ in $\mathcal{L}(X)$* (for details cf [GMS13]), then it was shown by Galé, Miana and Stinga [GMS13] that for $A \in \mathcal{I}_\alpha$ and $0 < s < 1$, the *Dirichlet problem associated with the Bessel-type operator* A_{1-2s},

$$\begin{cases} A_{1-2s}u(t) \ni 0 & \text{for almost every } t > 0,\\ u(0) = \varphi \end{cases} \tag{1.4}$$

admits a unique solution u and the fractional power operator A^s (in the Balakrishnan sense [Bal60]) coincides with the DtN operator Λ_s associated with the Bessel-type operator A_{1-2s} given by (1.1).

An important subclass of $\mathcal{I}_\alpha$ is given by the family of *m-sectorial operators* A defined on a Hilbert space H which are the restriction on H of a continuous, coercive, sesquilinear form $\mathfrak{a} : V \times V \to \mathbb{C}$. Here, V is another Hilbert space such that V is embedded into H by a linear bounded injection $j : V \hookrightarrow H$ with a dense image $j(V)$ in H. For this subclass of operators A on H, the identification of the fractional power A^s with the DtN operator Λ_s was recently revisited by Arendt, Ter Elst and Warma [ATEW18]. By using techniques from linear interpolation theory, they could characterise the domain $D(\Lambda_s)$ of Λ_s and establish an imporant *integration by parts* rule for solutions to the Dirichlet problem associated with A_{1-2s}.

Note, it is for the class of m-sectorial operators A associated with a real symmetric form $\mathfrak{a} : V \times V \to \mathbb{R}$ that the *linear* semigroup theory coincides with the *nonlinear* semigroup theory (cf [Bre73] or [CHK16]). We briefly recall from [Bre73] (see also [Bar10]), a (possibly multivalued) operator $A : D(A) \to 2^H$ is called *maximal monotone* in H, if A satisfies the *monotonicity property*: $(\hat{v} - v, \hat{u} - u)_H \geq 0$ for all (u, v), $(\hat{u}, \hat{v}) \in A$, and the so-called *range condition*: $Rg(I + A) = H$. Here, we follow the standard notation and usually identify an operator A with its *graph*

$$A = \left\{(u, v) \in H \times H \,\middle|\, v \in Au\right\} \quad \text{in } H \times H.$$

The set $D(A) := \{u \in H \,|\, Au \neq \emptyset\}$ is called the *domain* of A and $\mathrm{Rg}(A) := \bigcup_{u\in D(A)} Au \subseteq H$ the *range* of A.

In the framework of maximal monotone operators A in Hilbert spaces H, first results toward the fractional power $A^{1/2}$ were obtained by Barbu [Bar72]. More precisely, if $0 \in \mathrm{Rg}(A)$, then Barbu proved that the Dirichlet problem (1.4) associated with the Bessel-type operator A_0 is well-posedness, and then introduced the semigroup $\{T_{1/2}(t)\}_{t\geq 0}$ as the contractive extension on the closure $\overline{D(A)}^H$ of $T_{1/2}(t)\varphi := u(t)$, $(t \geq 0)$, for the unique solution u of the Dirichlet problem (1.4) with boundary value $\varphi \in D(A)$. The paper [Bar72] was quickly followed up by several authors. For example, Brezis [Bre72] provided a different proof of the well-posedness of Dirichlet problem (1.4), and Véron [V́76] established well-posedness of the following more general (but *non-singular*) Dirichlet problem

$$-p(t)u_{tt} - q(t)u_t + Au \ni 0, \qquad u(0) = \varphi$$

for $\varphi \in D(A)$ under the hypothesis that $p \in W^{2,\infty}(0,+\infty)$, $q \in W^{1,\infty}(0,+\infty)$, and there is an $\alpha > 0$ satisfying $p(t) \geq \alpha > 0$ for all $t \geq 0$. We emphasises that neither Barbu nor Brezis identified the (negative) infinitessimal generator of the semigroup $\{T_{1/2}(t)\}_{t\geq 0}$ by the DtN operator $\Lambda_{1/2}$ associated with A_0, but in stead denoted $\Lambda_{1/2}$ by $A_{1/2}$ in analogy to the fractional power $A^{1/2}$ of A as it is known from the linear theory (cf [Bal60] or [MCSA01, Chapter 6.3]).

More than twenty years later, Alraabiou & Bénilan [AB96] introduce the definition of the *square power*

$$A^2 := \liminf_{\lambda\to 0+} \frac{A - A_\lambda}{\lambda},$$

via the Yosida approximation $A_\lambda := (I - (I + \lambda A)^{-1})/\lambda$ and showed that if $A = \partial_H \mathcal{E}$ the subdifferential operator in H of a convex, proper, lower semicontinuous functional $\mathcal{E} : H \to [0, +\infty]$ with $0 \in \partial_H\mathcal{E}(0)$, then $A \subseteq (A_{1/2})^2$. This justifies (at least in the case $s = 1/2$) that the DtN operator Λ_s is a reasonable candidate for defining fractional powers A^s of maximal monotone operators A on Hilbert spaces.

The first aim of this paper is to extend the results obtained by Barbu [Bar72] and Brezis [Bre72] (in the case $s = 1/2$) to the complete range $0 < s < 1$, the well-posedness of Dirchlet problem (1.4) associated with Bessel-type operator A_{1-2s} for (possibly nonlinear) *maximal monotone operators* A on a Hilbert space H. Before stating our first main result, let us introduce the following notion of solutions of Dirchlet problem (1.4).

Definition 1.1. We call a function $u : [0, +\infty) \to H$ a *strong solution* of Bessel-type equation (1.1) if $u \in W^{2,2}_{loc}((0, +\infty); H)$ and for almost every $t > 0$, $u(t) \in D(A)$ and $\{t^{1-2s}u'(t)\}' \in t^{1-2s}A(u(t))$. Moreover, for given boundary value $\varphi \in H$, a function $u : [0, +\infty) \to H$ is called a *solution of Dirichlet problem* (1.4) if $u \in C([0, +\infty); H)$, $u(0) = \varphi$, and u is a strong solution of (1.1).

In the next theorem, we write $L^\infty(H)$ to denote $L^\infty(0, +\infty; H)$, $L^2_\star(H)$ for the L^2-space of all measurable functions $u : \mathbb{R}_+ \to H$ where $\mathbb{R}_+ := (0, +\infty)$ is equipped with the Haar-measure $\frac{\mathrm{d}t}{t}$, and $W^{1,2}_s(H)$ to denote the set of all

$u \in W^{1,1}_{loc}(H)$ such that $t^s u$ and $t^s u' \in L^2_\star(H)$ (for more details, see Section 2). With these preliminaries, our first main result reads as follows.

Theorem 1.2. *Let A be a maximal monotone operator on H with $0 \in Rg(A)$. Then, for every $0 < s < 1$, $\varphi \in \overline{D(A)}^H$ and $y \in A^{-1}(\{0\})$, there is a unique solution $u \in L^\infty(H)$ of Dirichlet problem (1.4) satisfying*

$$\|u(t) - y\|_H \le \|u(\hat{t}) - y\|_H \qquad \text{for all } t \ge \hat{t} \ge 0, \tag{1.5}$$

$$\|tu'\|_{L^2_\star(H)} \le \sqrt{s}\,\|\varphi - y\|_H \tag{1.6}$$

$$\|u'(t)\|_H \le 2s\frac{\|\varphi - y\|_H}{t} \qquad \text{for every } t > 0, \tag{1.7}$$

$$\left\|t^{1+2s}\{t^{1-2s}u'\}'\right\|_{L^2_\star(H)} \le \begin{cases} \sqrt{s}\,\|\varphi - y\|_H & \text{if } s \ge \frac{1}{2}, \\ \frac{\sqrt{s}\left(\frac{s}{1-2s}\frac{1}{2}+3\right)^{\frac{1}{2}}}{\sqrt{2}}\,\|\varphi - y\|_H & \text{if } 0 < s < \frac{1}{2}. \end{cases} \tag{1.8}$$

Moreover, if $\varphi \in D(A)$, then $t^{1-2s}u' \in W^{1,2}_s(H)$ and

$$\left\|\lim_{t\to 0+} t^{1-2s}u'(t)\right\|_H \le (2s)^{1-2s}\left(\|A^0\varphi\|_H^{\frac{1}{2}} + \|\varphi - y\|_H^{\frac{1}{2}}\right)^2, \tag{1.9}$$

$$\|t^s\{t^{1-2s}u'\}\|_{L^2_\star(H)} \le (2s)^{1-s}\left(\|A^0\varphi\|_H^{\frac{1}{2}} + \|\varphi - y\|_H\right), \tag{1.10}$$

$$\|t^s\{t^{1-2s}u'\}'\|_{L^2_\star(H)} \le (2s)^{1-s}\left(\|A^0\varphi\|_H + \|\varphi - y\|_H^{\frac{1}{2}}\,\|A^0\varphi\|_H^{\frac{1}{2}}\right). \tag{1.11}$$

In particular, for every two solutions u and $\hat{u}$ of (1.4) with boundary value φ and $\hat{\varphi} \in \overline{D(A)}$, one has that

$$\|u(t) - \hat{u}(t)\|_H \le \|u(\hat{t}) - \hat{u}(\hat{t})\|_H \qquad \text{for all } t \ge \hat{t} \ge 0. \tag{1.12}$$

According to Theorem 1.2, for every $\varphi \in D(A)$, the *outer unit normal derivative*

$$\Lambda_s\varphi := -\lim_{t\to 0+} t^{1-2s}u'(t) \qquad \text{exists in } H. \tag{1.13}$$

Thus, thanks to this well-posedness result, the DtN operator Λ_s assigning each Dirichlet boundary condition $\varphi \in D(A)$ to the Neumann derivative (1.13) of the unique solution u of (1.1) is a well-defined mapping $\Lambda_s : D(A) \to H$. More precisely, we can state the following theorem. Here, H_w denotes the space H equipped with the weak toplogy $\sigma(H, H')$.

Theorem 1.3. *Let A be a maximal monotone operator on H with $0 \in Rg(A)$. Then, for every $0 < s < 1$, the Dirichlet-to-Neumann operator*

$$\Lambda_s := \left\{ (\varphi, w) \in H \times H \;\middle|\; \begin{array}{l} \exists \text{ a strong solution } u \text{ of (1.1) satisfying (1.5)}, \\ u(0) = \varphi \text{ in } H, \;\&\; w = -\lim_{t\to 0+} t^{1-2s}u'(t) \text{ in } H \end{array} \right\}$$

is a monotone, well-defined mapping $\Lambda_s : D(\Lambda_s) \to H$ satisfying

$$D(A) \subseteq D(\Lambda_s) \subseteq \overline{D(A)}^H \qquad \text{and} \qquad D(A) \subseteq Rg(I_H + \lambda\,\Lambda_s) \qquad \text{for all } \lambda > 0.$$

If $\overline{D(A)}$ is dense in H, then closure $\overline{\Lambda}_s$ of Λ_s in $H \times H_w$ is maximal monotone and $-\overline{\Lambda}_s$ generates a strongly continuous semigroup $\{T_s(t)\}_{t\ge 0}$ of contractions

$T_s(t) : H \to H$. *Moreover, for every* $\varphi \in H$, *there is a function* $U(r,t)$ *satisfying*

$$\begin{cases} -\frac{1-2s}{r}U_r(r,t) - U_{rr}(r,t) + AU(r,t) \ni 0 & \text{for a.e. } r > 0, \text{ all } t > 0,\\ U(0,t) = T_s(t)\varphi & \text{for all } t \geq 0,\\ \lim_{r\to 0+} r^{1-2s}U_r(r,t) \in \frac{\mathrm{d}}{\mathrm{d}t}T_s(t)\varphi & \text{for all } t > 0. \end{cases}$$

Due to Theorem 1.3 and the results [Bar72, Bre72, AB96] (for $s = 1/2$), it makes sense to define the fractional power A^s of A for $0 < s < 1$ via the DtN operator Λ_s.

Definition 1.4. For a maximal monotone operator A on H with $0 \in \mathrm{Rg}(A)$, the *fractional power* A^s *of* A for $0 < s < 1$, is defined by

$$A^s = \overline{\Lambda}_s,$$

where $\overline{\Lambda}_s$ denotes the closure in $H \times H_w$ of the DtN operator Λ_s associated with the Bessel-type operator A_{1-2s} defined in Theorem 1.3.

It is the task of our forthcoming work in this direction to extend the definition of A^2 provided by Alraabiou & Bénilan [AB96] to the case A^k for every integer $k \geq 2$ and to show that $A \subseteq (A^{\frac{1}{k}})^k$.

In the case the Hilbert space $H = L^2(\Sigma,\mu)$ of a σ-finite measure space (Σ,μ), and the operator A is *completely accretive* on $L^2(\Sigma,\mu)$ then this property also holds for the DtN operator Λ_s for $0 < s < 1$. For the notion of *completely accretive* operators, *order preservability*, and *Orlicz spaces* $L^\psi(\Sigma,\mu)$, we refer to Definitions 2.3 and 2.4 in the subsequent section.

Theorem 1.5. *Let* A *be an m-completely accretive operator on the Hilbert space* $H = L^2(\Sigma,\mu)$ *of a* σ*-finite measure space* (Σ,μ), *and* $0 \in Rg(A)$. *Then for every* $0 < s < 1$, *the DtN operator* Λ_s *is also completely accretive on* $L^2(\Sigma,\mu)$. *In particular, if* $\{T_s(t)\}_{t\geq 0}$ *is the semigroup generated by* $-\overline{\Lambda}_s$ *on* $L^2(\Sigma,\mu)$, *then* $\{T_s(t)\}_{t\geq 0}$ *is order-preserving and every map* $T_s(t)$ *is* L^Ψ*-contractive on* $L^2(\Sigma,\mu)$ *for every right-continuous* N*-function* ψ.

As a byproduct of the theory developed in this paper, we obtain existence and uniqueness of the following abstract *Robin problem associated with the Bessel-type operator* A_{1-2s},

$$\begin{cases} A_{1-2s}u(t) \ni 0 & \text{in } H \text{ for almost every } t > 0,\\ -\lim_{t\to 0+} t^{1-2s}u'(t) + \lambda u(0) = \varphi & \text{on } H, \end{cases} \tag{1.14}$$

for any $\lambda > 0$ and $\varphi \in D(A)$. Here, we use the following notion of solutions of problem (1.14).

Definition 1.6. For given $\varphi \in H$, a function $u : [0,+\infty) \to H$ is called a *solution* of Robin problem (1.14) if $u \in C([0,+\infty); H)$ is a strong solution of (1.1) and

$$-\lim_{t\to 0+} t^{1-2s}u'(t) + \lambda u(0) = \varphi \qquad \text{exists in } H.$$

Now, our well-posedness result to the abstract Robin problem (1.14) reads as follows.

Theorem 1.7. *Let A be a maximal monotone operator on H with $0 \in Rg(A)$. Then, for every $0 < s < 1$, $\lambda > 0$, and $\varphi \in D(A)$, there is a unique solution $u \in L^{\infty}(H)$ of Robin problem (1.14) satisfying (1.5), (1.9)-(1.11) and (1.6)-(1.8). In particular, $t^{1-2s}u' \in W^{1,2}_s(H)$, the function $t \mapsto \|u(t)\|_H^2$ is convex, bounded and decreasing on $[0,+\infty)$, and for every two solutions u and $\hat{u}$ of (1.14) respectively with boundary value φ and $\hat{\varphi} \in D(A)$, one has that (1.12) holds.*

Remark 1.8. At the moment, we neither know how to obtain existence and uniqueness of solutions to the abstract Robin problem (1.14) for boundary data $\varphi \in \overline{D(A)}^H$ nor the continuous dependence of the solutions u of (1.14) on the boundary data φ. This result would not only complete the theory on studying the Robin problem, but also imply that the DtN map Λ_s restricted on the closure $\overline{D(A)}^H$ is a well-defined mapping.

This paper is organized as follows. In the subsequent section, we introduce the framework and notations used throughout this paper and state some preliminary results which will be useful for establishing existence and uniqueness of Dirichlet problem (1.4) and Robin problem (1.14). In Section 3, we then prove the existence and uniqueness of equation (1.1) equipped with a more abstract boundary-value problem for boundary data $\varphi \in D(A)$, which includes (1.4) and (1.14) as special cases (cf problem (3.1)). The well-posedness of Dirichlet problem (1.4) for boundary data $\varphi \in \overline{D(A)}^H$ follows from the inqualities (1.5)-(1.8) and (1.12). We provide a short proof of Theorem 1.2 and Theorem 1.7 at the beginning of Section 3. Our proof of Theorem 3.2 is based on the change of variable (1.2) which transforms equation (1.1) into (1.3). We emphasize that our proof demonstrates very well that the generalization of [Bre72] (case $s = 1/2$) to the full range $0 < s < 1$ is not trivial and uses techniques and spaces from interpolation theory. The statements of Theorem 1.3 follow from Corollary 3.7. In Section 4, we show that if for a given convex, proper, lower semicontinuous functional $\phi : H \to \mathbb{R} \cup \{+\infty\}$, the operator A is $\partial_H\phi$-monotone on H (for a definition see below in the next section), then the DtN operator Λ_s has also this property (see Theorem 4.1). This results has several applications; one can deduce invariance principles (cf [Bre73]), comparison principles and L^{ψ}-contractivity properties of the semigroup $\{T_s(t)\}_{t\ge 0}$ generated by $-\overline{\Lambda}_s$ on $L^2(\Sigma,\mu)$, for a σ-finite measure space (Σ,μ) (see Corollary 4.2). Here, L^{ψ} abbreviates the Orlicz space $L^{\psi}(\Sigma,\mu)$ for a given N-function ψ. Thus, the statement of Theorem 1.5 follows from Corollary 4.2. We conclude this paper with an application on the Leray-Lions operator $A = -\operatorname{div}(a(x,\nabla u))$ (see Section 5).

2. Preliminaries

Throughout this article, $(H,(\cdot,\cdot)_H)$ denotes a real Hilbert space with inner product $(\cdot,\cdot)_H$, and we use to write $\mathbb{R}_+$ to denote $(0,+\infty)$ and $\overline{\mathbb{R}}_+ := [0,+\infty]$.

For an operator A on H, the *minimal selection* A^0 of A is given by

$$A^0 := \Big\{(u,v) \in A \,\Big|\, \|v\|_H = \min_{w\in Au} \|w\|_H\Big\}$$

and for a convex functional $\mathcal{E} : H \to \mathbb{R} \cup \{+\infty\}$, the *subdifferential operator*

$$\partial_H\mathcal{E} := \Big\{(u,h) \in H \times H \,\Big|\, \mathcal{E}(u+v) - \mathcal{E}(u) \ge (h,v)_H \text{ for all } v \in H\Big\}.$$

The property that A being monotone is equivalent to that for every $\lambda > 0$, the *resolvent operator* $J_\lambda^A := (I + \lambda A)^{-1}$ of A is *contractive in* H:

$$\|J_\lambda^A u - J_\lambda^A \hat{u}\|_H \leq \|u - \hat{u}\|_H \qquad \text{for every } u,\, \hat{u} \in \mathrm{Rg}(I + \lambda A).$$

Further, we call a functional $j : H \to \overline{\mathbb{R}}_+$ *strongly coercive* if

$$\lim_{|v| \to +\infty} \frac{j(v)}{|v|} = +\infty. \tag{2.1}$$

If $j : H \to \mathbb{R} \cup \{+\infty\}$ is convex, proper, and lower semicontinuous on H, then an operator A on H is called $\partial_H j$*-monotone*, if for every $\lambda > 0$, the resolvent J_λ^A of A satisfies

$$j(J_\lambda^A u - J_\lambda^A \hat{u}) \leq j(u - \hat{u}) \qquad \text{for every } u,\, \hat{u} \in \mathrm{Rg}(I + \lambda A).$$

In addition, if the subdifferential operator $\partial_H j$ of j is a mapping $\partial_H j : D(\partial_H j) \to H$, then we call $\partial_H j$ *weakly continuous* if $\partial_H j$ maps weakly convergent sequences to weakly convergent sequences.

The notion of *completely accretive operators* was introduced in [BC91] by Crandall and Bénilan and further developed in [CH17]. Following the same notation as in these two references, $\mathcal{J}_0$ denotes the set of all convex, lower semicontinuous functions $j : \mathbb{R} \to \overline{\mathbb{R}}_+$ satisfying $j(0) = 0$. Let (Σ, μ) be a σ-finite measure space and $M(\Sigma, \mu)$ the space (of all classes) of measurable real-valued functions on Σ.

Definition 2.1. A mapping $S : D(S) \to M(\Sigma, \mu)$ with domain $D(S) \subseteq M(\Sigma, \mu)$ is called a *complete contraction* if

$$\int_\Sigma j(Su - S\hat{u})\, \mathrm{d}\mu \leq \int_\Sigma j(u - \hat{u})\, \mathrm{d}\mu$$

for all $j \in \mathcal{J}_0$ and every u, $\hat{u} \in D(S)$.

Choosing $j(\cdot) = |[\cdot]^+|^q \in \mathcal{J}_0$ if $1 \leq q < \infty$ and $j(\cdot) = [[\cdot]^+ - k]^+ \in \mathcal{J}_0$ for $k \geq 0$ large enough if $q = \infty$ shows that each complete contraction S is T-contractive in $L^q(\Sigma, \mu)$ for every $1 \leq q \leq \infty$. And by choosing $j(\cdot) = |[\cdot]^+|^q \in \mathcal{J}_0$ for any $1 \leq q < \infty$, a complete contraction S is *order preserving*, that is, for every u, $\hat{u} \in D(S)$ satisfying $u \leq \hat{u}$ a.e. on Σ, one has that $Su \leq S\hat{u}$. In fact, the following characterization holds.

Proposition 2.2 ([BC91])**.** *Suppose that the mapping $S : D(S) \to M(\Sigma, \mu)$ with domain $D(S) \subseteq M(\Sigma, \mu)$ satisfies the following: for every u, $\hat{u} \in D(S)$, $k \geq 0$, one has either*

$$\min\{u, (\hat{u} + k)\} \in D(S) \text{ or } \max\{(u - k), \hat{u}\} \in D(S).$$

Then, S is a complete contraction if and only if S is order preserving, and a L^1- and L^∞-contraction.

Definition 2.3. An operator A on $M(\Sigma, \mu)$ is called *completely accretive* if for every $\lambda > 0$, the resolvent operator J_λ of A is a complete contraction. An operator A on $M(\Sigma, \mu)$ is said to be *m-completely accretive* if A completely accretive and the range condition $\mathrm{Rg}(I + A) = L^2(\Sigma, \mu)$ holds. A semigroup $\{T_t\}_{t \geq 0}$ on a subset a closed subset C of $M(\Sigma, \mu)$ is called *order preserving* if each map T_t is order preserving.

Next, we first briefly recall the notion of *Orlicz spaces*. Following [RR91, Chapter 3], a continuous function $\psi : [0,+\infty) \to [0,+\infty)$ is an *N-function* if it is convex, $\psi(s) = 0$ if and only if $s = 0$, $\lim_{s\to 0+}\psi(s)/s = 0$, and $\lim_{s\to\infty}\psi(s)/s = \infty$.

Definition 2.4. Given an N-function ψ, the *Orlicz space*

$$L^{\psi}(\Sigma,\mu) := \left\{ u:\Sigma\to\mathbb{R} \;\middle|\; u \text{ measurable } \& \int_{\Sigma}\psi(\frac{|u|}{\alpha})\,\mathrm{d}\mu < \infty \text{ for some } \alpha > 0 \right\}$$

and equipped with the Orlicz-Minkowski norm

$$\|u\|_{L^{\psi}} := \inf\left\{ \alpha > 0 \;\middle|\; \int_{\Sigma}\psi(\frac{|u|}{\alpha})\,\mathrm{d}\mu \le 1 \right\}.$$

For $1 \le q \le \infty$, we write $L^q_{loc}(H)$ and $L^q(H)$ to denote the vector-valued Lebesgue spaces $L^q_{loc}(\mathbb{R}_+;H)$, $L^q(\mathbb{R}_+;H)$. The derivative u' of a function $u \in L^1_{loc}(H)$ is usually understood in the *distributional sense*. More precisely, a function $w \in L^1_{loc}(H)$ is called the *weak derivative* of $u \in L^1_{loc}(H)$ if

$$\int_0^{+\infty} u(t)\,\xi'(t)\,\mathrm{d}t = -\int_0^{+\infty} w(t)\,\xi(t)\,\mathrm{d}t$$

for all test functions $\xi \in C^{\infty}_c(\mathbb{R}_+)$. A function $w \in L^1_{loc}(H)$ satisfying the latter equation for all $\xi \in C^{\infty}_c(\mathbb{R}_+)$, is unique and so, one writes $u' = w$. We denote by $W^{1,1}_{loc}(H)$ the space of all $u \in L^1_{loc}(H)$ with a weak derivative $u' \in L^1_{loc}(H)$.

Next, let $0 < s < 1$. Then $L^2_s(H)$ denotes the space of all $u \in L^1_{loc}(H)$ satisfying $t^s u \in L^2_{\star}(H)$. We equip the *first order weighted Sobolev space*

$$W^{1,2}_s(H) = \left\{ u \in W^{1,1}_{loc}(H) \;\middle|\; u,\ u' \in L^2_s(H) \right\}.$$

with the inner product

$$(u,\hat{u})_{W^{1,2}_s(H)} := \int_0^{+\infty} \big(u(t)\,\hat{u}(t) + u'(t)\,\hat{u}'(t)\big)\,t^{2s}\,\frac{\mathrm{d}t}{t}.$$

Then, $W^{1,2}_s(H)$ is a Hilbert space and we denote by $\|\cdot\|_{W^{1,2}_s(H)}$ the induced norm of $W^{1,2}_s(H)$. Further, throughout this paper

$$\underline{s} := \frac{1-s}{2s} \qquad \text{and} \qquad \overline{s} := \frac{3s-1}{2s} \qquad \text{for every } 0 < s < 1.$$

Then, $L^2_{\underline{s}}(H)$ denotes the space of all $v \in L^1_{loc}(H)$ satisfying $z^{\frac{1-s}{2s}}v \in L^2_{\star}(H)$ equipped with the inner product

$$(v,w)_{L^2_{\underline{s}}(H)} := \int_0^{\infty} (z^{\frac{1-s}{2s}}v(z), z^{\frac{1-s}{2s}}w(z))\,\frac{\mathrm{d}z}{z} = \int_0^{\infty} (v(z),w(z))z^{\frac{1-2s}{s}}\,\mathrm{d}z \tag{2.2}$$

for every v, $w \in L^2_{\underline{s}}(H)$. Similarly, we write $L^2_{\overline{s}}(H)$ to denote the space of all $v \in L^1_{loc}(0,+\infty;H)$ satisfying $z^{\frac{3s-1}{2s}}v \in L^2_{\star}(H)$.

The spaces $W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$ and $W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)$ are *first order Sobolev spaces with mixed weights* defined by

$$W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H) = \left\{ v \in L^1_{loc}(H) \;\middle|\; z^{\frac{1-s}{2s}}v \in L^2_{\star}(H),\ z^{\frac{1}{2}}v' \in L^2_{\star}(H) \right\}$$

and

$$W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)=\Big\{v\in L^1_{loc}(H)\,\Big|\,z^{\frac{1}{2}}v\in L^2_\star(H),\ z^{\frac{3s-1}{2s}}v'\in L^2_\star(H)\Big\}.$$

In addition, we will employ the *second order Sobolev space with mixed weights*

$$W^{2,2}_{\frac{1-s}{2s},\frac{1}{2},\frac{3s-1}{2s}}(H):=\Big\{v\in W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)\,\Big|\,z^{\frac{3s-1}{2s}}v''\in L^2_\star(H)\Big\}.$$

Each of these spaces equipped with its natural inner product and the induced norms

$$\|v\|_{W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)}:=\Big(\|z^{\frac{1-s}{2s}}v\|^2_{L^2_\star(H)}+\|z^{\frac{1}{2}}v'\|^2_{L^2_\star(H)}\Big)^{1/2}$$

$$\|v\|_{W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)}:=\Big(\|z^{\frac{1}{2}}v\|^2_{L^2_\star(H)}+\|z^{\frac{3s-1}{2s}}v'\|^2_{L^2_\star(H)}\Big)^{1/2}$$

$$\|v\|_{W^{2,2}_{\frac{1-s}{2s},\frac{1}{2},\frac{3s-1}{2s}}(H)}:=\left(\|v\|^2_{W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)}+\|z^{\frac{3s-1}{2s}}v''\|^2_{L^2_\star(H)}\right)^{1/2}$$

is a Hilbert space.

The next proposition shows that for functions $u\in W^{1,2}_s(H)$, the initial value $u(0)\in H$. Note, this proposition is a special case of [ATEW18, Proposition 3.2.1] since for $X=Y=H$, the interpolation space $[X,Y]_\theta=H$ for every $\theta\in(0,1)$.

Proposition 2.5 ([Lun95, Proposition 1.2.10])**.** *Let $0<s<1$. Then for every $u\in W^{1,2}_s(H)$, the limit $u(0):=\lim_{t\to0+}u(t)$ exists in H. Moreover, the trace map*

$$Tr\colon W^{1,2}_s(H)\to H,\ u\mapsto u(0)\ \textit{is continuous, surjective}$$

and there are c_1, $c_2>0$ such that

$$c_1\,\|x\|_H\le\inf_{u\in W^{1,2}_s(H):u(0)=x}\|u\|_{W^{1,2}_s(H)}\le c_2\|x\|_H$$

for every $x\in H$.

To conclude this preliminary section, we state the following integration by parts rule from [ATEW18, Proposition 3.9].

Proposition 2.6. *Let $0<s<1$. For $u\in W^{1,2}_s(H)$ and $\xi\in W^{1,2}_{1-s}(H)$, the functions $t\mapsto(u'(t),\xi(t))_H$ and $t\mapsto(u(t),\xi'(t))_H$ belong to $L^1(0,+\infty)$. Moreover, the following integration by parts rule holds:*

$$-\int_0^{+\infty}(u'(t),\xi(t))_H\,\mathrm{d}t=\int_0^{+\infty}(u(t),\xi'(t))_H\,\mathrm{d}t+(u(0),\xi(0))_H.$$

3. Well-posedness of second order boundary value problems

In this section, we are concerned with establishing the well-posedness of the following more general abstract boundary-value problem

$$\begin{cases} u''(t)+\frac{1-2s}{t}u'(t)\in Au(t) & \text{for almost every } t>0,\\ \lim\limits_{t\to0+}t^{1-2s}u'(t)\in\partial j(u(0)-\varphi), \end{cases}\tag{3.1}$$

where $j : H \to \overline{\mathbb{R}}_+$ be a convex, strongly coercive, lower semicontinuous functional satisfying $j(0) = 0$, $\varphi \in \overline{D(A)}^H$, and $1 < s < 1$.

Definition 3.1. For $\varphi \in H$, a function $u : [0, +\infty) \to H$ is called a *solutions* of problem (3.1) if $u \in C([0, +\infty); H)$, u is a strong solution of (1.1), $\lim_{t\to 0+} t^{1-2s} u'(t)$ exists in H and

$$\lim_{t\to 0_+} t^{1-2s} u'(t) \in \partial j(u(0) - \varphi).$$

For our next theorem, we recall (cf [Bre73] or [Bar10]) that the *indicator function* $j : H \to \overline{\mathbb{R}}_+$ is defined by $j(v) := 0$ if $v = 0$ and $j(v) := +\infty$ if otherwise. The following theorem is the first main result of this section.

Theorem 3.2. *Let A be a maximal monotone operator on H with $0 \in Rg(A)$, and $j : H \to \overline{\mathbb{R}}_+$ be a convex, strongly coercive, lower semicontinuous functional satisfying $j(0) = 0$ and either $\partial_H j : D(\partial_H j) \to H$ is a weakly continuous mapping or j is the "indicator function". Assume further that A is ∂j-monotone. Then, for every $0 < s < 1$, $\varphi \in D(A)$ and $y \in A^{-1}(\{0\})$, there is a unique solution $u \in L^\infty(H)$ of problem (3.1) satisfying (1.5), (1.9)-(1.11) and (1.6)-(1.8). In particular, $t^{1-2s}u' \in W^{1,2}_s(H)$, the function $t \mapsto \|u(t)\|_H^2$ is convex, bounded and decreasing on $[0, +\infty)$, and for every two strong solutions u and $\hat{u} \in L^\infty(H)$ of equation (1.1), one has that (1.12) holds.*

Remark 3.3 (*The case $A = \partial_H \phi$*). Suppose $A = \partial_H \phi$ the subdifferential operator of a proper, convex, lower semicontinuous functional $\phi : H \to \mathbb{R} \cup \{+\infty\}$ attaining a global minimum $\min_{v\in H} \phi(v) = \phi(v_0)$ at some $v_0 \in H$. After possibly replacing ϕ by $\tilde{\phi}(v - v_0) - \phi(v_0)$, $(v \in H)$, we may assume without loss of generality that ϕ attains it minimum at $0 \in H$ and $\phi : H \to \overline{\mathbb{R}}_+$. Hence, for $\varphi \in D(A)$, the second-order boundary problem (3.1) can be rewritten as

$$0 \in \partial_{L^2_{1-s}(H)} \mathcal{E}(u), \tag{3.2}$$

or equivalently, as the minimization problem

$$\min_{u\in L^2_{1-s}(H)} \mathcal{E}(u)$$

for the functional $\mathcal{E} : L^2_{1-s}(H) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{E}(u) := \begin{cases} \displaystyle\int_0^{+\infty} t^{2(1-s)} \Big\{ \tfrac{1}{2} \|u'(t)\|_H^2 + \phi(u(t)) \Big\} \frac{\mathrm{d}t}{t} + j(u(0) - \varphi) & \text{if } u \in D(\mathcal{E}), \\ +\infty & \text{if otherwise,} \end{cases}$$

where $D(\mathcal{E}) := \{u \in W^{1,2}_{1-s}(H) \,|\, \phi(u) \in L^1_{1-s}(H),\ u(0) - \varphi \in D(j)\}$. It is not difficult to see that $\mathcal{E}$ is convex and by Proposition 2.5, $\mathcal{E}$ is proper and lower semicontinuous on $L^2_{1-s}(H)$ (cf [Bre73, Exemple 2.8.3]). But to see that $\mathcal{E}$ is coercive, stronger assumptions on ϕ are needed; for example, suppose there is an $\eta > 0$ such that

$$\phi(u) \ge \eta \, \|u\|_H^2, \qquad \text{for all } u \in D(\phi).$$

This is consistent with the linear theory of sectorial operators (cf [ATEW18, p 10 formula (4.4)]) and the case $s = 1/2$ in the nonlinear theory (cf [Bar76, Chapter V.2]).

Under the assumption that the statement of Theorem 3.2 holds, we outline how to deduce the statements of Theorem 1.2 and Theorem 1.7.

Proof of Theorem 1.2 and Theorem 1.7. By choosing j to be the indicator function, one sees that problem (3.1) reduces to Dirichlet problem (1.4). Therefore for $\varphi \in D(A)$, the statements of Theorem 1.2 follows from Theorem 3.2. Now, let $\varphi \in \overline{D(A)}^H$. Then there are sequences $(\varphi_n)_{n\ge 1}$ of $\varphi_n \in D(A)$ and $(v_n)_{n\ge 1}$ of corresponding solutions u_n of (1.4). By (1.12),

$$\sup_{t\ge 0}\|u_n(t) - u_m(t)\|_H \le \|\varphi_n - \varphi_m\|_H$$

for every n, $m \ge 0$. Moreover, by (1.5), every $u_n \in L^\infty(H)$. Therefore, $(u_n)_{n\ge 1}$ is a Cauchy sequence in $C^b([0,+\infty);H)$ and so, there is a function $u \in C^b([0,+\infty);H)$ such that

$$\lim_{n\to+\infty} u_n = u \qquad \text{in } C^b([0,+\infty);H).$$

From this limit, we can conclude that u satisfies (1.5) and (1.12). In particular, since $u_n(0) = \varphi_n \to \varphi$ in H, one has that $u(0) = \varphi$ in H. Further, by (1.6) and (1.8) applied to u_n, we can conclude that $u \in W^{2,2}_{loc}((0,+\infty);H)$ and after possibly passing to a subsequence of $(u_n)_{n\ge 1}$ and taking $\liminf_{n\to+\infty}$ in those inequalities, we obtain that u satisfies (1.6) and (1.8). To see that u is a strong solution of (1.1), one proceeds similarly to *step 5* in the proof of Theorem 3.6 (below). This proves the statement of Theorem 1.2.

Next, to see that also Theorem 1.7 holds, for given $\lambda > 0$ and $\varphi \in D(A)$, one chooses

$$j(\cdot) = \frac{\lambda}{2}\left\|\cdot - \left(\frac{1}{\lambda} - 1\right)\varphi\right\|_H^2$$

and applies Theorem 3.2. □

The key to prove Theorem 3.2 (respectively, Theorem 1.2) is via the change of variable (1.2), which transforms the abstract boundary-value problem (3.1) associated with the Bessel-type operator A_{1-2s} to the following abstract boundary-value problem

$$\begin{cases} z^{-\frac{1-2s}{s}}v''(z) \in Av(z) & \text{for almost every } z > 0,\\ \qquad v'(0) \in \partial\tilde{j}(v(0)-\varphi), \end{cases} \tag{3.3}$$

for given $\varphi \in \overline{D(A)}^H$ and where $\tilde{j} = (2s)^{-(1-2s)}j$.

Definition 3.4. We call a function $v : [0,+\infty) \to H$ a *strong solution* of equation (1.3) if $v \in W^{2,2}_{loc}((0,+\infty);H)$ and for almost every $z > 0$, $v(z) \in D(A)$ and $z^{-\frac{1-2s}{s}}v''(z) \in A(v(z))$. For given $\varphi \in H$ and $\tilde{j} = (2s)^{-(1-2s)}j$, a function v is called a *solution* of problem (3.3) if $v \in C^1([0,+\infty);H)$ is a strong solution of (1.3) satisfying $v'(0) \in \partial\tilde{j}(v(0)-\varphi)$.

Further, for given $\varphi \in H$, a function $v \in C([0,+\infty);H)$ is called a *solution of Dirichlet problem*

$$\begin{cases} z^{-\frac{1-2s}{s}}v''(z) \in Av(z) & \text{for almost every } z > 0,\\ \qquad v(0) = \varphi, \end{cases} \tag{3.4}$$

if $v \in C([0,+\infty);H)$, v is a strong solution of (1.3), and $v(0) = \varphi$.

Our first lemma outlines the equivalence between Definition (3.1) and Definition (3.4).

Lemma 3.5. *Let A be an operator on H, $j : H \mapsto \mathbb{R} \cup \{+\infty\}$ a proper functional, $\varphi \in H$, and $0 < s < 1$. For $u \in W^{2,2}_{loc}((0,+\infty);H)$, let $u(t) = v(z)$ for z given by the change of variable (1.2). Then, the following statements hold.*

(1) *u is a solution of (3.1) if and only if v is a solution of (3.3).*

(2) *A function $u \in W^{1,2}_{s}(H)$ if and only if $v \in W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)$, and a function $u \in W^{1,2}_{1-s}(H)$ if and only if $v \in W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$ with*

$$u'(t) = v'(z) z^{-\frac{1-2s}{2s}}. \tag{3.5}$$

Proof. If $z = z(t) = (\frac{t}{2s})^{2s}$, then $t = t(z) = 2s z^{\frac{1}{2s}}$ and so, $v(z) = u(t) = u(2sz^{\frac{1}{2s}})$. Then, for z and $t > 0$,

$$v'(z) = \frac{\mathrm{d}}{\mathrm{d}z} v(z) = \frac{\mathrm{d}u}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}z} = u'(t) z^{\frac{1-2s}{2s}},$$

proving (3.5). By using (3.5), one easily verifies that $u \in W^{1,2}_{1-s}(H)$ if and only if $v \in W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$, and $u \in W^{1,2}_{s}(H)$ if and only if $v \in W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)$. Thus claim (2) holds. Further,

$$v''(z) = \frac{\mathrm{d}}{\mathrm{d}z} v'(z) = u''(t) z^{\frac{1-2s}{s}} + u'(t)\, \frac{1-2s}{2s} z^{\frac{1-2s}{2s}-1}$$

for z, $t > 0$. Multiplying this equation by $z^{-\frac{1-2s}{s}}$, one sees that

$$z^{-\frac{1-2s}{s}} v''(z) = u''(t) + \frac{1-2s}{2s} z^{-\frac{1}{2s}} u'(t) = u''(t) + \frac{1-2s}{t} u'(t)$$

for almost every z, $t > 0$. Therefore, $z^{-\frac{1-2s}{s}} v''(z) \in Av(z)$ for a.e. $z > 0$ if and only if $u''(t) + \frac{1-2s}{t} u'(t) \in Au(t)$ for a.e. $t > 0$, showing that u is a strong solution of (1.1) if and only if v is a strong solution of (1.3). Moreover, $t^{1-2s} u' \in W^{1,2}_{s}(H)$ if and only if $z^{\frac{1}{2}} v'$ and $z^{-\frac{1-3s}{2s}} v'' \in L^2_{\star}(H)$. Multiplying $v'(z)$ by $(2s)^{1-2s}$ shows that

$$(2s)^{1-2s} v'(z) = (2s)^{1-2s} u'(t) z^{\frac{1-2s}{2s}} = t^{1-2s} u'(t).$$

Note that $\lim_{z\to 0+} t(z) = 0$ and $\lim_{t\to 0+} z(t) = 0$. Thus

$$\lim_{t\to 0+} t^{1-2s} u'(t) \text{ exists in } H \text{ if and only if } \lim_{z\to 0+} v'(z) \text{ exists in } H.$$

By Proposition 2.5, $t^{1-2s} u' \in W^{1,2}_{s}(H)$ implies that $\lim_{t\to 0+} t^{1-2s} u'(t)$ exists in H. Similarly, one has that $u(0) := \lim_{t\to 0+} u(t)$ exists in H if and only if $v(0) = \lim_{z\to 0+} v(z)$ exists in H, and by Proposition 2.5, $u(0) := \lim_{t\to 0+} u(t) = \varphi$ exists in H. Therefore and by the definition of the subdifferential $\partial_H j$, one has that $\lim_{t\to 0_+} t^{1-2s} u'(t) \in \partial_H j(u(0) - \varphi)$ if and only if $(1-\alpha)^{\alpha} v'(0) \in \partial_H j(v(0) - \varphi)$, which completes the proof of showing that u is a solution of (3.1) if and only if v is a solution of (3.3). $\square$

Our next theorem provides the existence and uniqueness of problem (3.3).

Theorem 3.6. *Under the hypothesis of Theorem 3.2, let* $\tilde{j} := (2s)^{-(1-2s)} j$. *Then, for every* $\varphi \in D(A)$ *and* $y \in A^{-1}\{0\}$, *there is a unique solution*

$$v \in L^{\infty}(H) \cap C^1([0,+\infty);H) \qquad \text{with} \qquad v' \in W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)$$

of problem (3.3) *satisfying*

$$\|v(z)-y\|_H \le \|v(\hat{z})-y\|_H \qquad \text{for all } z \ge \hat{z} \ge 0, \tag{3.6}$$

$$\|v'(z)\|_H \le \|v'(\hat{z})\|_H \qquad \text{for all } z \ge \hat{z} \ge 0, \tag{3.7}$$

$$\|z\, v'(z)\|_{L^2_\star(H)} \le \frac{\|\varphi - y\|_H}{\sqrt{2}} \tag{3.8}$$

$$\|v'(z)\|_H \le \frac{\|\varphi - y\|_H}{z} \qquad \text{for every } z > 0, \tag{3.9}$$

$$\|z^2 v''\|_{L^2_\star(H)} \le \begin{cases} \dfrac{\|\varphi-y\|_H}{\sqrt{2}} & \text{if } s \ge \frac{1}{2},\\[2mm] \dfrac{1}{2}\left(\dfrac{s}{1-2s}\dfrac{1}{2}+3\right)^{\frac{1}{2}} \|\varphi - y\|_H, & \text{if } 0 < s < \frac{1}{2}. \end{cases} \tag{3.10}$$

$$\|v'(0)\|_H^{\frac{1}{2}} \le \|A^0\varphi\|_H^{\frac{1}{2}} + \|\varphi - y\|_H^{\frac{1}{2}}, \tag{3.11}$$

$$\|z^{\frac{1}{2}} v'\|_{L^2_\star(H)} \le \|A^0\varphi\|_H^{\frac{1}{2}} + \|\varphi - y\|_H, \tag{3.12}$$

$$\|v''\|_{L^2_s} \le \|A^0\varphi\|_H + \|\varphi - y\|_H^{\frac{1}{2}}\, \|A^0\varphi\|_H^{\frac{1}{2}}. \tag{3.13}$$

For every two strong solutions v *and* $\hat{v} \in L^{\infty}(H)$ *of* (1.3), *one has that*

$$\|v(z) - \hat{v}(z)\|_H \le \|v(\hat{z}) - \hat{v}(\hat{z})\|_H \qquad \text{for every } z \ge \hat{z} \ge 0. \tag{3.14}$$

Further, for every $\varphi \in \overline{D(A)}^H$ *and* $y \in A^{-1}\{0\}$, *there is a unique solution* $v \in L^{\infty}(H)$ *of Dirichlet problem* (3.4) *satisfying* (3.6)-(3.10), *and* (3.14).

Thanks to Lemma 3.5, Theorem 3.6 implies that the statement of Theorem 3.2 holds. In particular, by Theorem 3.6, if the boundary value $\varphi \in D(A)$, then the unique solution v of Dirichlet problem (3.4) satisfies $v \in W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)$, implying that the Neumann derivative

$$-(2s)^{1-2s} v'(0) =: \Theta_s \varphi \qquad \text{exists in } H.$$

This allows us to define the DtN operator Θ_s associated with $\tilde{A}_{1-2s}$ as given in the following Corollary.

Corollary 3.7. *Let* A *be a maximal monotone operator on* H *with* $0 \in Rg(A)$. *Then, for every* $0 < s < 1$, *the Dirichlet-to-Neumann operator* Θ_s *associated with* $\tilde{A}_{1-2s}$ *defined by*

$$\Theta_s = \left\{ (\varphi, w) \in H \times H \,\middle|\, \begin{array}{l} \exists \text{ a solution } v \text{ of Dirichlet problem (1.3)},\\ \text{with } v(0) = \varphi \text{ and } w = -(2s)^{1-2s} v'(0) \text{ in } H. \end{array} \right\}$$

is a monotone, well-defined mapping $\Theta_s : D(\Theta_s) \to H$ *satisfying*

$$D(A) \subseteq D(\Theta_s) \subseteq \overline{D(A)}^H, \qquad \text{and} \qquad D(A) \subseteq Rg(I_H + \lambda\, \Theta_s) \qquad \text{for all } \lambda > 0.$$

The closure $\overline{\Theta}_s$ of Θ_s in $H\times H$ is characterized by

$$\overline{\Theta}_s := \left\{ (\varphi, w) \in H \times H \;\middle|\; \begin{array}{l} \exists\, (\varphi_n, w_n) \in \Theta_s \text{ s.t. } \lim_{n\to+\infty} (\varphi_n, w_n) = (\varphi, w) \\ \text{in } H\times H \text{ \& a strong solution } v \text{ of (1.3)} \\ \text{satisfying } v(0) = \varphi \text{ in } H. \end{array} \right\}$$

with domain $D(\overline{\Theta}_s) = \overline{D(A)}^H$ and

$$\overline{D(A)}^H \subseteq Rg(I_H + \lambda\,\overline{\Theta}_s) \qquad \text{for all } \lambda > 0.$$

If $D(A)$ is dense in H, then $\overline{\Theta}_s$ is maximal monotone.

By Lemma 3.5, Corollary 3.7 implies that Theorem 1.3 holds. The rest of this section is dedicated to the proof of Theorem 3.6

By using the change of variable (1.2), Proposition 2.5 can be rewritten for functions $v \in W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$. Moreover, by [ATEW18], we have that the following *integration by parts* holds.

Lemma 3.8. *Let $0 < s < 1$. Then, the following statements hold.*

(1) *(**Trace theorem on** $W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$) For $v \in W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$, the limit $v(0) := \lim_{z\mapsto 0+} v(z)$ exists in H. In particular, the trace operator*

$$Tr : W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H) \to H, \quad v \mapsto v(0)$$

is continuous and surjective.

(2) *(**Trace theorem on** $W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)$) For $v \in W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)$, the limit $v(0) := \lim_{z\mapsto 0+} v(z)$ exists in H. In particular, the trace operator*

$$Tr : W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H) \to H, \quad v \mapsto v(0)$$

is continuous and surjective, and there are c_1, $c_2 > 0$ such that

$$c_1\,\|x\|_H \le \inf_{\xi\in W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H):\xi(0)=x} \|\xi\|_{W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)} \le c_2\|x\|_H \tag{3.15}$$

for every $x \in H$.

(3) *For $w \in W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)$ and $\xi \in W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$, the functions*

$$z \mapsto (w'(z), \xi(z))_H \text{ and } z \mapsto (w(z), \xi'(z))_H$$

belong to $L^1(0,+\infty)$ and the following integration by parts rule holds:

$$\int_0^{+\infty} (w'(z), \xi(z))_H \,\mathrm{d}z = -(w(0), \xi(0))_H - \int_0^{+\infty} (w(z), \xi'(z))_H \,\mathrm{d}z. \tag{3.16}$$

Proof. As mentioned in the proof of Lemma 3.5, $u \in W^{1,2}_{1-s}(H)$ if and only if $v \in W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$, and since $\lim_{z\to 0+} t(z) = 0$ and $\lim_{t\to 0+} z(t) = 0$, one has that the fact that $u(0) := \lim_{t\to 0+} u(t)$ exists in H is equivalent to $v(0) := \lim_{z\to 0+} v(z)$ exists in H. Therefore by Proposition 2.5, for every $v \in W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$, the limit $v(0) := \lim_{z\mapsto 0+} v(z)$ exists in H and the trace operator $\mathrm{Tr} : W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H) \to H$ is surjective. To see that the operator Tr is continuous, it suffices to note that

$$\|u\|_{W^{1,2}_{1-s}(H)} = (2s)^{\frac{1-2s}{2}}\,\|v\|_{W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)}.$$

Thus, claim (1) of this proposition holds and since by Lemma 3.5, $u \in W^{1,2}_{s}(H)$ if and only if $v \in W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)$, one shows similarly that claim (2) as well holds true. Next, let $w \in W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)$ and $\xi \in W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$. Since $z = z^{\frac{3s-1}{2s}}\, z^{\frac{1-s}{2s}}$, Hölder's inequality yields that

$$\begin{aligned}\int_0^\infty |(w'(z),\xi(z))_H|\,\mathrm{d}z &= \int_0^\infty |(z^{\frac{3s-1}{2s}}w'(z), z^{\frac{1-s}{2s}}\xi(z))_H|\,\frac{\mathrm{d}z}{z}\\ &\le \|z^{\frac{3s-1}{2s}}w'\|_{L^2_*(H)}\,\|z^{\frac{1-s}{2s}}\xi\|_{L^2_*(H)}\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty |(w(z),\xi'(z))_H|\,\mathrm{d}z &= \int_0^\infty |(z^{\frac{1}{2}}w(z), z^{\frac{1}{2}}\xi'(z))_H|\,\frac{\mathrm{d}z}{z}\\ &\le \|z^{\frac{1}{2}}w\|_{L^2_*(H)}\,\|z^{\frac{1}{2}}\xi'\|_{L^2_*(H)},\end{aligned}$$

proving that $(w',\xi)_H$ and $(w,\xi')_H \in L^1(0,+\infty)$. Finally, to see that integration by parts (3.16) holds, one applies [ATEW18, Proposition 3.9] to $w(z) = u(t)$ and $\xi(z) = v(t)$ with the change of variable (1.2). □

With these preliminaries, we can now prove the uniqueness of solutions to problem (3.3). Here, our proof adapts an idea by Brezis [Bre72] to the more general case $0 < s < 1$.

Proof of Theorem 3.6 (uniqueness and proof of inquality (3.14)). Suppose v_1 and $v_2 \in L^\infty(H)$ are two strong solutions of (1.3) and set $w = v_1 - v_2$. Then, by the monotonicity of A and by (1.3),

$$z^{-\frac{1-2s}{s}}(w''(z),w(z))_H = (z^{-\frac{1-2s}{s}}v_1''(z) - z^{-\frac{1-2s}{s}}v_2'(z), v_1(z)-v_2(z))_H \ge 0$$

for almost every $z > 0$.Thus, $(w''(z),w(z))_H \ge 0$ for almost every $z > 0$ and so,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\|w(z)\|_H^2 &= \frac{\mathrm{d}}{\mathrm{d}z}(w'(z),w(z))_H\\ &= (w''(z),w(z))_H + \|w'(z)\|_H^2 \ge \|w'(z)\|_H^2 \ge 0\end{aligned} \tag{3.17}$$

for a.e. $z > 0$. Therefore, the function $z \to \|w(z)\|_H^2$ is convex and since by hypothesis, w is bounded on $\mathbb{R}_+$ with values in H, the function $z \to \|w(z)\|_H^2$ is necessarily monotonically decreasing on $\mathbb{R}_+$. In particular, this argument shows that for every two strong solutions v_1, $v_2 \in L^\infty(H)$ of (1.3), one as that (3.14) holds. Further, from this, we can deduce that

$$(w'(z),w(z))_H = \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{2}\|w(z)\|_H^2 \le 0$$

for every $z \in \mathbb{R}_+$. Now, note that for $0 < s < 1$, $\alpha = 1-2s > 0$. Thus, for $\tilde{j} := (2s)^{-(1-2s)}\,j$, $\partial_H\tilde{j}$ is monotone. Hence, if v_1 and v_2 are solutions of problem (3.3) for the same $\varphi \in H$, then by the condition $v_i'(0) \in \partial_H\tilde{j}(v(0)-\varphi)$ for $i = 1,2$, one has that

$$0 \ge (w'(0),w(0))_H = (v_1'(0)-v_2'(0),(v_1(0)-\varphi)-(v_2(0)-\varphi))_H \ge 0.$$

Combining this with (3.17), one finds

$$0 \ge (w'(z),w(z))_H = \int_0^z \frac{\mathrm{d}}{\mathrm{d}r}(w'(r),w(r))_H\mathrm{d}s \ge \int_0^z \|w'(r)\|_H^2\mathrm{d}r,$$

implying that $w'(z) = 0$ in H for all $z \geq 0$. Thus, $v_1'(0) = v_2'(0)$ and since

$$\tilde{j}(v_1(0) - \varphi) - \tilde{j}(v_2(0) - \varphi) \geq (v_2'(0), v_1(0) - v_2(0))$$

and

$$\tilde{j}(v_2(0) - \varphi) - \tilde{j}(v_1(0) - \varphi) \geq (v_1'(0), v_2(0) - v_1(0)),$$

it follows that

$$\tilde{j}(v_2(0) - \varphi) - \tilde{j}(v_1(0) - \varphi) = (v_1'(0), v_2(0) - v_1(0)). \tag{3.18}$$

Now, if $v_1(0) \neq v_2(0)$, then by the strict convexity of $\tilde{j}$ and (3.18),

$$\begin{aligned}\frac{1}{2}\tilde{j}(v_1(0) - \varphi) + \frac{1}{2}\tilde{j}(v_2(0) - \varphi) &> \tilde{j}\Big(\frac{v_1(0) + v_2(0)}{2} - \varphi\Big)\\ &\geq \tilde{j}(v_1(0) - \varphi) + \Big(v_1'(0), \frac{v_2(0) - v_1(0)}{2}\Big)\\ &= \frac{1}{2}\tilde{j}(v_1(0) - \varphi) + \frac{1}{2}\tilde{j}(v_2(0) - \varphi),\end{aligned}$$

which is a contradiction. Therefore, $v_1(0) = v_2(0)$, implying that $v_1 = v_2$. This completes the proof of uniqueness. □

For proving existence of strong solutions of problem (3.3), we need the following proposition.

Proposition 3.9. *Suppose $j : H \to \overline{\mathbb{R}}_+$ is a convex, strongly coercive, lower semicontinuous functional satisfying $j(0) = 0$, let $\varphi \in H$ and for $0 < s < 1$, $\tilde{j} := (2s)^{-(1-2s)} j$. Further, let $\mathcal{E}_1$ and $\mathcal{E}_2 : L^2_{\underline{s}}(H) \to \mathbb{R} \cup \{+\infty\}$ be given by*

$$\mathcal{E}_1(v) := \begin{cases} \frac{1}{2}\displaystyle\int_0^{+\infty} \|v'(z)\|_H^2 \,\mathrm{d}z & \text{if } v \in \mathcal{W}^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H),\\ +\infty & \text{if otherwise.}\end{cases}$$

and

$$\mathcal{E}_2(v) := \begin{cases} \tilde{j}(v(0) - \varphi) & \text{if } v \in \mathcal{W}^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H),\\ +\infty & \text{if otherwise.}\end{cases}$$

Then, the functional $\mathcal{E} : L^2_{\underline{s}}(H) \to \mathbb{R} \cup \{+\infty\}$ defined by $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$ is proper, convex and lower semicontinuous on $L^2_{\underline{s}}(H)$. In particular, the subdifferential $\partial_{L^2_{\underline{s}}}\mathcal{E}$ of $\mathcal{E}$ is a mapping $\partial_{L^2_{\underline{s}}}\mathcal{E} : D(\partial\mathcal{E}) \to L^2_{\underline{s}}(H)$ given by

$$\partial_{L^2_{\underline{s}}}\mathcal{E} = \Big\{(v, -z^{-\frac{1-2s}{s}} v'') \in L^2_{\underline{s}}(H) \times L^2_{\underline{s}}(H) \;\Big|\; v'(0) \in \partial_H \tilde{j}(v(0) - \varphi)\Big\}.$$

In particular, for every $v \in D(\partial_{L^2_{\underline{s}}}\mathcal{E})$, one has that

$$v \in W^{2,2}_{\frac{1-s}{2s},\frac{1}{2},\frac{3s-1}{2s}}(H) \cap C^1([0, +\infty); H).$$

Proof of Proposition 3.9. It is clear that $\mathcal{E}$ is convex, and $\mathcal{E}$ is proper since by Lemma 3.8, for every $\varphi \in H$, there is a $v \in W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$ satisfying $v(0) = \varphi$ and $\mathcal{E}(v) = \mathcal{E}_1(v)$ is finite. To see that $\mathcal{E}$ is lower semicontinuous on $L^2_{\underline{s}}(H)$, let $c \in \mathbb{R}$ and $(v_n)_{n\geq 1}$ be a sequence in $L^2_{\underline{s}}(H)$ such that $v_n \to v$ in $L^2_{\underline{s}}(H)$ for some $v \in L^2_{\underline{s}}(H)$ and satisfying $\mathcal{E}(v_n) \leq c$ for all $n \geq 1$. Since $\tilde{j} \geq 0$, this implies that $(v_n)_{n\geq 1}$ is bounded in $W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$. Since $W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$ is reflexive, one can conclude that $v \in W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$ and there is a subsequence of $(v_n)_{n\geq 1}$, which we denote, for simplicity, again by $(v_n)_{n\geq 1}$ such that v_n' converges weakly to v' in $L^2(H)$. By Lemma 3.8, the trace map $\mathrm{Tr} : W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H) \to H$ is linearly bounded and so, $v_n(0)$ converges weakly to $v(0)$ in H. Therefore, and since $\mathcal{E}$ is convex, we can conclude that $v \in D(\mathcal{E})$ and $\mathcal{E}(v) \leq c$. It remains to characterize the subdifferential

$$\partial_{L^2_{\underline{s}}}\mathcal{E} := \left\{ (v,w) \in L^2_{\underline{s}}(H) \times L^2_{\underline{s}}(H) \;\middle|\; \begin{array}{r} \mathcal{E}(\hat{v}) - \mathcal{E}(v) \geq (w, \hat{v} - v)_{L^2_{\underline{s}}(H)} \\ \text{for all } \hat{v} \in D(\mathcal{E}) \end{array} \right\}.$$

For every $v \in D(\mathcal{E})$, the weak derivative $v' \in L^2(H)$. Hence $v \in C([0,+\infty);H)$. Now, let $(v,w) \in \partial_{L^2_{\underline{s}}}\mathcal{E}$ and take $\hat{v} = v + \varepsilon\xi$ for $\varepsilon \in \mathbb{R}$ and $\xi \in D(\mathcal{E})$. Then,

$$\mathcal{E}(v + \varepsilon\xi) - \mathcal{E}(v) \geq \varepsilon\,(w,\xi)_{L^2_{\underline{s}}(H)}.$$

Suppose first that $\varepsilon > 0$. Then, dividing the above inequality by ε gives

$$\begin{aligned} &\int_0^{+\infty} \frac{\frac{1}{2}\|v'(z) + \varepsilon\xi'(z)\|_H^2 - \frac{1}{2}\|v'(z)\|_H^2}{\varepsilon}\,\mathrm{d}z \\ &\quad + \frac{\tilde{j}(v(0) + \varepsilon\xi(0) - \varphi) - \tilde{j}(v(0) - \varphi)}{\varepsilon} \geq \int_0^{+\infty} (w(z),\xi(z))_H z^{\frac{1-2s}{s}}\,\mathrm{d}z. \end{aligned} \tag{3.19}$$

Since

$$\lim_{\varepsilon\to 0+} \int_0^{+\infty} \frac{\frac{1}{2}\|v'(z) + \varepsilon\xi'(z)\|_H^2 - \frac{1}{2}\|v'(z)\|_H^2}{\varepsilon}\,\mathrm{d}z = \int_0^{+\infty} (v',\xi')_H\,\mathrm{d}z,$$

and $\tilde{j}$ is convex, we can conclude by sending $\varepsilon \to 0+$ in (3.19) that

$$\begin{aligned} \int_0^{+\infty} (v',\xi')_H\,\mathrm{d}z + \inf_{\varepsilon>0} &\frac{\tilde{j}(v(0) + \varepsilon\xi(0) - \varphi) - \tilde{j}(v(0) - \varphi)}{\varepsilon} \\ &\geq \int_0^{+\infty} (w(z),\xi(z))_H z^{\frac{1-2s}{s}}\,\mathrm{d}z. \end{aligned}$$

In particular, we have that

$$\begin{aligned} \int_0^{+\infty} (v',\xi')_H\,\mathrm{d}z + \tilde{j}(v(0) + \xi(0) - \varphi) - \tilde{j}(v(0) - \varphi)& \\ \geq \int_0^{+\infty} (w(z),\xi(z))_H z^{\frac{1-2s}{s}}\,\mathrm{d}z.& \end{aligned} \tag{3.20}$$

Note, for any $\xi \in C_c^1((0,+\infty);H)$, the sum $v + \varepsilon\xi$ belongs to $D(\mathcal{E})$ and

$$\frac{\tilde{j}(v(0) + \epsilon\xi(0) - \varphi) - \tilde{j}(v(0) - \varphi)}{\epsilon} = 0.$$

Thus, proceeding as before with $\xi \in C_c^1((0,+\infty);H)$ and $\varepsilon < 0$, one obtains

$$\begin{aligned} \int_0^{+\infty} (v',\xi')_H\,\mathrm{d}z &= \int_0^{+\infty} (w(z),\xi(z))_H z^{\frac{1-2s}{s}}\mathrm{d}z \\ &= -\int_0^{+\infty} (-z^{\frac{1-2s}{s}} w(z),\xi(z))_H \mathrm{d}z. \end{aligned}$$

Since this equality holds for all $\xi \in C_c^1((0,+\infty);H)$, we have thereby shown that $-z^{\frac{1-2s}{s}}w = v''$ and $v \in W^{2,2}_{\frac{1-s}{2s},\frac{1}{2},\frac{3s-1}{2s}}(H)$. Moreover, $W^{2,2}_{\frac{1-s}{2s},\frac{1}{2},\frac{3s-1}{2s}}(H)$ is a linear subspace of $W^{2,2}_{loc}((0,+\infty);H)$. Thus, one also has that

$$v \in C^1((0,+\infty);H) \cap W^{2,2}_{loc}((0,+\infty);H).$$

In addition, since $v' \in W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)$, Lemma 3.8 says that $v'(0) := \lim_{z\to 0+} v'(z)$ exists in H. Thus, $v \in C^1([0,+\infty);H)$.

Further, by Lemma 3.8, for given $\xi_0 \in H$, there is a $\xi \in W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$ satisfying $\xi(0) = \xi_0$ in H. Suppose $v(0) + \xi_0 \in D(\mathcal{E}_2)$ (otherwise, inequality (3.21) below always holds). Then, inserting $w = -z^{-\frac{1-2s}{s}}v''$ into (3.20) and integrating by parts (Lemma 3.8) on the right hand side of the same inequality yields that

$$\begin{aligned}
\int_0^{+\infty} (v'(z),\xi'(z))_H \,\mathrm{d}z + \tilde{j}(v(0)+\xi_0-\varphi) - \tilde{j}(v(0)-\varphi)&\\
&\geq -\int_0^{+\infty} (v''(z),\xi(z))_H \,\mathrm{d}z\\
&= -(v'(0),\xi(0))_H + \int_0^{+\infty} (v'(z),\xi'(z))_H \,\mathrm{d}z\\
&= (v'(0),\xi_0)_H + \int_0^{+\infty} (v'(z),\xi'(z))_H \,\mathrm{d}z,
\end{aligned}$$

from where we obtain that

$$\tilde{j}(v(0)+\xi_0-\varphi) - \tilde{j}(v(0)-\varphi) \geq (v'(0),\xi_0)_H. \tag{3.21}$$

Since inequality (3.21) holds for arbitrary $\xi_0 \in H$, we have thereby shown that $v'(0) \in \partial_H \tilde{j}(v(0)-a)$. This completes the proof of this proposition. □

With these preliminaries in mind, we focus now on proving existence of solutions of problem (3.3) (2nd part of Theorem 3.6). First, we give a briefly sketch of the existence proof.

Let $\varphi \in D(A)$. Then the strategy of proving existence of solutions to (3.3) is lifting equation (1.3) in H to the following abstract equation

$$\mathcal{A}_{loc}v - z^{-\frac{1-2s}{s}}v'' \ni 0 \qquad \text{in } L^2_{loc}(H), \tag{3.22}$$

where

$$\mathcal{A}_{loc} := \Big\{(v,w) \in L^2_{loc}(H) \times L^2_{loc}(H) \,\Big|\, w(z) \in A(v(z)) \text{ for a.e. } z \geq 0\Big\}.$$

Existence of a solutions v of (3.22) satisfying

$$v'(0) \in \partial_H \tilde{j}(v(0)-\varphi) \tag{3.23}$$

is shown in two steps: first, let

$$\mathcal{A} := \Big\{(v,w) \in L^2_{\underline{s}}(H) \times L^2_{\underline{s}}(H) \,\Big|\, w(z) \in A(v(z)) \text{ for a.e. } z \geq 0\Big\}. \tag{3.24}$$

Then, one shows that for every λ, $\delta > 0$, the following regularized equation

$$\mathcal{A}_\lambda v_\lambda + \delta v_\lambda + \partial_{L^2_{\underline{s}}(H)}\mathcal{E}(v_\lambda) = 0 \qquad \text{in } L^2_{\underline{s}}(H) \tag{3.25}$$

admits a (unique) solution v_λ. Here, $\mathcal{A}_\lambda := \frac{1}{\lambda}(I_{L^2_{\underline{s}}(H)} - J^{\mathcal{A}}_\lambda)$ denotes the Yosida approximation of $\mathcal{A}$ in $L^2_{\underline{s}}(H)$. After establishing *a priori* estimates on $(v_\lambda)_{\lambda>0}$, one can conclude that for every $\delta > 0$, there is a subsequence of $(v_\lambda)_{\lambda>0}$ converging to a (unique) solution v_δ of

$$\mathcal{A}v_\delta + \delta v_\delta + \partial_{L^2_{\underline{s}}(H)}\mathcal{E}(v_\delta) \ni 0 \qquad \text{in } L^2_{\underline{s}}(H). \tag{3.26}$$

After establishing *a priori* estimates on $(v_\delta)_{\delta\in(0,1]}$, one shows that there is a subsequence of $(v_\delta)_{\delta\in(0,1]}$ converging to a solution v of (3.22) satisfying (3.23). This method generalizes an idea by Brezis [Bre72] to the general fractional power case $0 < s < 1$.

Proof of Theorem 3.6 (existence). We begin by taking $\varphi \in D(A)$. By hypothesis, A is a maximal monotone operator on H satisfying $0_H \in \mathrm{Rg}(A)$. For simplicity, we may assume without loss of generality that $0 \in A0$, otherwise we replace A by $\tilde{A} := A(\cdot + y)$ for some $y \in A^{-1}(\{0\})$. Then, the corresponding operator $\mathcal{A}$ on $L^2_{\underline{s}}(H)$ given by (3.24) is maximal monotone (cf [Bre73]). Moreover, the Yosida approximation $\mathcal{A}_\lambda$ of $\mathcal{A}$ is a maximal monotone and Lipschitz continuous mapping on $L^2_{\underline{s}}(H)$. Since $\partial_{L^2_{\underline{s}}}\mathcal{E}$ is also maximal monotone operators on $L^2_{\underline{s}}(H)$, [Bre73, Lemme 2.4] implies that for every λ and $\delta > 0$, problem (3.25) has a strong solution $v_\lambda \in L^2_{\underline{s}}(H)$. By Proposition 3.9,

$$v_\lambda \in W^{2,2}_{\frac{1-s}{2s},\frac{1}{2},\frac{3s-1}{2s}}(H) \cap C^1([0,+\infty); H).$$

In addition, since the Yosida approximation A_λ of A is Lipschitz continuous on H, we can conclude from (3.25) that $v_\lambda \in C^2((0,+\infty); H)$.

1. <u>*A priori* estimates on $(v_\lambda)_{\lambda>0}$.</u> The following estimates hold uniformly for all $\lambda > 0$ and $0 < \delta \le 1$:

$$\|v_\lambda(z)\|_H \le \|v_\lambda(\hat{z})\|_H \quad \text{for all } z \ge \hat{z} \ge 0, \tag{3.27}$$

$$\frac{1}{\sqrt{2}}\|v_\lambda(0)\|_H \le C, \tag{3.28}$$

$$\|v'_\lambda(0)\|_H^{\frac{1}{2}} \le \Big(\|A^0\varphi\|_H + \delta\,\|\varphi\|_H\Big)^{\frac{1}{2}} + \|\varphi\|_H^{\frac{1}{2}}, \tag{3.29}$$

$$\|z^{\frac{1}{2}}v'_\lambda\|_{L^2_\star(H)} \le \left(\Big(\|A^0\varphi\|_H + \delta\,\|\varphi\|_H\Big)^{\frac{1}{2}} + \|\varphi\|_H^2\right)\|\varphi\|_H^{\frac{1}{2}}, \tag{3.30}$$

$$\|v''_\lambda\|_{L^2_{\overline{s}}(H)} \le \Big(\|A^0\varphi\|_H + \delta\,\|\varphi\|_H\Big) + \|\varphi\|_H^{\frac{1}{2}}\Big(\|A^0\varphi\|_H + \delta\,\|\varphi\|_H\Big)^{\frac{1}{2}}, \tag{3.31}$$

$$\|\mathcal{A}_\lambda v_\lambda\|_{L^2_{\underline{s}}(H)} \le \Big(\|A^0\varphi\|_H + \delta\,\|\varphi\|_H\Big) + \|\varphi\|_H^{\frac{1}{2}}\Big(\|A^0\varphi\|_H + \delta\,\|\varphi\|_H\Big)^{\frac{1}{2}}, \tag{3.32}$$

$$\|v'_\lambda(z)\|_H \le \|v'_\lambda(\hat{z})\|_H \quad \text{for all } z \ge \hat{z} \ge 0, \tag{3.33}$$

where C is a constant independent of λ. To show that these inequalities hold, we first multiply (3.25) by v_λ with respect to the $L^2_{\underline{s}}(H)$-inner product (2.2). Then,

$$(\mathcal{A}_\lambda v_\lambda, v_\lambda)_{L^2_{\underline{s}}(H)} + \delta\,\|v_\lambda\|^2_{L^2_{\underline{s}}(H)} + (\partial_{L^2_{\underline{s}}(H)}\mathcal{E}(v_\lambda), v_\lambda)_{L^2_{\underline{s}}(H)} = 0.$$

Since $\partial_{L^2_{\underline{s}}}\mathcal{E}(v_\lambda) = -z^{-\frac{1-2s}{s}} v_\lambda''$, the last equation is equivalent to

$$0 = (\mathcal{A}_\lambda v_\lambda, v_\lambda)_{L^2_{\underline{s}}(H)} + \delta\, \|v_\lambda\|^2_{L^2_{\underline{s}}(H)} - (z^{-\frac{1-2s}{s}} v_\lambda'', v_\lambda)_{L^2_{\underline{s}}(H)} \tag{3.34}$$

By Cauchy-Schwarz's inequality,

$$|(z^{-\frac{1-2s}{s}} v_\lambda'', v_\lambda)_{L^2_{\underline{s}}(H)}| \le \|z^{-\frac{1-2s}{s}} v_\lambda''\|_{L^2_{\underline{s}}(H)}\, \|v_\lambda\|_{L^2_{\underline{s}}(H)} = \|v_\lambda''\|_{L^2_{\overline{s}}}\, \|v_\lambda\|_{L^2_{\underline{s}}(H)},$$

and since

$$(z^{-\frac{1-2s}{s}} v_\lambda'', v_\lambda)_{L^2_{\underline{s}}(H)} = \int_0^\infty (v_\lambda''(z), v_\lambda(z))_H\,\mathrm{d}z = (v_\lambda'', v_\lambda)_{L^2(H)},$$

one has that

$$(v_\lambda'', v_\lambda)_H \in L^1(\mathbb{R}_+). \tag{3.35}$$

On the other hand, (3.25) is equivalent to

$$v_\lambda''(z) = z^{\frac{1-2s}{s}} A_\lambda v_\lambda(z) + z^{\frac{1-2s}{s}}\,\delta\, v_\lambda(z) \qquad \text{for every } z > 0. \tag{3.36}$$

Multiplying (3.36) by v_λ with respect to the H-inner product applying the monotonicity of A_λ and that $0 \in A_\lambda 0$, one sees that

$$\begin{aligned} (v_\lambda''(z), v_\lambda(z))_H &= z^{\frac{1-2s}{s}} (A_\lambda v_\lambda(z), v_\lambda(z))_H + z^{\frac{1-2s}{s}}\,\delta\,\|v_\lambda(z)\|_H^2 \\ &\ge z^{\frac{1-2s}{s}}\,\delta\,\|v_\lambda(z)\|_H^2 \end{aligned} \tag{3.37}$$

for every $z > 0$. Thus,

$$\int_z^\infty (v_\lambda''(r), v_\lambda(r))_H\,\mathrm{d}r \ge \delta \int_z^\infty \|v_\lambda(r)\|_H^2\, r^{\frac{1-2s}{s}}\,\mathrm{d}r \ge 0 \tag{3.38}$$

for every $z \ge 0$. Further, since

$$\frac{\mathrm{d}}{\mathrm{d}z}(v_\lambda'(z), v_\lambda(z))_H = (v_\lambda''(z), v_\lambda(z))_H + \|v_\lambda'(z)\|_H^2$$

for every $z > 0$ and since $v_\lambda \in D(\mathcal{E})$ requires that $\|v_\lambda'\|_H^2 \in L^1(\mathbb{R}_+)$, it follows from (3.35) that the function $z \mapsto (v_\lambda'(z), v_\lambda(z))_H$ is continuous on $[0, +\infty)$ and by (3.38) that

$$-(v_\lambda'(z), v_\lambda(z))_H = \int_z^{+\infty} (v_\lambda''(r), v_\lambda(r))_H\,\mathrm{d}r + \int_z^{+\infty} \|v_\lambda'(r)\|_H^2\,\mathrm{d}r \ge 0 \tag{3.39}$$

for every $z \ge 0$. Therefore,

$$(v_\lambda'(z), v_\lambda(z))_H \le 0 \qquad \text{for every } z \ge 0 \tag{3.40}$$

and since

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{2}\|v_\lambda(z)\|_H^2 = (v_\lambda'(z), v_\lambda(z))_H,$$

the function $z \mapsto \frac{1}{2}\|v_\lambda(z)\|_H^2$ is decreasing on $[0, +\infty)$, implying that (3.27) holds. Next, by (3.37), one has that

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\frac{1}{2}\|v_\lambda(z)\|_H^2 = (v_\lambda''(z), v_\lambda(z))_H + \|v_\lambda'(z)\|_H^2 \ge \|v_\lambda'(z)\|_H^2 \ge 0$$

for all $z > 0$. Hence, the function $z \mapsto \|v_\lambda(z)\|_H^2$ is convex on $[0, +\infty)$. Taking $z = 0$ in (3.39) and applying (3.38), one finds

$$(v_\lambda'(0), v_\lambda(0))_H + \int_0^{+\infty} \|v_\lambda'(z)\|_H^2\,\mathrm{d}z = -(v_\lambda'', v_\lambda)_{L^2(H)} \le 0.$$

Therefore, and since $v'_\lambda(0) \in \partial_H\tilde{j}(v_\lambda(0)-\varphi)$ and $\partial_H\tilde{j}$ is monotone with $0 \in \partial_H\tilde{j}(0)$, we get that

$$\begin{aligned}\|z^{\frac{1}{2}}v'_\lambda\|^2_{L^2_\star(H)} &= \int_0^{+\infty}\|v'_\lambda(z)\|^2_H\,\mathrm{d}z\\ &\le -(v'_\lambda(0),v_\lambda(0))_H\\ &= -(v'_\lambda(0)-0,(v_\lambda(0)-\varphi)-0)_H-(v'_\lambda(0),\varphi)_H\\ &\le 0-(v'_\lambda(0),\varphi)_H\end{aligned}$$

and so, by Cauchy-Schwarz's inequality, one obtains

$$\|z^{\frac{1}{2}}v'_\lambda\|_{L^2_\star(H)} \le \|v'_\lambda(0)\|_H^{\frac{1}{2}}\,\|\varphi\|_H^{\frac{1}{2}}. \tag{3.41}$$

By the Lipschitz continuity of $A_\lambda : H \to H$, the function

$$w_\lambda(z) := A_\lambda v_\lambda(z) + \delta v_\lambda(z)$$

is differentiable at almost everywhere $z \in \mathbb{R}_+$ with weak derivative

$$w'_\lambda(z) = \frac{\mathrm{d}}{\mathrm{d}z}A_\lambda(v_\lambda(z)) + \delta\, v'_\lambda(z) \qquad \text{for almost every } z \in \mathbb{R}_+,$$

where $\frac{\mathrm{d}}{\mathrm{d}z}A_\lambda(v_\lambda)$ is the weak derivative of $z \mapsto A_\lambda(v_\lambda(z))$. On the other hand, by (3.36),

$$w_\lambda(z) = z^{-\frac{1-2s}{s}}v''_\lambda(z).$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}z}(w_\lambda(z),v'_\lambda(z))_H = (w'_\lambda(z),v'_\lambda(z))_H + \|z^{-\frac{3s-1}{2s}}v''_\lambda(z)\|^2_H\frac{1}{z} \tag{3.42}$$

for almost every $z \in \mathbb{R}_+$. Since the Yosida approximation A_λ is Lipschitz continuous with constant $1/\lambda$ (cf [Bre73, Proposition 2.6]), one has that

$$\left\|\frac{\mathrm{d}}{\mathrm{d}z}A_\lambda(v_\lambda(z))\right\|_H \le \frac{1}{\lambda}\,\|v'_\lambda(z)\|_H \qquad \text{for a.e. } z \in \mathbb{R}_+.$$

Therefore and since $v_\lambda \in W^{2,2}_{\frac{1-s}{2s},\frac{1}{2},\frac{3s-1}{2s}}(H)$, (3.42) means that the function $z \mapsto (w_\lambda(z),v'_\lambda(z))_H$ is absolutely continuous on $[0,+\infty)$ and

$$-(w_\lambda(z),v'_\lambda(z))_H = \int_z^{+\infty}\frac{\mathrm{d}}{\mathrm{d}r}(w_\lambda(r),v'_\lambda(r))_H\,\mathrm{d}r \tag{3.43}$$

for every $z \ge 0$. Moreover, by the monotonicity of A_λ

$$\begin{aligned}&\left(\frac{\mathrm{d}}{\mathrm{d}z}A_\lambda(v_\lambda(z)),v'_\lambda(z)\right)_H\\ &\quad= \lim_{h\to 0}\left(\frac{A_\lambda(v_\lambda(z+h))-A_\lambda(v_\lambda(z))}{h},\frac{v_\lambda(z+h)-v_\lambda(z+h)}{h}\right)_H \ge 0\end{aligned}$$

for almost every $z \in \mathbb{R}_+$. Therefore, one has that

$$(w'_\lambda(z),v'_\lambda(z))_H = \left(\frac{\mathrm{d}}{\mathrm{d}z}A_\lambda(v_\lambda(z)),v'_\lambda(z)\right)_H + \delta\,\|v'_\lambda(z)\|^2_H \ge 0 \tag{3.44}$$

for almost every $z \in \mathbb{R}_+$. Applying this inequality to (3.42) and subsequently, inserting (3.42) into (3.43) yields

$$(w_\lambda(z),v'_\lambda(z))_H \le 0 \qquad \text{for every } z \ge 0.$$

From this, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{2}\|v_\lambda'(z)\|_H^2 = (v_\lambda''(z), v_\lambda'(z))_H = z^{\frac{1-2s}{s}}(w_\lambda(z), v_\lambda'(z))_H \le 0,$$

implying that $z \mapsto \|v_\lambda'(z)\|_H^2$ is non-increasing on $[0,+\infty)$ and, in particular, (3.33) holds. Moreover, applying (3.44) to (3.42), gives

$$\frac{\mathrm{d}}{\mathrm{d}z}(w_\lambda(z), v_\lambda'(z))_H \ge \|v_\lambda''(z)\|_H^2 z^{-\frac{1-2s}{s}} \qquad \text{for a.e. } z \in \mathbb{R}_+$$

and by integrating this inequality over $\mathbb{R}_+$, one finds that

$$\|v_\lambda''\|^2_{L^2_{\frac{2}{s}\star}(H)} \le -(w_\lambda(0), v_\lambda'(0))_H.$$

Since $w_\lambda = A_\lambda v_\lambda + \delta v_\lambda$ and by applying [Bre73, Proposition 4.7(iii)]) to $u_\lambda'(0) \in \partial\tilde{j}(u_\lambda(0)-\varphi)$, one sees that

$$\begin{aligned}\|v_\lambda''\|^2_{L^2_{\frac{2}{s}\star}(H)} &\le -(w_\lambda(0), v_\lambda'(0))_H\\ &= -(A_\lambda v_\lambda(0) + \delta v_\lambda(0), v_\lambda'(0))_H\\ &\le -(A_\lambda\varphi + \delta\varphi, v_\lambda'(0))_H,\end{aligned}$$

and if A^0 is the *minimal selection* of A, then

$$\|v_\lambda''\|_{L^2_{\frac{2}{s}}} \le \|v_\lambda'(0)\|_H^{1/2}\left(\|A^0\varphi\|_H + \delta\,\|\varphi\|_H\right)^{1/2}. \tag{3.45}$$

Next, given $x \in H$ satisfying $\|x\|_H \le 1$. By Lemma 3.8, there is a $\xi \in W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$ such that $\xi(0) = x$ in H. Then, by $v' \in W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)$, Cauchy-Schwarz's inequality gives

$$\begin{aligned}(v_\lambda'(0), x)_H &= -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}(v_\lambda'(r), \xi(r))_H \mathrm{d}r\\ &= -\int_0^\infty (v_\lambda''(r), \xi(r))_H \mathrm{d}r - \int_0^\infty (v_\lambda'(r), \xi'(r))_H \mathrm{d}r\\ &\le \|v_\lambda''\|_{L^2_{\frac{2}{s}}}\|\xi\|_{L^2_{\frac{s}{2}}(H)} + \|z^{\frac{1}{2}}v'\|_{L^2_\star(H)}\|z^{\frac{1}{2}}\xi'\|_{L^2_\star(H)}\\ &\le \left(\|v_\lambda''\|_{L^2_{\frac{2}{s}}} + \|z^{\frac{1}{2}}v'\|_{L^2_\star(H)}\right)\|\xi\|_{W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)}.\end{aligned}$$

Moreover, in the latter inequality, taking the infimum over all $\xi \in W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$ satisfying $\xi(0) = x$ and subsequently applying (3.15), it follows that

$$(v_\lambda'(0), x)_H \le \left(\|v_\lambda''\|_{L^2_{\frac{2}{s}}} + \|z^{\frac{1}{2}}v'\|_{L^2_\star(H)}\right) C\,\|x\|_H.$$

Now, taking the supremum over all $x \in H$ satisfying $\|x\|_H \le 1$, yields

$$\|v_\lambda'(0)\|_H \le C\left(\|v_\lambda''\|_{L^2_{\frac{2}{s}}} + \|z^{\frac{1}{2}}v'\|_{L^2_\star(H)}\right).$$

Applying (3.45) and (3.41) to this inequality, one obtains

$$\|v_\lambda'(0)\|_H \le \left(\|v_\lambda'(0)\|_H^{1/2}\left(\|A^0\varphi\|_H + \delta\,\|\varphi\|_H\right)^{1/2} + \|v_\lambda'(0)\|_H^{1/2}\,\|\varphi\|_H^{1/2}\right),$$

from where we can conclude that (3.29) holds. Now, inserting (3.29) into (3.45), one obtains (3.31), and inserting (3.29) into (3.41), one sees that (3.30) holds.

Next, by hypothesis, $\tilde{j}$ satisfies (2.1), which is equivalent to $(\partial_H\tilde{j})^{-1}$ maps bounded sets into bounded sets (cf [Bre73, Proposition 2.14]). Since each $v_\lambda(0) \in (\partial_H\tilde{j})^{-1}(v'_\lambda(0)) + \varphi$ and by (3.29), the sequence $(v'_\lambda(0))_{\lambda>0}$ is bounded, we have that there is a constant $C > 0$ such that (3.28) holds.

To see that *a priori* estimate (3.32) holds, we multiply (3.25) by $\mathcal{A}_\lambda v_\lambda$ with respect to the $L^2_{\underline{s}^\star}(H)$-inner product. Then, by the monotonicity of $\mathcal{A}_\lambda$ and since $\partial_{L^2_{\underline{s}}}\mathcal{E}(v_\lambda) = -z^{-\frac{1-2s}{s}}v''_\lambda$, one sees that

$$\begin{aligned}\|\mathcal{A}_\lambda v_\lambda\|^2_{L^2_{\underline{s}}(H)} &\le \|\mathcal{A}_\lambda v_\lambda\|^2_{L^2_{\underline{s}}(H)} + \delta\,(v_\lambda, \mathcal{A}_\lambda v_\lambda)_{L^2_{\underline{s}}(H)}\\ &= (z^{-\frac{1-2s}{s}}v''_\lambda, \mathcal{A}_\lambda v_\lambda)_{L^2_{\underline{s}}(H)}\\ &\le \|v''_\lambda\|_{L^2_{\underline{s}}(H)}\,\|\mathcal{A}_\lambda v_\lambda\|_{L^2_{\underline{s}}(H)}.\end{aligned}$$

Therefore,

$$\|\mathcal{A}_\lambda v_\lambda\| \le \|v''_\lambda\|_{L^2_{\underline{s}}(H)}$$

for all $\lambda > 0$ and so, by (3.31), one gets (3.32).

2. For every $\delta > 0$, there is a unique solution v_δ of (3.26) and $v_\lambda \to v_\delta$. To establish the existence of a solution v_δ of (3.26), we begin with the following convergence result.

Lemma 3.10. *For every $\delta > 0$, the sequence $(v_\lambda)_{\lambda>0}$ of solutions v_λ of (3.25), is a Cauchy sequence in $L^2_{\underline{s}}(H)$. In particular, there is a $v_\delta \in L^2_{\underline{s}}(H)$ such that*

$$\lim_{\lambda\to 0+} v_\lambda = v_\delta \qquad \text{in } L^2_{\underline{s}}(H). \tag{3.46}$$

Proof of Lemma 3.10. For λ, $\hat{\lambda} > 0$, let v_λ and $v_{\hat{\lambda}}$ be two solutions of (3.25). Then, multiplying

$$\delta(v_\lambda - v_{\hat{\lambda}}) = -(\mathcal{A}_\lambda v_\lambda - \mathcal{A}_{\hat{\lambda}}v_{\hat{\lambda}}) - (\partial_{L^2_{\underline{s}}}\mathcal{E}(v_\lambda) - \partial_{L^2_{\underline{s}}}\mathcal{E}(v_{\hat{\lambda}}))$$

by $v_\lambda - v_{\hat{\lambda}}$ with respect to the $L^2_{\underline{s}}(H)$-inner product and using that $\partial_{L^2_{\underline{s}}}\mathcal{E}$ is monotone, shows that

$$\delta\,\|v_\lambda - v_{\hat{\lambda}}\|^2_{L^2_{\underline{s}}(H)} \le -(\mathcal{A}_\lambda v_\lambda - \mathcal{A}_{\hat{\lambda}}v_{\hat{\lambda}}, v_\lambda - v_{\hat{\lambda}})_{L^2_{\underline{s}}(H)}.$$

We recall from [Bre73, p 28] that for the resolvent operator $J^{\mathcal{A}}_\lambda$ of $\mathcal{A}$, one has that $\mathcal{A}_\lambda u \in \mathcal{A}J^{\mathcal{A}}_\lambda u$. Thus, by the monotonicity of $\mathcal{A}$, one has that

$$\begin{aligned}\delta\,\|v_\lambda - v_{\hat{\lambda}}\|^2_{L^2_{\underline{s}}(H)} &\le -(\mathcal{A}_\lambda v_\lambda - \mathcal{A}_{\hat{\lambda}}v_{\hat{\lambda}}, v_\lambda - v_{\hat{\lambda}})_{L^2_{\underline{s}}(H)}\\ &= -(\mathcal{A}_\lambda v_\lambda - \mathcal{A}_{\hat{\lambda}}v_{\hat{\lambda}}, (v_\lambda - J^{\mathcal{A}}_\lambda v_\lambda) - (v_{\hat{\lambda}} - J^{\mathcal{A}}_{\hat{\lambda}}v_{\hat{\lambda}}))_{L^2_{\underline{s}}(H)}\\ &\qquad - (\mathcal{A}_\lambda v_\lambda - \mathcal{A}_{\hat{\lambda}}v_{\hat{\lambda}}, J^{\mathcal{A}}_\lambda v_\lambda - J^{\mathcal{A}}_{\hat{\lambda}}v_{\hat{\lambda}})_{L^2_{\underline{s}}(H)}\\ &\le -(\mathcal{A}_\lambda v_\lambda - \mathcal{A}_{\hat{\lambda}}v_{\hat{\lambda}}, \lambda\,\mathcal{A}_\lambda v_\lambda - \hat{\lambda}\,\mathcal{A}_{\hat{\lambda}}v_{\hat{\lambda}})_{L^2_{\underline{s}}(H)}\\ &= -\lambda\,\|\mathcal{A}_\lambda v_\lambda\|^2_{L^2_{\underline{s}}(H)} + (\lambda + \hat{\lambda})\,(\mathcal{A}_\lambda u_\lambda, \mathcal{A}_{\hat{\lambda}}v_{\hat{\lambda}})_{L^2_{\underline{s}}(H)}\\ &\qquad - \hat{\lambda}\,\|\mathcal{A}_{\hat{\lambda}}v_{\hat{\lambda}}\|^2_{L^2_{\underline{s}}(H)}.\end{aligned}$$

By Cauchy-Schwarz's and Young's inequality,

$$
\begin{aligned}
&(\lambda+\hat\lambda)\,(\mathcal{A}_\lambda v_\lambda,\mathcal{A}_{\hat\lambda}v_{\hat\lambda})_{L^2_{\underline{s}}(H)}\\
&\quad\le(\lambda+\hat\lambda)\,\|\mathcal{A}_\lambda v_\lambda\|_{L^2_{\underline{s}}(H)}\,\|\mathcal{A}_{\hat\lambda}v_{\hat\lambda}\|_{L^2_{\underline{s}}(H)}\\
&\quad\le\lambda\,\|\mathcal{A}_\lambda v_\lambda\|^2_{L^2_{\underline{s}}(H)}+\frac{\lambda}{4}\|\mathcal{A}_{\hat\lambda}v_{\hat\lambda}\|^2_{L^2_{\underline{s}}(H)}+\hat\lambda\,\|\mathcal{A}_{\hat\lambda}v_{\hat\lambda}\|^2_{L^2_{\underline{s}}(H)}+\frac{\hat\lambda}{4}\|\mathcal{A}_\lambda v_\lambda\|^2_{L^2_{\underline{s}}(H)}.
\end{aligned}
$$

Hence,

$$\delta\,\|v_\lambda-v_{\hat\lambda}\|^2_{L^2_{\underline{s}}(H)}\le\frac{\lambda}{4}\|\mathcal{A}_{\hat\lambda}v_{\hat\lambda}\|^2_{L^2_{\underline{s}}(H)}+\frac{\hat\lambda}{4}\|\mathcal{A}_\lambda v_\lambda\|^2_{L^2_{\underline{s}}(H)},$$

which by (3.32) shows that

$$\delta\,\|v_\lambda-v_{\hat\lambda}\|^2_{L^2_{\underline{s}}(H)}\le\frac{\lambda+\hat\lambda}{4}\left[\Big(\|A^0\varphi\|_H+\delta\,\|\varphi\|_H\Big)+\|\varphi\|_H^{\frac12}\Big(\|A^0\varphi\|_H+\delta\,\|\varphi\|_H\Big)^{\frac12}\right]^2.$$

Therefore, for every $\delta>0$, $(v_\lambda)_{\lambda>0}$ is a Cauchy sequence in $L^2_{\underline{s}}(H)$. This proves the claim of this lemma. □

Continuation of the proof of Theorem 3.2. By Lemma 3.10, there is a $v_\delta\in L^2_{\underline{s}}(H)$ such that (3.46) holds. Now, the *a priori* estimates (3.30) and (3.31) imply that $v_\delta\in W^{2,2}_{\frac{1-s}{2s},\frac12,\frac{3s-1}{2s}}(H)$ and after possibly passing to a subsequence of $(v_\lambda)_{\lambda>0}$, which we denote again by $(v_\lambda)_{\lambda>0}$, one has that

$$\lim_{\lambda\to0+}v'_\lambda=v'_\delta\qquad\text{weakly in }L^2(H),\tag{3.47}$$

$$\lim_{\lambda\to0+}v''_\lambda=v''_\delta\qquad\text{weakly in }L^2_{\underline{s}}(H).\tag{3.48}$$

Moreover, since $\partial_{L^2_{\underline{s}}}\mathcal{E}(v_\lambda)=-z^{-\frac{1-2s}{s}}v''_\lambda$ and by [Bre73, Proposition 2.5], the limits (3.46) and (3.48) imply that $v_\delta\in D(\partial_{L^2_{\underline{s}}}\mathcal{E})$ and $-z^{-\frac{1-2s}{s}}v''_\delta=\partial_{L^2_{\underline{s}}}\mathcal{E}(v_\delta)$. Next, by (3.32), there is a $\chi\in L^2_{\underline{s}}(H)$ and a subsequence of $(v_\lambda)_{\lambda>0}$, which we denote again by $(v_\lambda)_{\lambda>0}$ such that

$$\lim_{\lambda\to0+}\mathcal{A}_\lambda v_\lambda=\chi\qquad\text{weakly in }L^2_{\underline{s}}(H).\tag{3.49}$$

Moreover,

$$
\begin{aligned}
\lim_{\lambda\to0}(\mathcal{A}_\lambda v_\lambda,v_\lambda)_{L^2_{\underline{s}}(H)}&=\lim_{\lambda\to0}(\mathcal{A}_\lambda v_\lambda,v_\lambda-v_\delta)_{L^2_{\underline{s}}(H)}+\lim_{\lambda\to0}(\mathcal{A}_\lambda v_\lambda,v_\delta)_{L^2_{\underline{s}}(H)}\\
&=(\chi,v_\delta)_{L^2_{\underline{s}}(H)}
\end{aligned}
$$

Note, $J^{\mathcal{A}}_\lambda v_\delta\to v_\delta$ in $L^2_{\underline{s}}(H)$ as $\lambda\to0+$. Thus, and since

$$
\begin{aligned}
\|J^{\mathcal{A}}_\lambda v_\lambda-v_\delta\|_{L^2_{\underline{s}}(H)}&\le\|J^{\mathcal{A}}_\lambda v_\lambda-J^{\mathcal{A}}_\lambda v_\delta\|_{L^2_{\underline{s}}(H)}+\|J^{\mathcal{A}}_\lambda v_\delta-v_\delta\|_{L^2_{\underline{s}}(H)}\\
&\le\|v_\lambda-v_\delta\|_{L^2_{\underline{s}}(H)}+\|J^{\mathcal{A}}_\lambda v_\delta-v_\delta\|_{L^2_{\underline{s}}(H)},
\end{aligned}
$$

one has that

$$\lim_{\lambda\to0+}J^{\mathcal{A}}_\lambda v_\lambda=v_\delta\qquad\text{in }L^2_{\underline{s}}(H).$$

Therefore and since $\mathcal{A}_\lambda v_\lambda\in\mathcal{A}J^{\mathcal{A}}_\lambda v_\lambda$, [Bre73, Proposition 2.5] implies that $v_\delta\in D(\mathcal{A})$ and $\chi\in\mathcal{A}v_\delta$. Now, by (3.46), (3.48), and (3.49), taking the $L^2_{\underline{s}}(H)$-weak limit in (3.25) yields that for every $\delta>0$, v_δ is a solution of (3.26), which by Proposition 3.9 has the regularity

$$v_\delta \in W^{2,2}_{\frac{1-s}{2s},\frac{1}{2},\frac{3s-1}{2s}}(H) \cap C^1([0,+\infty);H) \cap W^{2,2}_{loc}((0,+\infty);H)$$

Now, let $x \in H$ and for $\rho \in C^\infty([0,+\infty))$ satisfying $0 \le \rho \le 1$ on $[0,+\infty)$, $\rho \equiv 1$ on $[0,1]$ and $\rho \equiv 0$ on $[2,+\infty)$, set $\xi(z) = \rho(z)\,x$ for every $z \ge 0$. By (3.46), (3.27) and (3.28), one has that

$$\lim_{\lambda\to 0+} v_\lambda = v_\delta \qquad \text{in } L^2(0,T;H), \text{ for every } T > 0. \tag{3.50}$$

Thus, by (3.47) and since v_δ and v belong to $C^1([0,+\infty);H)$, one has that

$$\begin{aligned}(v_\lambda(0),x)_H &= -\int_0^{+\infty} \frac{\mathrm{d}}{\mathrm{d}z}(v_\delta(z),\xi(z))_H\,\mathrm{d}z\\ &= -\int_0^{+\infty}(v_\delta'(z),\xi(z))_H\,\mathrm{d}z - \int_0^{+\infty}(v_\delta(z),\xi'(z))_H\,\mathrm{d}z\\ &\to -\int_0^{+\infty}(v'(z),\xi(z))_H\,\mathrm{d}z - \int_0^{+\infty}(v(z),\xi'(z))_H\,\mathrm{d}z = (v(0),x)_H\end{aligned}$$

as $\lambda \to 0+$. Since $x \in H$ was arbitrary, this means that $v_\lambda(0) \rightharpoonup v_\delta(0)$ weakly in H as $\lambda \to 0+$ and hence,

$$v_\lambda(z) = v_\lambda(0) + \int_0^z v_\lambda'(r)\,\mathrm{d}r \rightharpoonup v_\delta(0) + \int_0^z v_\delta'(r)\,\mathrm{d}r = v_\delta(z)$$

weakly in H as $\lambda \to 0+$ for every $z > 0$. In addition, by (3.50) and (3.47), one has that

$$\begin{aligned}\frac{1}{2}\|v_\lambda(0)\|_H^2 &= -\int_0^2 \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{2}\|v_\lambda(z)\rho(z)\|_H^2\,\mathrm{d}z\\ &= -\int_0^2 (v_\lambda'(z),v_\lambda(z))_H\,\rho^2(z)\,\mathrm{d}z - \int_0^2 \|v_\lambda(z)\|_H^2\,\rho'(z)\rho(z)\,\mathrm{d}z\\ &\to -\int_0^2 (v_\delta'(z),v_\delta(z))_H\,\rho^2(z)\,\mathrm{d}z - \int_0^2 \|v_\delta(z)\|_H^2\,\rho'(z)\rho(z)\,\mathrm{d}z\\ &= \frac{1}{2}\|v_\delta(0)\|_H^2\end{aligned}$$

as $\lambda \to 0+$. Therefore, by the weak limit $v_\lambda(0) \rightharpoonup v_\delta(0)$ in and since H is a Hilbert space, we have that

$$\lim_{\lambda\to 0+} v_\lambda(0) = v_\delta(0) \qquad \text{in } H.$$

By this, (3.50), and (3.47), one has that

$$\begin{aligned}\frac{1}{2}\|v_\lambda(z)\|_H^2 &= \frac{1}{2}\|v_\lambda(0)\|_H^2 + \int_0^z \frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{2}\|v_\lambda(r)\|_H^2\,\mathrm{d}r\\ &= \frac{1}{2}\|v_\lambda(0)\|_H^2 + \int_0^z (v_\lambda'(r),v_\lambda(r))_H\,\mathrm{d}r\\ &\to \frac{1}{2}\|v_\lambda(0)\|_H^2 + \int_0^z (v_\lambda'(r),v_\lambda(r))_H\,\mathrm{d}r = \frac{1}{2}\|v_\delta(z)\|_H^2.\end{aligned}$$

Therefore, and since H is a Hilbert space, we have that

$$\lim_{\lambda\to 0+} v_\lambda(z) = v_\delta(z) \qquad \text{in } H \text{ for every } z \ge 0. \tag{3.51}$$

3. *A priori*-estimates on $(v_\delta)_{\delta>0}$. One has that the following inequalities hold for all $0<\delta\le 1$:

$$\|z\, v_\delta'\|_{L^2_\star(H)} \le \frac{\|v_\delta(0)\|_H^2}{2} \tag{3.52}$$

$$\|v_\delta(z)\|_H \le \|v_\delta(\hat z)\|_H \le \|v_\delta(0)\|_H \le C \quad (\text{for all } z\ge \hat z\ge 0,) \tag{3.53}$$

$$\|v_\delta'(0)\|_H^{\frac{1}{2}} \le \Big(\|A^0\varphi\|_H + \|\varphi\|_H\Big)^{\frac{1}{2}} + \|\varphi\|_H^{\frac{1}{2}}, \tag{3.54}$$

$$\|z^{\frac{1}{2}} v_\delta'\|_{L^2_\star(H)} \le \Big(\Big(\|A^0\varphi\|_H + \delta\,\|\varphi\|_H\Big)^{\frac{1}{2}} + \|\varphi\|_H^{\frac{1}{2}}\Big)\,\|\varphi\|_H^{\frac{1}{2}}, \tag{3.55}$$

$$\|v_\delta''\|_{L^2_{\frac{3}{8}}} \le \Big(\|A^0\varphi\|_H + \delta\,\|\varphi\|_H\Big) + \|\varphi\|_H^{\frac{1}{2}}\Big(\|A^0\varphi\|_H + \delta\,\|\varphi\|_H\Big)^{\frac{1}{2}}, \tag{3.56}$$

$$\|v_\delta'(z)\|_H \le \|v_\delta'(\hat z)\|_H \quad \text{for all } z\ge\hat z>0, \tag{3.57}$$

$$(v_\delta'(z), v_\delta(z))_H \le 0 \qquad \text{for all } z\ge 0. \tag{3.58}$$

where the constant C is independent of δ.

Due to the limits (3.48), and (3.49), sending $\lambda\to 0+$ in (3.30) and (3.31) shows that (3.55) and (3.56) hold. Next, inequality (3.57) follows from (3.33) and by the limit

$$\lim_{\lambda\to 0+} v_\lambda'(z) = v_\delta'(z) \qquad \text{strongly in } H \text{ for every } z>0. \tag{3.59}$$

The convergence (3.59) follows by the same reasoning as shown to prove (3.65) in Lemma 3.11 below. To avoid repetitive arguments, we outline this method only once. By the two limits (3.59) and (3.51), sending $\lambda\to 0+$ in (3.40) yields (3.58) for all $z>0$ and by continuity of v_δ and v_δ', one has that (3.58) also holds for $z=0$. To see that (3.52) holds, we note that by (3.26) and since A is monotone,

$$\int_0^T z\,(v_\delta''(z), v_\delta(z))_H\,\mathrm{d}z \ge 0$$

for every $T>0$. By this estimate, one sees that

$$\begin{aligned}
T(v_\delta'(T), v_\delta(T))_H &= \int_0^T \frac{\mathrm{d}}{\mathrm{d}z}\Big(z\,(v_\delta'(z), v_\delta(z))_H\Big)\,\mathrm{d}z\\
&= \int_0^T (v_\delta'(z), v_\delta(z))_H\,\mathrm{d}z + \int_0^T z\,(v_\delta''(z), v_\delta(z))_H\,\mathrm{d}z\\
&\qquad + \int_0^T z\,\|v_\delta'(z)\|_H^2\,\mathrm{d}z\\
&\ge \int_0^T \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{2}\|v_\delta(z)\|_H^2\,\mathrm{d}z + \int_0^T z\,\|v_\delta'(z)\|_H^2\,\mathrm{d}z\\
&= \frac{1}{2}\|v_\delta(T)\|_H^2 - \frac{1}{2}\|v_\delta(0)\|_H^2 + \int_0^T z\|v_\delta'(z)\|_H^2\,\mathrm{d}z\\
&\ge -\frac{1}{2}\|v_\delta(0)\|_H^2 + \int_0^T z\|v_\delta'(z)\|_H^2\,\mathrm{d}z
\end{aligned}$$

Rearranging this inequality and applying by (3.58), one gets

$$\int_0^T z\|v_\delta'(z)\|_H^2\,\mathrm{d}z \le T(v_\delta'(T), v_\delta(T))_H + \frac{1}{2}\|v_\delta(0)\|_H^2 \le \frac{1}{2}\|v_\delta(0)\|_H^2.$$

Sending $T \to +\infty$ in this estimate shows that (3.52) holds.

By (3.29), we can extract another subsequence of $(v_\lambda)_{\lambda>0}$ such that $v'_\lambda(0) \rightharpoonup v'_\delta(0)$ in H and so, sending $\lambda \to 0+$ in (3.29), one finds that (3.54) holds. By (3.54) and since $v_\delta(0) \in (\partial_H \tilde{j})^{-1}(v'_\delta(0)) + \varphi$, we can conclude that the sequence $(v_\delta(0))_{\delta>0}$ is bounded in H, showing that the right hand side inequality in (3.53) holds. Since $-z^{-\frac{1-2s}{s}} v''_\delta = \partial_{L^2_{\underline{s}}} \mathcal{E}(v_\delta)$ and A is monotone with $0 \in A0$, multiplying (3.26) by $v_\delta(z)$ gives

$$z^{-\frac{1-2s}{s}} (v''_\delta(z), v_\delta(z))_H = \delta \, \|v_\delta(z)\|^2_H + (w_\delta(z), v_\delta(z))_H \geq 0$$

for almost every $z > 0$, where $w_\delta(z) \in Av_\delta(z)$ satisfies $w_\delta(z) + \delta v_\delta(z) = z^{-\frac{1-2s}{s}} v''_\delta(z)$. Therefore,

$$(v''_\delta(z), v_\delta(z))_H \geq 0 \qquad \text{for almost every } z > 0. \tag{3.60}$$

Since $v_\delta \in W^{2,2}_{\frac{1-s}{2s},\frac{1}{2},\frac{3s-1}{2s}}(H)$, the integration by parts rule in Lemma 3.8 yields that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{2}\|v_\delta(z)\|^2_H &= (v'_\delta(z), v_\delta(z))_H \\
&= -\int_z^{+\infty} \frac{\mathrm{d}}{\mathrm{d}r}(v'_\delta(r), v_\delta(r))_H \,\mathrm{d}r \\
&= -\int_z^{+\infty} (v''_\delta(r), v_\delta(r))_H \,\mathrm{d}s - \int_z^{+\infty} \|v'_\delta(r)\|^2_H \,\mathrm{d}r
\end{aligned}$$

for every $z \geq 0$. Thus and by (3.60), $z \mapsto \|v_\delta(z)\|^2_H$ is decreasing on $[0, +\infty)$ and, in particular, the first two inequalities in (3.53) hold.

4. There is a function $v \in C^1([0, +\infty); H)$ such that $v'_\delta \to v'$ as $\delta \to 0+$. We need the following convergence results.

Lemma 3.11. *There is a function $v \in C^1([0, +\infty); H) \cap L^\infty(H)$ with $v' \in W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)$ such that after passing to a subsequence, one has that*

$$\lim_{\delta\to 0+} v_\delta(z) = v(z) \qquad \textit{weakly } H \textit{ for every } z \geq 0, \tag{3.61}$$

$$\lim_{\delta\to 0+} v_\delta = v \qquad \textit{weakly in } L^2_{loc}(H), \tag{3.62}$$

$$\lim_{\delta\to 0+} v'_\delta = v' \qquad \textit{strongly in } L^2(H), \tag{3.63}$$

$$\lim_{\delta\to 0+} v'_\delta(0) = v'(0) \qquad \textit{weakly in } H, \tag{3.64}$$

$$\lim_{\delta\to 0+} v'_\delta(z) = v'(z) \qquad \textit{strongly in } H \textit{ for every } z > 0, \textit{ and} \tag{3.65}$$

$$\lim_{\delta\to 0+} v''_\delta = v'' \qquad \textit{weakly in } L^2_{\underline{s}^\star}(H). \tag{3.66}$$

Before proceeding with *step 4*, we give the proof of this lemma.

Proof of Lemma 3.11. For δ, $\hat{\delta} \in (0, 1]$, let v_δ and $v_{\hat{\delta}}$ be two strong solutions of (3.26) and set $w = v_\delta - v_{\hat{\delta}}$. Then, by (3.26),

$$z^{-\frac{1-2s}{s}} w''(z) \in Av_\delta(z) - Av_{\hat{\delta}} + \delta v_\delta(z) - \hat{\delta} v_{\hat{\delta}}(z) \qquad \text{for almost every } z > 0$$

and hence, the monotonicity of A implies that

$$\begin{aligned} & z^{-\frac{1-2s}{s}}(w''(z),w(z))_H \\ &\quad \geq (\delta v_\delta(z)-\hat{\delta}v_{\hat{\delta}}(z),v_\delta(z)-v_{\hat{\delta}}(z))_H \\ &\quad = \delta\|v_\delta(z)\|_H^2-\delta\,(v_\delta,v_{\hat{\delta}}(z))_H-\hat{\delta}\,(v_\delta(z),v_{\hat{\delta}}(z))_H+\hat{\delta}\,\|v_\delta\|_H^2. \end{aligned}$$

By Young's inequality,

$$\delta\,(v_\delta,v_{\hat{\delta}})_H \leq \delta\|v_\delta\|_H^2+\frac{\delta}{4}\|v_{\hat{\delta}}\|_H^2 \quad \text{and} \quad \hat{\delta}\,(v_\delta,v_{\hat{\delta}})_H \leq \hat{\delta}\,\|v_{\hat{\delta}}\|_H^2+\frac{\hat{\delta}}{4}\|v_\delta\|_H^2.$$

Therefore and by (3.53),

$$z^{-\frac{1-2s}{s}}(w''(z),w(z))_H \geq -\frac{\delta}{4}\|v_{\hat{\delta}}(z)\|_H^2-\frac{\hat{\delta}}{4}\|v_\delta(z)\|_H^2 \geq -\frac{\delta+\hat{\delta}}{4}\,C^2.$$

Integrating this inequality over $[0,T]$, for $T>0$, gives

$$\begin{aligned} & (w'(T),w(T))_H-(w'(0),w(0))_H-\int_0^T\|w'(z)\|_H^2\,\mathrm{d}z \\ &\quad = \int_0^T\frac{\mathrm{d}}{\mathrm{d}z}(w'(z),w(z))_H\,\mathrm{d}z-\int_0^T\|w'(z)\|_H^2\,\mathrm{d}z \\ &\quad = \int_0^T\Big((w''(z),w(z))_H+\|w'(z)\|_H^2\Big)\,\mathrm{d}z-\int_0^T\|w'(z)\|_H^2\,\mathrm{d}z \\ &\quad \geq -\frac{(\gamma+\delta)}{4}C^2\int_0^T z^{\frac{1-2s}{s}}\,\mathrm{d}z. \end{aligned}$$

Since $v_\delta'(0)\in\partial_H\tilde{j}(v_\delta(0)-\varphi)$ and $v_{\hat{\delta}}'(0)\in\partial_H\tilde{j}(v_{\hat{\delta}}(0)-\varphi)$, the monotonicity of $\partial_H\tilde{j}$ implies that $(w'(0),w(0))_H\geq 0$ and so, we can conclude from the previous inequality that

$$\int_0^T\|w'(z)\|_H^2\,\mathrm{d}z \leq (w'(T),w(T))_H+\frac{(\gamma+\delta)}{4}C^2\int_0^T z^{\frac{1-2s}{s}}\,\mathrm{d}z.$$

Applying Cauchy-Schwarz's inequality and (3.53) to the right hand side of this inequality, gives

$$\int_0^T\|w'(z)\|_H^2\,\mathrm{d}z \leq C\|w'(T)\|_H+\frac{(\gamma+\delta)}{4}C^2\int_0^T z^{\frac{1-2s}{s}}\,\mathrm{d}z. \tag{3.67}$$

On the other hand, for every $\delta\in(0,1]$ and $T>0$,

$$T\int_T^{+\infty}\|v_\delta'(z)\|_H^2\,\mathrm{d}z \leq \int_T^{+\infty}\|z\,v_\delta'(z)\|_H^2\,\frac{\mathrm{d}z}{z}$$

and by (3.57), the function $z\mapsto\|v_\delta'(z)\|_H^2$ is decreasing on $(0,+\infty)$. Thus,

$$\int_0^T\|zv_\delta'(z)\|_H^2\,\frac{\mathrm{d}z}{z} \geq \frac{T^2}{2}\|v_\delta'(T)\|_H^2$$

and by (3.52) and (3.53),

$$\begin{aligned} & \frac{T^2}{2}\|v_\delta'(T)\|_H^2+T\int_T^{+\infty}\|v_\delta'(z)\|_H^2\,\mathrm{d}z \\ &\quad \leq \int_0^T\|zv_\delta'(z)\|_H^2\,\mathrm{d}z+\int_T^{+\infty}\|z\,v_\delta'(z)\|_H^2\,\frac{\mathrm{d}z}{z} = \|z\,v_\delta'(z)\|_{L^2_\star(H)} \leq \frac{C^2}{2}. \end{aligned}$$

Hence,

$$\int_T^{+\infty} \|v_\delta'(z)\|_H^2\,\mathrm{d}z \le \frac{C^2}{2T} \qquad \text{for every } \delta\in(0,1],$$

and

$$\|v_\delta'(T)\|_H \le \frac{C}{T} \qquad \text{for every } \delta\in(0,1].$$

Now, applying these estimates to (3.67). Then, we obtain

$$\begin{aligned}\int_0^T \|w'(z)\|_H^2\,\mathrm{d}z &\le C\|w'(T)\|_H + \frac{(\gamma+\delta)}{4}C^2\int_0^T z^{\frac{1-2s}{s}}\,\mathrm{d}z\\ &\le 2\frac{C^2}{T} + \frac{(\delta+\hat{\delta})}{4}C^2\frac{s}{1-s}T^{\frac{1-s}{s}}\end{aligned}$$

and so,

$$\begin{aligned}\int_0^{+\infty}\|w'(z)\|_H^2\,\mathrm{d}z &\le \int_0^T \|w'(z)\|_H^2\,\mathrm{d}z + 2\int_0^T \|v_\delta'(z)\|_H^2\,\mathrm{d}z + 2\int_0^T \|v_{\hat{\delta}}'(z)\|_H^2\,\mathrm{d}z\\ &\le 2\frac{C^2}{T} + \frac{(\delta+\hat{\delta})}{4}C^2\frac{s}{1-s}T^{\frac{1-s}{s}} + 4\frac{C^2}{2T}.\end{aligned}$$

Choosing $T := 1/(\delta+\hat{\delta})^s$ and inserting $w = v_\delta - v_{\hat{\delta}}$, then we get

$$\int_0^{+\infty}\|v_\delta'(z)-v_{\hat{\delta}}'(z)\|_H^2\,\mathrm{d}z \le \left(2C^2 + \frac{C^2}{4}\frac{s}{1-s} + 4\frac{C^2}{2}\right)(\delta+\hat{\delta})^s.$$

Therefore, $(v_\delta')_{\delta>0}$ is a Cauchy sequence in $L^2(H)$, implying that there is a $\hat{v}\in L^2(H)$ such that

$$\lim_{\delta\to 0+} v_\delta' = \hat{v} \qquad \text{in } L^2(H).$$

In addition, by (3.53), $(v_\delta)_{\delta>0}$ is bounded in $L^2(0,T;H)$ for every $T>0$. Therefore, there is a function $v\in L^2_{loc}(H)$ with weak derivative $v'=\hat{v}$ in $L^2(H)$ and after possibly passing to a subsequence of $(v_\delta)_{\delta>0}$, one has that (3.62) and (3.63) hold. Moreover, by (3.56), the sequence $(v_\delta'')_{\delta\in(0,1]}$ is bounded in $L^2_{\frac{1}{s}^\star}(H)$. Thus, v has a second weak derivative $v''\in L^2_{\frac{1}{s}^\star}(H)$ and by possibly replacing $(v_\delta)_{\delta\in(0,1]}$ again by a subsequence, one obtains (3.66). Further,

$$\|v(z)-v(\hat{z})\|_H \le \left|\int_{\hat{z}}^z \|v'(r)\|_H\,\mathrm{d}r\right| \le |z-\hat{z}|^{1/2}\,\|v'\|_{L^2(H)}$$

for every z, $\hat{z}\ge 0$, showing that $v:[0,+\infty)\to H$ is uniformly continuous. In particular, since $v'\in W^{1,2}_{\frac{1}{2},\frac{3s-1}{s}}(H)$, Lemma 3.8 and (3.53) imply that $v\in C^1([0,+\infty);H)\cap L^\infty(H)$. By the limits (3.63) and (3.66), the trace theorem on $W^{1,2}_{\frac{1}{2},\frac{3s-1}{s}}(H)$ (Lemma 3.8) implies that (3.64) holds. For $z>0$, Cauchy-Schwarz's inequality gives

$$\begin{aligned}z^{-\frac{1-2s}{2s}}\frac{1}{2}\|v_\delta'(z)-v'(z)\|_H^2 &= \int_z^{+\infty} r(r)^{-\frac{1-2s}{2s}}\,(v_\delta''(r)-v''(r), v_\delta'(r)-v'(r))_H\,\mathrm{d}r\\ &\qquad + \tfrac{2s-1}{2s}\int_z^{+\infty} r^{-\frac{1}{2s}}\frac{1}{2}\|v_\delta'(r)-v'(r)\|_H^2\,\mathrm{d}r\\ &\le \|z^{\frac{3s-1}{2s}}(v_\delta''-v'')\|_{L^2_\star(H)}\,\|v_\delta'-v'\|_{L^2(H)},\end{aligned}$$

$$
\begin{aligned}
&\qquad + \left|\tfrac{2s-1}{2s}\right| \tfrac{1}{2} z^{-\frac{1}{2s}} \, \|v_\delta' - v'\|^2_{L^2(H)}\\
&= \|v_\delta'' - v''\|_{L^2_{\frac{s}{s^\star}}(H)} \, \|v_\delta' - v'\|_{L^2(H)}\\
&\qquad + \left|\tfrac{2s-1}{2s}\right| \tfrac{1}{2} z^{-\frac{1}{2s}} \, \|v_\delta' - v'\|^2_{L^2(H)}.
\end{aligned}
$$

Thus, and by (3.63) and (3.56), one has that (3.65). By (3.53), there is a $v_0 \in H$ and a subsequence of $(v_\delta)_{\delta \in (0,1]}$ such that $v_\delta(0) \rightharpoonup v_0$ weakly in H as $\delta \to 0+$. Now, let $x \in H$ and for $\rho \in C^\infty([0,+\infty))$ satisfying $0 \le \rho \le 1$, $\rho \equiv 1$ on $[0,1]$ and $\rho \equiv 0$ on $[2,+\infty)$, set $\xi(z) = \rho(z)\, x$ for every $z \ge 0$. Since v_δ and v belong to $C^1([0,+\infty);H)$ and by the limits (3.62) and (3.65), one has that

$$
\begin{aligned}
(v_\delta(0), x)_H &= -\int_0^2 \frac{\mathrm{d}}{\mathrm{d}z}(v_\delta(z), \xi(z))_H \,\mathrm{d}z\\
&= -\int_0^2 (v_\delta'(z), \xi(z))_H \,\mathrm{d}z - \int_0^2 (v_\delta(z), \xi'(z))_H \,\mathrm{d}z\\
&\to -\int_0^{+\infty} (v'(z), \xi(z))_H \,\mathrm{d}z - \int_0^2 (v(z), \xi'(z))_H \,\mathrm{d}z\\
&= (v(0), x)_H
\end{aligned}
$$

as $\delta \to 0+$. Since $x \in H$ was arbitrary, this shows that $v_0 = v(0)$ and

$$
\lim_{\delta \to 0+} v_\delta(0) = v(0) \qquad \text{weakly in } H.
$$

Moreover, since for every $z > 0$,

$$
v_\delta(z) = v_\delta(0) + \int_0^z v_\delta'(r)\,\mathrm{d}r \qquad \text{and} \qquad v(z) = v(0) + \int_0^z v'(r)\,\mathrm{d}r,
$$

the previous limit together with (3.63) yields that (3.61) holds. □

Continuation of the proof of Theorem 3.2.

5. The limit v is a solution of (3.3). For a compact interval $K := [a,b]$ with $0 < a < b < +\infty$, let $L^2_K(H)$ denote the set of all L^2-integrable functions $v : K \to H$ and $\mathcal{A}_K$ denote the operator

$$
\mathcal{A}_K := \Big\{ (v,w) \in L^2_K(H) \times L^2_K(H) \,\Big|\, w(z) \in Av(z) \text{ for a.e. } z \ge 0 \Big\}.
$$

Then, since A is maximal monotone on H and K has finite measure, it follows that $\mathcal{A}_K$ is maximal monotone on $L^2_K(H)$. Moreover, since the restrictions on K of functions $v \in L^2_{\frac{s}{s^\star}}(H)$, $L^2(H)$, and $L^2_{\frac{s}{s^\star}}(H)$ belong to $L^2_K(H)$, the equation (3.26) can be rewritten as

$$
\mathcal{A}_K v_\delta + \delta v_\delta \ni z^{-\frac{1-2s}{s}} v_\delta'' \qquad \text{in } L^2_K(H).
$$

By (3.53) and (3.66), $f_\delta := z^{-\frac{1-2s}{s}} v_\delta'' - \delta\, v_\delta$ satisfies $f_\delta \in \mathcal{A}_K v_\delta$ and

$$
\lim_{\delta \to 0+} f_\delta = z^{-\frac{1-2s}{s}} v'' \text{ weakly in } L^2_K(H). \tag{3.68}
$$

By (3.62), if

$$
\lim_{\delta \to 0+} (v_\delta, f_\delta)_{L^2_K(H)} \le (v, z^{-\frac{1-2s}{s}} v'')_{L^2_K(H)}, \tag{3.69}
$$

then by [Bre73, Proposition 2.15], we have that

$$
v \in D(\mathcal{A}_K) \quad \text{and} \quad z^{-\frac{1-2s}{s}} v'' \in \mathcal{A}_K v.
$$

To see that (3.69) holds, we write

$$(v_\delta,f_\delta)_{L^2_K(H)}-(v,z^{-\frac{1-2s}{s}}v'')_{L^2_K(H)}=(v_\delta-v,f_\delta)_{L^2_K(H)}+(v,f_\delta-z^{-\frac{1-2s}{s}}v'')_{L^2_K(H)}$$

and note that by (3.68), one has that

$$(v,f_\delta-z^{-\frac{1-2s}{s}}v'')_{L^2_K(H)}=(v,f_\delta)_{L^2_K(H)}-(v,z^{-\frac{1-2s}{s}}v'')_{L^2_K(H)}\to 0$$

as $\delta\to 0+$. In addition, since $K=[a,b]$, and by the limits (3.61), (3.62), (3.63), (3.65) and since by (3.53), $\delta v_\delta\to 0$ strongly in $L^2_{loc}(H)$, we have that

$$\begin{aligned}
(v_\delta-v,f_\delta)_{L^2_K(H)}&=(v_\delta-v,z^{-\frac{1-2s}{s}}v''_\delta-\delta v_\delta)_{L^2_K(H)}\\
&=(v_\delta-v,z^{-\frac{1-2s}{s}}(v''_\delta-v''))_{L^2_K(H)}\\
&\qquad\qquad+(v_\delta-v,z^{-\frac{1-2s}{s}}v''-\delta v_\delta)_{L^2_K(H)}\\
&=\int_a^b(v_\delta(z)-v(z),v''_\delta(z)-v''(z))_H\,z^{-\frac{1-2s}{s}}\,\mathrm{d}z\\
&\qquad\qquad+(v_\delta-v,z^{-\frac{1-2s}{s}}v''-\delta v_\delta)_{L^2_K(H)}\\
&=\int_a^b\frac{\mathrm{d}}{\mathrm{d}z}(v_\delta(z)-v(z),v'_\delta(z)-v'(z))_H\,z^{-\frac{1-2s}{s}}\,\mathrm{d}z\\
&\qquad-\int_a^b\|v'_\delta(z)-v'(z)\|_H^2\,z^{-\frac{1-2s}{s}}\,\mathrm{d}z\\
&\qquad\qquad+(v_\delta-v,z^{-\frac{1-2s}{s}}v''-\delta v_\delta)_{L^2_K(H)}\\
&=(v_\delta(z)-v(z),v'_\delta(z)-v'(z))_H\,z^{-\frac{1-2s}{s}}\Big|_a^b\\
&\quad+\frac{1-2s}{s}\int_a^b(v_\delta(z)-v(z),v'_\delta(z)-v'(z))_H\,z^{-\frac{1-s}{s}}\,\mathrm{d}z\\
&\qquad-\int_a^b\|v'_\delta(z)-v'(z)\|_H^2\,z^{-\frac{1-2s}{s}}\,\mathrm{d}z\\
&\qquad\qquad+(v_\delta-v,z^{-\frac{1-2s}{s}}v''-\delta v_\delta)_{L^2_K(H)}\\
&\to 0\qquad\text{as }\delta\to 0+,
\end{aligned}$$

This show that (3.69) holds and since the compact sub-interval $K=[a,b]$ of $\mathbb{R}_+$ was arbitrary, we have thereby shown that v is a solution of (3.22) or equivalently, a strong solution of (1.3). It remains to show that $v'(0)\in\partial_H\tilde{j}(v(0)-\varphi)$. To see this, note first that if j is the indicator function, then condition (3.23) reduces to the condition $v(0)=\varphi$. Since $v_\delta(0)=\varphi$ for all $\delta>0$, we have by (3.61) that $v(0)=\varphi$. Now, suppose $\partial_H\tilde{j}:D(\partial_H\tilde{j})\to H$ is a weakly continuous mapping. Then, since $v'_\delta(0)=\partial_H\tilde{j}(v_\delta(0)-\varphi)$ for every $\delta\in(0,1]$, the weak continuity of $\partial_H\tilde{j}$ together with (3.61) and (3.64) imply that $v(0)-\varphi\in D(\partial_H\tilde{j})$ and $v'(0)=\partial_H\tilde{j}(v(0)-\varphi)$.

6. The solution v of (3.3) satisfies $v\in L^\infty(H)$ and (3.6)-(3.13). Thanks to (3.65), we can send $\delta\to 0+$ in (3.57) and obtain (3.7). Due to (3.65) and (3.61), sending $\delta\to 0+$ in (3.58) yields that

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{2}\|v(z)\|_H^2=(v'(z),v(z))_H\le 0\qquad\text{for all }z\ge 0.$$

Hence, (3.6) holds and, in particular, $v \in L^\infty(H)$. By (3.65), sending $\delta \to 0+$ in (3.57) and using that $v \in C^1([0,+\infty);H)$, one obtains (3.7). To see that (3.8) holds, we first note that by (3.65) and (3.61), sending $\delta \to 0+$ in (3.58) and using that $v \in C^1([0,+\infty);H)$ yields

$$(v'(z), v(z))_H \le 0 \qquad \text{for every } z \ge 0. \tag{3.70}$$

Moreover, since v is a strong solution of (1.3) and since A is monotone,

$$\int_\varepsilon^T z\,(v''(z), v(z))_H \,\mathrm{d}z \ge 0$$

for every $T > \varepsilon > 0$. Using this estimate, one sees that

$$\begin{aligned}
&T(v'(T), v(T))_H - \varepsilon(v'(\varepsilon), v(\varepsilon))_H\\
&\quad\ge \int_\varepsilon^T \frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{2}\|v(z)\|_H^2\,\mathrm{d}z + \int_\varepsilon^T z\,\|v'(z)\|_H^2\,\mathrm{d}z\\
&\quad= \frac{1}{2}\|v(T)\|_H^2 - \frac{1}{2}\|v(\varepsilon)\|_H^2 + \int_\varepsilon^T z\|v'(z)\|_H^2\,\mathrm{d}z\\
&\quad\ge -\frac{1}{2}\|v(\varepsilon)\|_H^2 + \int_\varepsilon^T z\|v'(z)\|_H^2\,\mathrm{d}z
\end{aligned}$$

Rearranging this inequality and applying (3.70) and (3.6), one gets

$$\begin{aligned}
\int_\varepsilon^T z\|v'(z)\|_H^2\,\mathrm{d}z &\le T(v'(T), v(T))_H - \varepsilon(v'(\varepsilon), v(\varepsilon))_H + \frac{1}{2}\|v(\varepsilon)\|_H^2\\
&\le -\varepsilon\,(v'(\varepsilon), v(\varepsilon))_H + \frac{1}{2}\|v(0)\|_H^2.
\end{aligned}$$

Since $v \in C^1([0,+\infty);H)$, sending first $\varepsilon \to 0+$ and then $T \to +\infty$ in the resulting inequality shows that (3.8) holds. By (3.7) and (3.8), we see that

$$\frac{z^2}{2}\|v'(z)\|_H^2 \le \int_0^z r\|v'(r)\|_H^2\,\mathrm{d}r \le \frac{1}{2}\|v(0)\|_H^2$$

for every $z > 0$, which shows that (3.9) holds. The estimates (3.11)-(3.13) are obtained from (3.54)-(3.56) by taking advantage of the underlying weak limit. Finally, we want to show that (3.10) holds. To do this, let $h > 0$. Then, by the monotonicity of A, one has that

$$\Big((z+h)^{\frac{1-2s}{s}} v''(z+h) - z^{\frac{1-2s}{s}} v''(z), v(z+h) - v(z)\Big)_H \ge 0$$

for almost every $z > 0$. Let $\xi \in C^2([0,+\infty))$ be such that

$$\psi(z) := \xi(z)(z+h)^{\frac{1-2s}{s}}$$

is increasing and satisfies $\psi(0) = 0$. (For example, take

$$\xi(z) = z^\beta (z+h)^{-\frac{1-2s}{s}} \qquad \text{for every } z \ge 0, \tag{3.71}$$

for some $\beta > 0$.) Then,

$$\begin{aligned}
&\int_0^z \psi(r)\,(v''(r+h) - v''(r), v(r+h) - v(r))_H\,\mathrm{d}r\\
&\qquad + \int_0^z \xi(r)\left((r+h)^{\frac{1-2s}{s}} - r^{\frac{1-2s}{s}}\right)(v''(r), v(r+h) - v(r))_H\,\mathrm{d}r \ge 0
\end{aligned}$$

and hence

$$
\begin{aligned}
&\psi(z)\,(v'(z+h)-v'(z),v(z+h)-v(z))_H \\
&\qquad -\int_0^z \psi'(r)\,(v'(r+h)-v'(r),v(r+h)-v(r))_H\,\mathrm{d}r \\
&\qquad\qquad -\int_0^z \psi(r)\|v'(r+h)-v'(r)\|_H^2\,\mathrm{d}r \\
&+\int_0^z \xi(r)\left((r+h)^{\frac{1-2s}{s}}-r^{\frac{1-2s}{s}}\right)(v''(r),v(r+h)-v(r))_H\,\mathrm{d}r \ge 0.
\end{aligned}
$$

Therefore, and since $(v'(z+h)-v'(z),v(z+h)-v(z))_H \le 0$,

$$
\begin{aligned}
&\int_0^z \psi(r)\|v'(r+h)-v'(r)\|_H^2\,\mathrm{d}r+\int_0^z \psi'(r)\,\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{2}\|v(r+h)-v(r)\|_H^2\,\mathrm{d}r \\
&\quad\le \psi(z)\,(v'(z+h)-v'(z),v(z+h)-v(z))_H \\
&\qquad +\int_0^z \xi(r)\left((r+h)^{\frac{1-2s}{s}}-r^{\frac{1-2s}{s}}\right)(v''(r),v(r+h)-v(r))_H\,\mathrm{d}r \\
&\quad\le \int_0^z \xi(r)\left((r+h)^{\frac{1-2s}{s}}-r^{\frac{1-2s}{s}}\right)(v''(r),v(r+h)-v(r))_H\,\mathrm{d}r.
\end{aligned}
$$

Now, integrating by parts, yields

$$
\begin{aligned}
&\int_0^z \psi(r)\|v'(r+h)-v'(r)\|_H^2\,\mathrm{d}r+\psi'(z)\,\frac{1}{2}\|v(z+h)-v(z)\|_H^2 \\
&\quad\le \int_0^z \xi(r)\left((r+h)^{\frac{1-2s}{s}}-r^{\frac{1-2s}{s}}\right)(v''(r),v(r+h)-v(r))_H\,\mathrm{d}r \\
&\qquad +\int_0^z \psi''(r)\,\frac{1}{2}\|v(r+h)-v(r)\|_H^2\,\mathrm{d}r+\psi'(0)\,\frac{1}{2}\|v(0+h)-v(0)\|_H^2.
\end{aligned}
$$

Dividing this inequality by h^2 and sending $h\to 0+$, yields

$$
\begin{aligned}
&\int_0^z \psi(r)\|v''(r)\|_H^2\,\mathrm{d}r+\psi'(z)\,\frac{1}{2}\|v'(z)\|_H^2 \\
&\quad\le \frac{1-2s}{s}\int_0^z \xi(r)\,r^{\frac{1-3s}{s}}\,(v''(r),v'(r))_H\,\mathrm{d}r \\
&\qquad +\int_0^z \psi''(r)\,\frac{1}{2}\|v'(r)\|_H^2\,\mathrm{d}r+\psi'(0)\,\frac{1}{2}\|v'(0)\|_H^2.
\end{aligned}
$$

Inserting ξ from (3.71), then

$$
\begin{aligned}
&\int_0^z r^{\beta}\|v''(r)\|_H^2\,\mathrm{d}r+\beta\,z^{\beta-1}\,\frac{1}{2}\|v'(z)\|_H^2 \\
&\quad\le \frac{1-2s}{s}\int_0^z r^{\beta-1}\,(v''(r),v'(r))_H\,\mathrm{d}r \\
&\qquad +\beta(\beta-1)\int_0^z r^{\beta-2}\,\frac{1}{2}\|v'(r)\|_H^2\,\mathrm{d}r+\beta\,z^{\beta-1}_{|z=0}\,\frac{1}{2}\|v'(\delta)\|_H^2.
\end{aligned}
$$

Choosing $\beta=3$ in this estimate, then one finds,

$$
\begin{aligned}
&\int_0^z r^{3}\|v''(r)\|_H^2\,\mathrm{d}r+3\,z^{2}\,\frac{1}{2}\|v'(z)\|_H^2 \\
&\quad\le \frac{1-2s}{s}\int_0^z r^{2}\,(v''(r),v'(r))_H\,\mathrm{d}r+3\int_0^z r\,\|v'(r)\|_H^2\,\mathrm{d}r.
\end{aligned}
$$

Thus, if $s \geq 1/2$, then by applying (3.7), one sees that

$$\int_0^z r^3 \|v''(r)\|_H^2 \, \mathrm{d}r \leq 3\, \frac{\|\varphi\|_H^2}{2}$$

and hence the first part of (3.10) holds by sending $z \to +\infty$. If $0 < s < 1/2$, then $\frac{1-2s}{s} > 0$ and so, by Young's inequality, we have for every $\varepsilon > 0$ that

$$\begin{aligned}\int_0^z & r^3 \|v''(r)\|_H^2 \, \mathrm{d}r + 3\, z^2\, \frac{1}{2} \|v'(z)\|_H^2 \\ &\leq \varepsilon \int_0^z \|r^{\frac{3}{2}} v''(r)\|_H^2 \, \mathrm{d}r + \frac{s}{1-2s} \frac{1}{4\varepsilon} \int_0^z \|r^{\frac{1}{2}} v'(r)\|_H^2 \, \mathrm{d}r + 3 \int_0^z r\, \|v'(r)\|_H^2 \, \mathrm{d}r.\end{aligned}$$

Choosing $\varepsilon = 1/2$, one finds

$$\frac{1}{2} \int_0^z r^3 \|v''(r)\|_H^2 \, \mathrm{d}r \leq \left(\frac{s}{1-2s} \frac{1}{2} + 3 \right) \int_0^z \|r^{\frac{1}{2}} v'(r)\|_H^2 \, \mathrm{d}r$$

and hence by applying (3.7) and sending $z \to +\infty$, one sees that the second part of (3.10) holds.

To see that for boundary data $\varphi \in \overline{D(A)}^H$, there is a solution v of Dirichlet problem (3.4) satisfying (3.6)-(3.10), and (3.14), one proceeds as in the proof of Theorem 1.2 and Theorem 1.7. This completes the proof of this theorem. □

Next, we outline the proof that the DtN operator Θ_{1-2s} is monotone and establish the characterization of the closure $\overline{\Theta}_s$.

Proof of Corollary 3.7. For given φ_1 and $\varphi_2 \in D(A)$, let v_1, $v_2 \in L^\infty(H)$ be two strong solutions of (3.3) respectively to boundary data φ_1 and φ_2. Then by the monotonicity of A,

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}z^2} \frac{1}{2} \|v_1(z) - v_2(z)\|_H^2 &= (v_1''(z) - v_2''(z), v_1(z) - v_2(z))_H \\ &\qquad + \|v_1'(z) - v_2'(z)\|_H^2 \geq 0.\end{aligned} \tag{3.72}$$

Thus, the map $z \mapsto \frac{1}{2}\|w(z)\|_H^2$ is convex on $[0, +\infty)$. Since $v_1 - v_2 \in L^\infty(H)$ and since every bounded convex function is necessarily decreasing, we have that $z \mapsto \frac{1}{2}\|w(z)\|_H^2$ is decreasing on $[0, +\infty)$. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{2} \|v_1(z) - v_2(z)\|_H^2 = (v_1'(z) - v_2'(z), v_1(z) - v_2(z))_H \leq 0$$

for every $z \geq \hat{z} \geq 0$, which shows that $\Theta_s \varphi := -(2s)^{1-2s} v'(0)$ is a monotone operator. It follows from the definition of Θ_s and by Theorem 3.6 that $D(A)$ is a subset of $D(\Theta_s)$. On the other hand, for every $\varphi \in D(\Theta_s)$, there is a solution $v \in C^1([0, +\infty); H)$ of Dirichlet problem (3.4) with $v(0) = \varphi$. Since $v(t) \in D(A)$ for a.e. $t > 0$, it follows that $\varphi \in \overline{D(A)}^H$, showing that $D(\Theta_s) \subseteq \overline{D(A)}^H$. Further, by the regularity $v \in C^1([0, +\infty); H)$ and the uniqueness of the solutions to Dirichlet problem (3.4), the operator Θ_s is a well-defined mapping from $D(\Theta_s)$ to H.

Next, we show that the closure $\overline{\Theta}_s$ of Θ_s in $H \times H_w$ coincides with the set

$$B := \left\{ (\varphi, w) \in H \times H \;\middle|\; \begin{array}{l} \exists\, (\varphi_n, w_n) \in \Theta_s \text{ s.t. } \lim\limits_{n \to +\infty} (\varphi_n, w_n) = (\varphi, w) \\ \text{ in } H \times H_w \text{ \& a strong solution } v \text{ of } (1.3) \\ \text{satisfying } (3.6) \text{ \& } v(0) = \varphi \text{ in } H. \end{array} \right\}$$

Obviously, the set B is cointained in $\overline{\Theta}_s$. So, let $(\varphi, w) \in \overline{\Theta}_s$. Then, there is a sequence $((\varphi_n, w_n))_{n\ge 1} \subseteq \Theta_s$ such that $(\varphi_n, w_n) \to (\varphi, w)$ in $H \times H_w$ as $n \to +\infty$. By definition of Θ_s, there are solutions $v_n \in C^1([0,+\infty); H)$ of Dirichlet problem (3.4) with $v_n(0) = \varphi_n$ and satisfying $\Theta_s v_n(0) = w_n$. Since every $\varphi_n \in \overline{D(A)}^H$, Theorem 3.6 yields that each v_n satisfies (3.6) and by (3.14), $(v_n)_{n\ge 0}$ is a Cauchy sequence in $C^b([0,+\infty); H)$. Hence, there is a $v \in C^b([0,+\infty); H)$ such that $v_n \to v$ in $C^b([0,+\infty); H)$ as $n \to +\infty$ and so, $v(0) = \varphi$. Since each $\varphi_n \in \overline{D(A)}^H$, we also have that $\varphi \in \overline{D(A)}^H$. Hence, Theorem 3.6 yields the existence of a unique strong solution v of (3.4), proving that $(\varphi, w) \in B$.

Now, for given $\lambda > 0$, the functional $j : H \to [0,+\infty)$ given by $j(v) = \frac{1}{2\lambda}\|v\|_H^2$, $(v \in H)$, is strictly coercive, continuously differentiable on H and its Fréchet derivative j' coincides with the subdifferential $\partial_H j$ given by $\partial_H j(v) = \frac{\mathrm{d}}{\mathrm{d}v} j(v) = \frac{1}{\lambda} v$, $(v \in H)$. Note, $\partial_H j$ is a bounded linear operator on H and hence, in particular, weakly continuous, and A is $\partial_H j$ monotone. Thus, by Theorem 3.6, for every $\varphi \in D(A)$, there is a unique solutions $v \in C^1([0,+\infty); H)$ of

$$\begin{cases} z^{-\frac{1-2s}{s}} v''(z) \in A(v(z)) & \text{for a.e. } z > 0, \\ -(2s)^{1-2s} v'(0) + \lambda v(0) = \varphi. \end{cases} \tag{3.73}$$

By definition of Θ_s, we have thereby shown that for every $\varphi \in D(A)$ and $\lambda > 0$, there is a $v(0) \in D(\Theta_s)$ satisfying

$$v(0) + \lambda \Theta_s v(0) = \varphi. \tag{3.74}$$

Thus, $D(A) \subseteq \mathrm{Rg}(I_H + \lambda\Theta_s)$ for every $\lambda > 0$. To see that also $\overline{D(A)}^H \subseteq \mathrm{Rg}(I_H + \lambda\overline{\Theta}_s)$, take $\varphi \in \overline{D(A)}^H$. Then, there is a sequence $(\varphi_n)_{n\ge 1} \subseteq D(A)$ such that $\varphi_n \to \varphi$ in H as $n \to +\infty$. Moreover, since $D(A) \subseteq \mathrm{Rg}(I_H + \lambda\Theta_s)$, for each φ_n, there is a unique solution $v_n \in C^1([0,+\infty); H)$ of Dirichlet problem (3.4) satisfying $\Theta_s v_n(0) + \lambda v_n(0) = \varphi_n$. Thus by the monotonicity of Θ_s, one has that

$$\begin{aligned} \|v_n(0) - v_m(0)\|_H^2 &= \frac{1}{\lambda}(v_n(0) - v_m(0), \varphi_n - \varphi_m)_H \\ &\qquad - \frac{1}{\lambda}(v_n(0) - v_m(0), \Theta_s v_n(0) - \Theta_s v_m(0))_H \\ &\le \frac{1}{\lambda}\|v_n(0) - v_m(0)\|_H \, \|\varphi_n - \varphi_m\|_H \end{aligned}$$

and so,

$$\|v_n(0) - v_m(0)\|_H \le \frac{1}{\lambda}\,\|\varphi_n - \varphi_m\|_H. \tag{3.75}$$

Since each $v_n(0) \in \overline{D(A)}^H$, v_n satisfies (3.6) and (3.14)holds. Combining this with (3.75), one sees that $(v_n)_{n\ge 0}$ is a Cauchy sequence in $C^b([0,+\infty); H)$ and so, there is a $v \in C^b([0,+\infty); H)$ such that $v_n \to v$ in $C^b([0,+\infty); H)$ as $n \to +\infty$. In particular, $v_n(0) \to v(0)$ in H as $n \to +\infty$. Thus, $v(0) \in \overline{D(A)}^H$ and so, Theorem 3.6 yields that $v \in C^1([0,+\infty); H)$ and v is a solution of Dirichlet problem (3.4). Moreover, we have that

$$w := \lim_{n\to+\infty} \Theta_s v_n(0) = \varphi - \lambda v(0) \qquad \text{exists in } H,$$

showing that $(v(0), w) \in \overline{\Theta}_s$ and $\overline{\Theta}_s v(0) + \lambda v(0) = \varphi$. This proves that $\overline{D(A)}^H \subseteq \mathrm{Rg}(I_H + \lambda\overline{\Theta}_s)$. Thus, if $\overline{D(A)}^H = H$, then $\overline{\Theta}_s$ is maximal monotone on H. This completes the proof of this corollary. □

4. Interpolation properties

Our first theorem of this section, allows us to establish interpolation properties and comparison principles of the semigroup $\{\tilde{T}_s(t)\}_{t\ge 0}$ generated by $-\overline{\Theta}_s$.

Theorem 4.1. *Let A be a maximal monotone operator on H with $0 \in Rg(A)$. Suppose, $\phi : H \to \mathbb{R}\cup\{+\infty\}$ is convex, proper and lower semicontinuous such that A is $\partial_H\phi$-monotone. Then for every for $0 < s < 1$, Θ_s is $\partial_H\phi$-monotone.*

In the case $H = L^2(\Sigma, \mu)$ the Lebesgue space of 2-integrable functions defined on a σ-finite measure space (Σ, μ), then Theorem 4.1 provides the following interpolation properties of the semigroup $\{\tilde{T}_r\}_{r\ge 0}$ generated by $-\overline{\Theta}_s$.

Corollary 4.2. *Let (Σ, μ) be a σ-finite measure space and A a maximal monotone operator on $L^2(\Sigma, \mu)$ with $0 \in Rg(A)$. If A is completely accretive, then for every $0 < s < 1$, the DtN operator Θ_s is completely accretive on $L^2(\Sigma, \mu)$. In particular, if the closure $\overline{\Theta}_s$ of Θ_s is maximal monotone, then the semigroup $\{\tilde{T}_s(t)\}_{t\ge 0}$ generated by $-\overline{\Theta}_s$ is L^ψ-contractive on $L^2(\Sigma, \mu)$ for any N-function psi, $\{\tilde{T}_s(t)\}_{t\ge 0}$ is L^1- and L^∞-contractive on $L^2(\Sigma, \mu)$, and order preserving.*

The proof of Corollary 4.2 follows immediately from Theorem 4.1 since if A is completely accretive on $L^2(\Sigma, \mu)$, then for every $\lambda > 0$, the resolvent J_λ^A of A is a complete contraction, which by Proposition 2.2 means that for every convex, lower semicontinous function $j : \mathbb{R} \to \overline{\mathbb{R}}_+$ satisfying $j(0) = 0$, A is $\partial_H\phi_j$-monotone for $\phi_j : L^2(\Sigma, \mu) \to \overline{\mathbb{R}}_+$ given by

$$\phi_j(u) = \begin{cases} \int_\Sigma j(u)\,\mathrm{d}\mu & \text{if } j(u) \in L^1(\Sigma; \mu), \\ +\infty & \text{if otherwise,} \end{cases}$$

for every $u \in L^2(\Sigma, \mu)$. In particular, for every N-function ψ and $\alpha > 0$, A is $\partial_H\phi_j$-monotone for $j = \psi(\frac{\cdot}{\alpha})$, and A is $\partial_H\phi_j$-monotone for $j = \psi(\frac{\max\{\cdot,0\}}{\alpha})$. Thus, by Theorem 4.1, the semigroup $\{\tilde{T}_s(t)\}_{t\ge 0}$ generated by $-\overline{\Theta}_s$ is L^ψ-contractive on $L^2(\Sigma, \mu)$ for any N-function *psi* and order preserving. Choosing $\psi = s^q$ for $q \in (1, +\infty)$. Letting $q \to 1$ or $q \to +\infty$ yields that $\{\tilde{T}_s(t)\}_{t\ge 0}$ is L^1- and L^∞-contractive on $L^2(\Sigma, \mu)$.

Remark 4.3 (Interpolation of $\{\tilde{T}_s(t)\}_{t\ge 0}$ on $L^\psi(\Sigma, \mu)$). By Corollary 4.2, if there is a $u_0 \in L^1 \cap L^\infty(\Sigma, \mu)$ such that the orbit $\tilde{T}_s(\cdot)u_0 := \{\tilde{T}_s(t)u_0 \,|\, t \ge 0\}$ is locally bounded on $[0, +\infty)$ with values in $L^1 \cap L^\infty(\Sigma, \mu)$, then for every N-function ψ, every $\tilde{T}_s(t)$ of the semigroup $\{\tilde{T}_s(t)\}_{t\ge 0}$ generated by $-\overline{\Theta}_s$ has unique contractive extension on $L^\psi(\Sigma, \mu)$, $L^1(\Sigma, \mu)$ and on $\overline{L^2 \cap L^\infty(\Sigma, \mu)}^{L^\infty}$, which we denote again by $\tilde{T}_s(t)$.

We continue by giving the proof of the $\partial_H\phi$-monotonicity of the DtN operator Θ_s.

Proof of Theorem 4.1. For $\mu > 0$, let $\phi_\mu : H \to \mathbb{R}$ be the regularization of ϕ defined by

$$\phi_\mu(v) = \min_{w\in H} \frac{1}{\mu 2}\|w - v\|_H^2 + \phi(w) \qquad \text{for all } v \in H.$$

By [Bre73, Proposition 2.11], $\phi_\mu \in C^1(H;\mathbb{R})$ and the Fréchet derivative $\phi'_\mu : H \to H$ is Lipschitz continuous. Thus, for every $w \in W^{1,1}_{loc}((0,+\infty);H)$, $\phi'_\mu(w(z))$ is weakly differentiable at almost every $z > 0$ and by the monotonicity of ϕ'_μ, one has that

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}z}\phi'_\mu(w(z)), w'(z)\right)_H \geq 0 \qquad \text{for a.e. } z > 0. \tag{4.1}$$

Now, let φ, $\hat{\varphi} \in D(\Theta_s)$ and v and $\hat{v}$ two solutions of Dirichlet problem (3.3) with initial value $v(0) = \varphi$ and $\hat{v}(0) = \hat{\varphi} \in D(A)$. Since v and $\hat{v}$ satisfy

$$z^{-\frac{1-2s}{s}} v''(z) \in Av(z) \qquad \text{and} \qquad z^{-\frac{1-2s}{s}} \hat{v}''(z) \in A\hat{v}(z)$$

for almost every $z > 0$ and since A is $\partial_H\phi$-monotone (cf [Bre73, Proposition 4.7]), one has that

$$z^{-\frac{1-2s}{s}}(v''(z) - \hat{v}''(z), \phi'_\mu(v(z) - \hat{v}(z)))_H \geq 0$$

for almost every $z > 0$. Thus, if we set $w = v - \hat{v}$, then

$$(w''(z), \phi'_\mu(w(z)))_H \geq 0 \qquad \text{for almost every } z > 0. \tag{4.2}$$

Moreover, since $v(0)$ and $\hat{v}(0) \in D(A)$, one has $w' \in W^{1,2}_{\frac{1}{2},\frac{3s-1}{2s}}(H)$ and $w \in W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$, and by the Lipschitz continuity of ϕ'_μ, $\phi'_\mu \circ w \in W^{1,2}_{\frac{1-s}{2s},\frac{1}{2}}(H)$. Therefore, we can apply the integration by parts rule (3.16) of Lemma 3.8. This together with (4.1) and (4.2), shows that

$$\begin{aligned} -(w'(0), \phi'_\mu(w(0)))_H = \int_0^{+\infty} &(w''(z), \phi'_\mu(w(z)))_H \,\mathrm{d}z \\ &+ \int_0^{+\infty} \left(w'(z), \frac{\mathrm{d}}{\mathrm{d}z}\phi'_\mu(w(z))\right)_H \mathrm{d}z \geq 0. \end{aligned}$$

Since $w = v - \hat{v}$ and φ, $\hat{\varphi} \in D(\Theta_s)$ were arbitrary, this implies by [Bre73, Proposition 4.7]) that Θ_s is $\partial_H\phi$-monotone. □

5. Applications

In this section, we outline an application of Theorem 1.2 - Theorem 1.7.

Let Σ be an open subset of $\mathbb{R}^d$, $(d \geq 1)$, and $\mu = \mathcal{L}^d$ be the d-dimensional Lebesgue measure. For $1 < p < +\infty$, suppose that $a : \Sigma \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying the following *p-coercivity*, *growth* and *monotonicity* conditions

$$a(x,\xi)\xi \geq \eta|\xi|^p \tag{5.1}$$

$$|a(x,\xi)| \leq c_1|\xi|^{p-1} + h(x) \tag{5.2}$$

$$(a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2) > 0 \tag{5.3}$$

for a.e. $x \in \Sigma$ and all ξ, ξ_1, $\xi_2 \in \mathbb{R}^d$ with $\xi_1 \neq \xi_2$, where $h \in L^{p'}(\Sigma)$ and c_1, $\eta > 0$ are constants independent of $x \in \Sigma$ and $\xi \in \mathbb{R}^d$. Under these assumptions, the second order quasi linear operator

$$\mathcal{B}u := -\operatorname{div}(a(x, \nabla u)) \qquad \text{in } \mathcal{D}'(\Sigma)$$

for $u \in W^{1,p}_{loc}(\Omega)$ belongs to the class of *Leray-Lions operators* (cf. [LL65]), of which the *p-Laplace operator* $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is a classical prototype. Here, we write $L^q(\Sigma)$ for $L^q(\Sigma, \mu)$ and if $L^q_0(\Sigma)$ is the closed subspace of $L^q(\Sigma)$ of all $u \in L^q(\Sigma)$ satisfying $\int_\Sigma u \, \mathrm{d}x = 0$.

Then, the operator $\mathcal{B}$ either equipped with homogeneous *Dirichlet boundary conditions*

$$u = 0 \quad \text{on } \partial\Sigma \times (0, \infty) \qquad \text{if } \Sigma \subseteq \mathbb{R}^d, \tag{5.4}$$

homogeneous *Neumann boundary conditions*

$$a(x, \nabla u) \cdot \nu = 0 \quad \text{on } \partial\Sigma \times (0, \infty) \qquad \text{if } |\Sigma| < \infty, \tag{5.5}$$

or, homogeneous *Robin boundary conditions*

$$a(x, \nabla u) \cdot \nu + b(x)|u|^{p-1}u = 0 \quad \text{on } \partial\Sigma \times (0, \infty) \quad \text{if } |\Sigma| < \infty \tag{5.6}$$

is an m-completely accretive, single-valued, operator A on $L^2(\Sigma)$ (respectively, on $L^2_0(\Sigma)$ if $\mathcal{B}$ is equipped with (5.5)) with dense domain $D(A)$ in $L^2(\Sigma)$ (respectively, in $L^2_0(\Sigma)$). In particular, A is a maximal monotone on $L^2(\Sigma)$ (respectively, in $L^2_0(\Sigma)$) satisfying $A0 = 0$.

Thus by Theorem 1.2, the following Dirichlet problem is well-posed. For every $\varphi \in L^2(\Sigma)$ (respectively, in $L^2_0(\Sigma)$) and $0 < s < 1$, there is a unique solution $u : \Sigma \times [0, +\infty) \to \mathbb{R}$ of

$$\begin{cases} -\frac{1-2s}{r}u_r - u_{rr} - \operatorname{div}_x(a(x, \nabla_x u)) = 0 & \text{for } (x, r) \in \Sigma \times \mathbb{R}_+, \\ u \text{ satisfies (5.4), ((5.5) for } \varphi \in L^2_0(\Sigma)), \text{ or (5.6)} & \text{for } (x, r) \in \partial\Sigma \times \mathbb{R}_+, \\ u(\cdot, 0) = \varphi(\cdot) & \text{on } \Sigma. \end{cases}$$

Further by Theorem 1.3, for every $0 < s < 1$, there is a strongly continuous semigroup $\{T_s(t)\}_{t\geq 0}$ generated by $-A^s$ on $L^2(\Sigma)$ (respectively, on $L^2_0(\Sigma)$). The semigroup $\{T_s\}_{t\geq 0}$ is order preserving, and each $T_s(t)$ has a unique contractive extension on $L^\psi(\Sigma)$ (respectively, on $L^\psi_0(\Sigma)$) for any N-function, on $L^1(\Sigma)$ and on $\overline{L^2 \cap L^\infty(\Sigma)}^{L^\infty}$. In particular, for every $\varphi \in L^2(\Sigma)$ (respectively, $\varphi \in L^2_0(\Sigma)$) and $0 < s < 1$, there is a unique solution $U : \Sigma \times [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ of boundary-value problem

$$\begin{cases} -\frac{1-2s}{r}U_r - U_{rr} - \operatorname{div}_x(a(x, \nabla_x U)) = 0 & \text{for } (x, r, t) \in \Sigma \times \mathbb{R}_+ \times \mathbb{R}_+, \\ U \text{ either satisfies (5.4), (5.5), or (5.6)} & \text{for } (x, r, t) \in \partial\Sigma \times \mathbb{R}_+ \times \mathbb{R}_+, \\ U(0, t) = T_s(t)\varphi & \text{on } \Sigma \text{ for every} t \geq 0, \\ \lim_{t\to 0+} t^{1-2s}U_r(r, t) \in \frac{\mathrm{d}}{\mathrm{d}t}T_s(t) & \text{on } \Sigma, \text{ for every } t > 0, \\ U(\cdot, 0, 0) = \varphi(\cdot) & \text{on } \Sigma. \end{cases}$$

Finally, due to Theorem 1.7, for every $\varphi \in D(A)$, $\lambda > 0$, and $0 < s < 1$, there is a unique solution $u : \Sigma \times [0, +\infty) \to \mathbb{R}$ of the Robin problem

$$\begin{cases} -\frac{1-2s}{r} u_r - u_{rr} - \mathrm{div}_x(a(x, \nabla_x u)) = 0 & \text{for } (x,r) \in \Sigma \times \mathbb{R}_+, \\ u \text{ either satisfies (5.4), (5.5), or (5.6)} & \text{for } (x,r) \in \partial\Sigma \times \mathbb{R}_+, \\ \lim_{r \to 0+} r^{1-2s} u_r(r) + \lambda u(\cdot, 0) = \varphi(\cdot) & \text{on } \Sigma. \end{cases}$$

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