Adams' formula revised

Richard Cowan
CSIRO Division of Mathematics and Statistics

February 10, 2015

This is the personal archive version of a paper that I published in 1984 — in the journal *Traffic Engineering and Control*, pages 272–274. Because of age, my only printed copy of the original paper has become illegible in places. So in 2015, in order to have a copy to send out when requested, I have constructed a modern LaTeX file with redrawn figures. An important typographical error in the 1984 paper has also been corrected in the new file.

My address of 2015 is: School of Mathematics and Statistics, University of Sydney, NSW, 2006, Australia. *e-mail: richard.cowan@sydney.edu.au*

Introduction

One of the oldest formulae in the traffic engineering literature is that derived by Adams [1]. He derived the expected delay to a pedestrian wishing to cross a traffic stream given that he needs a gap of at least β in the stream before he can cross. Adams used the Poisson process as a statistical model for the traffic stream. If the flow rate is q, then each headway of the Poisson process is exponentially distributed with mean $1/q$. Adams derived the average delay to pedestrians as[1]

$$\mathbb{E}(D) = \frac{e^{q\beta} - 1}{q} - \beta. \tag{1}$$

Observations of traffic have usually confirmed that the upper tail of headways is well fitted by an exponential distribution (see for example Buckley [2], Branston [3], Wasielewski [4] and Cowan [5]). The lower tail is clearly influenced by the safety headway (of around 2 seconds) that cars leave between each other. Furthermore, real traffic streams have a tendency to form bunches of vehicles with gaps between bunches. The Poisson process does not model these features, so one would expect Adams' formula to behave poorly at the higher flows (where bunching and minimum spacing constraints are significant).

One model which has been used extensively by the author in traffic modelling studies ([6]–[9]) encapsulates these realistic features whilst retaining most of the tractable mathematics that one obtains with Poisson process assumptions. In this paper a revised version of Adams' formula, based on this powerful model, is presented and a small comparative study given.

The traffic stream model

The model is presented in Fig 1. Behind each vehicle there is a headway of at least one time unit. Vehicles within a bunch have this minimum headway between each other. At the end of a bunch the headway exceeds one time unit by a random quantity X which is assumed to be exponentially distributed (in conformity with the empirical observation of exponential upper tail for headways). The number of vehicles per bunch is random, with any sensible probability distribution on the positive integers. Both the mean and variance of bunch size affect the results, so we denote these by μ and σ^2. The mean of X is denoted by g (for 'gap'). The flow rate q is linked to these parameters by

$$q = \frac{\mu}{\mu + g} \tag{2}$$

[1] This formula was mistyped in the original 1984 paper. Another good reason to have a 2015 revision of the paper! The figures and comparative remarks in the original paper were based on the correct Adams formula.

Any two of μ, g and q determine the third via (2). The two chosen, together with σ^2, can be considered for the moment as unrelated variables. Later we consider additional, empirically-based links between these four variables. Note that the time units are in terms of the minimal headway which will typically be around 2 seconds. Thus q will be between 0 and 1 in these units; $q = 1$ indicates capacity flow.

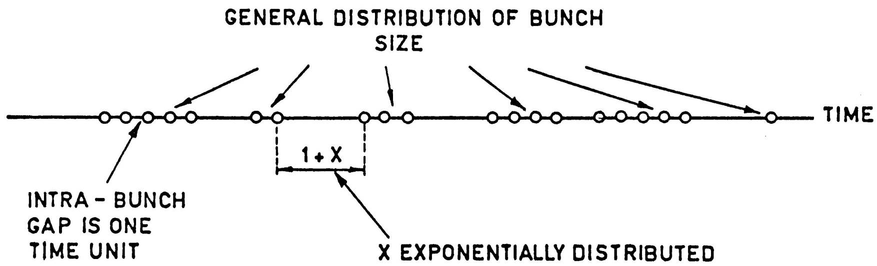

Figure 1: The stochastic arrival process for the traffic stream.

This stochastic model for a traffic stream has many interesting theoretical features. Two or more such streams merging into one generate a single stream with the same stochastic structure of bunches and exponentially distributed gaps. Enforcing a larger intra-bunch gap creates a new stream with similarly-preserved structure.

Feeding a single-lane road with such a stream and allowing vehicles to take differing speeds initially destroys the bunches. Eventually, however, bunches with exponentially distributed gaps reform. Thus there are some 'forces' prevalent in traffic which tend to preserve the stochastic structure of this simple model.

The new delay formula

A pedestrian arrives at an arbitrary time independent of the traffic stream. This epoch may be within a traffic gap whereupon the pedestrian compares the residual gap with β. If it is greater than β he crosses immediately with zero delay. If it is less than β he must wait until the next inter-bunch gap (precisely until one time unit after the last vehicle in the bunch) before making another gap comparison with β. Similarly, if he arrives within a bunch he must wait at least until the end of the bunch before comparing gaps. He will eventually cross at the beginning of a gap which is greater than β.

The new average delay formula, which is derived in the Appendix is given here in (3):

$$\mathbb{E}(D) = (e^{\beta/g} - 1)(\mu + g) - \beta + \tfrac{1}{2}q(\mu + \sigma^2/\mu). \tag{3}$$

Note that as flow rate $q \longrightarrow 0$, implying $g \longrightarrow \infty$, then $\mathbb{E}(D) \longrightarrow 0$. Also as capacity flow is approached ($q \longrightarrow 1$), then $g \longrightarrow 0$ and $\mathbb{E}(D) \longrightarrow \infty$. (Note that there is a sensible concept of capacity flow here, unlike with the Poisson process model). Representative curves are shown in Figs 2 and 3. They demonstrate that average delay is sensitive to the bunch gap structure of traffic. Importantly, the new results are sufficiently different from those given by Adams' original formula that a revision of practical traffic warrants involving pedestrian facilities seems desirable. Finally note that, since the three variables μ, g and q are linked by equation (2), equation (3) can be reorganised in a variety of forms.

Remarks and generalisations

1. The Appendix also shows other results concerning delay. In particular, the chance that a pedestrian is undelayed is $(1 - q)e^{-\beta/g}$. The Laplace transform of delay, which embodies full distributional information, is also presented.

2. The assumption that intra-bunch headways are constant is not one which seriously reduces the realism of the model, since in real traffic intra-bunch headway variance is small compared with inter-

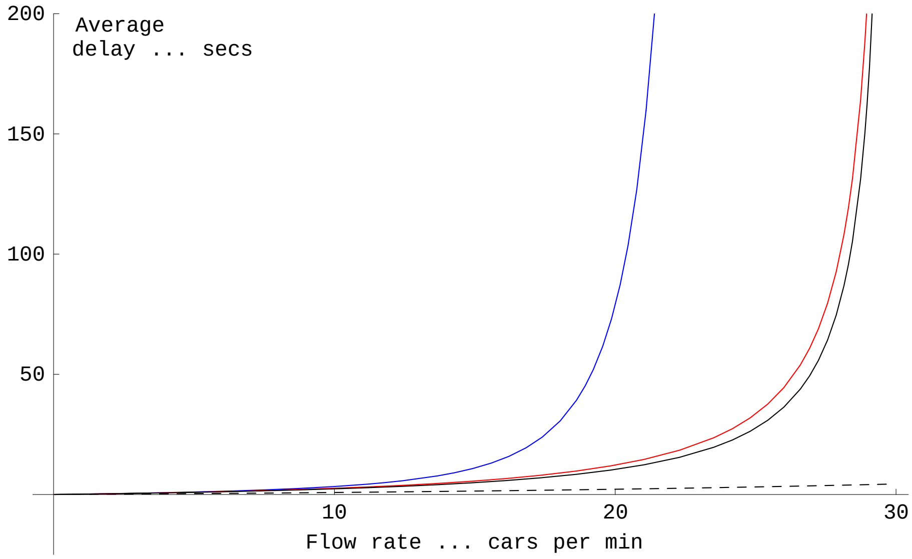

Figure 2: A comparison of average delay for three cases using the present model (solid curves) and Adams' original model (dashed curve). For the three solid curves geometrically distributed bunches have been assumed. The red and black curves incorporate realistic relationships between mean bunch size and flow rate (see Remarks 3 and 4). The blue curve has mean bunch size equal to one (and therefore all bunches have only one car.) The critical gap β has been chosen as 4 seconds compared with an assumed intra-bunch headway of 2 seconds, both realistic values.

bunch headway variance. It is possible, however, to relax this assumption without complicating the formulae unduly (see Appendix).

3. The probability distribution of bunch size is required in some studies of this type. The most sensible choices are the geometric distribution (sometimes mathematically convenient because the arrival process becomes a 'renewal' process) or the Borel distribution which has a theoretical traffic rationale ([5], [7], [10]–[12]). Our curves are based upon these choices to the extent that Fig 2 uses $\sigma^2 = \mu^2 - \mu$ (a property of the geometric distribution) whilst Fig 3 uses $\sigma^2 = \mu^3 - \mu^2$ (a property of the Borel distribution).

   We note that, under the assumption that bunches are geometrically distributed, our formula can be derived by the methods of Mayne [13] who generalised Adams' result for renewal processes.

4. Our model for arrivals does not necessarily constrain the mean bunch size μ to be functionally dependent upon flow rate q. Indeed, the blue curve in Figs 2 and 3 fixes μ to be 1 for all values of q. This is somewhat unrealistic, though the curve is of interest as a datum for comparison with Adams' formula. In practice μ rises with q; thus some sensible empirical relationship between them should be used in combination with our new formula. This study has used (for the red curve) a theoretically-derived relationship $\mu = 1/(1-q)$ which will be appropriate for streams of traffic which have (a) just merged or (b) have had ample overtaking opportunities upstream. The solid black curve uses an *ad hoc* relationship $\mu = (1+2q)/(1-q)$ which models the situation where bunch sizes have grown somewhat due to upstream constraints on overtaking.

5. Note that in Fig 2 the average delay decreases as μ increases for any fixed q. This occurs because g increases too, significantly raising the chance that a gap exceeds β; in effect the $(e^{\beta g}-1)$ contribution decreases faster than other terms rise. On the other hand if an increase in μ produces a more substantial rise in σ^2 (as in Fig 3) the last term plays a more dominant role. This explains the fact in Fig 3 that the black curve is higher than the red curve.

6. Our concepts of bunches and gaps should not be confused with those of *blocks* and *anti-blocks* employed by Raff [14] and Oliver [15] using Poisson and renewal processes respectively. It is possible, however, to link their concepts with our model. The *anti-blocks* are the periods during which crossing is possible; these periods occupy a proportion $(1-q)e^{-\beta/g}$ of the time and are exponentially

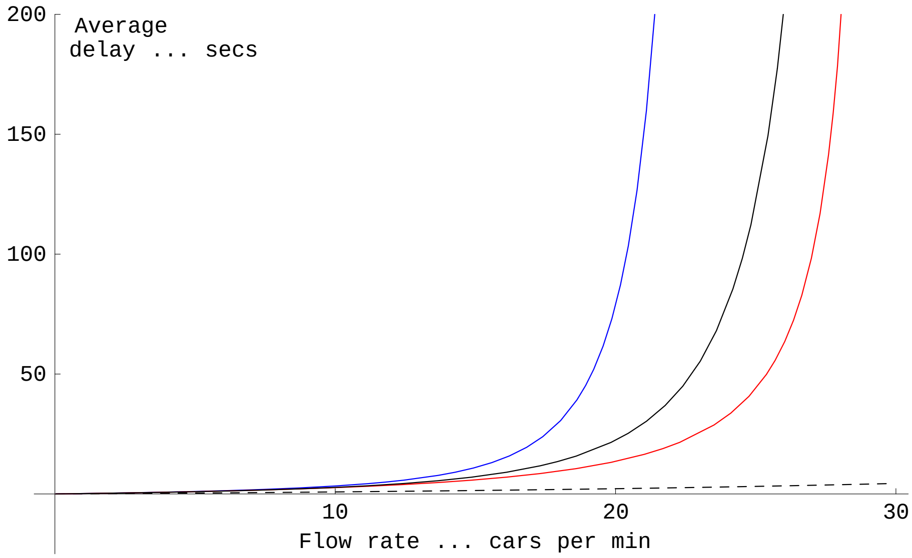

Figure 3: This figure differs from Fig 2 only in the choice of bunch size distribution. Here Borel-distributed bunches are used (see Remark 3).

distributed with mean g. (In short, any gap of length $X > \beta$ creates an anti-block of length $X - \beta$.) The remaining time periods are *blocks* which may comprise a number of bunches and any intervening gaps which are less than β. The relationship between $\mathbb{E}(D)$ and the mean and variance of block durations, together with a formula analogous to Raff's Equation (8) (in [14]), are given in the Appendix.

Acknowledgement

In the original 1984 paper, Figs 2 and 3 were prepared by Elaine Smith. The author has prepared the figures in the current reconstruction of the paper.

APPENDIX

- Let D be the delay to a random pedestrian arriving at an arbitrary time. Let $\mathcal{G}$ be the event that he arrives within a gap of the traffic stream (that is, the most recent car passed by at least one time unit previously). $\mathcal{G}'$ is the complement. It is easy to show that $\mathbb{P}\{\mathcal{G}\} = 1 - q$.

  Given $\mathcal{G}$, the time until the first car is distributed exponentially with mean g. This will exceed the critical gap β with probability $e^{-\beta/g}$. Thus using the more convenient notation $\gamma := 1/g$

$$\mathbb{P}\{D = 0\} = (1 - q)e^{-\beta\gamma}.$$

  If the first gap is less than β, the pedestrian will be delayed at least until the start of the next full gap, whereupon he will be confronted with another (stochastically-identical) decision. Thus we have the regenerative argument

$$\begin{aligned}\mathbb{E}(D|\mathcal{G}) &= \int_0^\beta [x + \mu + E(D|\mathcal{G})]\gamma e^{-\gamma x}\, dx \\ &= (e^{\beta\gamma} - 1)(\mu + g) - \beta.\end{aligned}$$

  Now given $\mathcal{G}'$, the event that the pedestrian arrives during a traffic bunch (that is, within one time-unit of a vehicle arrival), he waits at least until that bunch passes and then faces a decision which is stochastically identical to those mentioned above. Given $\mathcal{G}'$, the time until the start of the next gap, where the decision is faced, has mean $\frac{1}{2}(\mu + \sigma^2/\mu)$. This is a result of the well-known fact that given $\mathcal{G}'$ the size (and time duration) of the bunch encountered is size-biassed having mean $\mu + \sigma^2/\mu$. The residual time is on average half of this size-biassed time duration. Therefore $E(D) = (1-q)E(D|\mathcal{G}) + qE(D|\mathcal{G}')$ where $E(D|\mathcal{G}') = \frac{1}{2}(\mu+\sigma^2/\mu) + E(D|\mathcal{G})$. This yields Equation (3).

- Similar arguments yield the complete Laplace transform of D

$$
\begin{aligned}
E(e^{-sD}|\mathcal{G}) &= \frac{(s+\gamma)[1-\exp(-s\beta)]}{s+\gamma-\gamma f^*(s)+\gamma f^*(s)\exp[\beta(s+\gamma)]}, \\
E(e^{-sD}|\mathcal{G}') &= \frac{1-f^*(s)}{\mu s}\, E(e^{-sD}|\mathcal{G}) \qquad \text{and} \\
E(e^{-sD}) &= (1-q)E(e^{-sD}|\mathcal{G}) + qE(e^{-sD}|\mathcal{G}')
\end{aligned}
$$

  Here $f^*(s)$ is the Laplace transform of bunch duration (and size).

- In fact it is bunch duration (rather than size) that is the important quantity. Thus the reader wishing to generalise the theory need only replace μ, σ^2 and f^* in the equation above by newly-defined quantities μ_d, σ_d^2 and f_d^*. These are the equivalent quantities for bunch *duration*rather than size. Such a consideration allows one to relax the assumption that intra-bunch headways are constant. For example, if intra-bunch headways are assumed random and independent with mean μ_0, variance σ_0^2 and transform f_0^*, with μ, σ^2 and f^* still referring to bunch size, then

$$
\begin{aligned}
\mu_d &= \mu\mu_0 \\
\sigma_d^2 &= \mu\sigma_0^2 + \sigma^2\mu_0^2 \\
f_d^*(s) &= f^*(-\ln f_0^*(s)).
\end{aligned}
$$

- Let μ_B and σ_B^2 be the mean and variance of *blocks* (see Remark 6). A pedestrian arrives in a block period with probability $\theta_B = 1-(1-q)e^{-\beta/g}$. If so, the block duration will be size-biassed in the usual manner: its residual duration R will have mean $\frac{1}{2}(\mu_B + \sigma_B^2/\mu_B)$ and distribution function

$$\mathbb{P}\{R \le r\} = \frac{1}{\mu_B}\int_0^r [1-B(u)]du$$

  where B is the distribution function of a typical block duration.

  Clearly $\mathbb{E}(D)$ and $\mathbb{P}\{D \le t\}$ can be expressed in terms of block properties.

$$
\begin{aligned}
\mathbb{E}(D) &= \tfrac{1}{2}\theta_B(\mu_B + \sigma_B^2/\mu_B) \\
\mathbb{P}\{D \le t\} &= 1-\theta_B + \frac{\theta_B}{\mu_B}\int_0^t [1-B(u)]\,du. \qquad (4)
\end{aligned}
$$

  Equation (4) is analogous to Raff's equation (8).

References

[1] Adams, W. F. (1936). Road traffic considered as a random series. *J. Inst. Civ. Engrs.* **4,** 121–130.

[2] Buckley, D. J. (1968). A semi-Poisson model of traffic flow. *Transpn. Sci.* **2 (2),** 107–133.

[3] Branston, D. (1976). Models of single-lane time headway distributions. *Transpn. Sci.* **10,** 125–148.

[4] Wasielewski (1979). Car-following headways on freeways interpreted by the semi-Poisson headway distribution model. *Transpn. Sci.* **13,** 36–55.

[5] Cowan, R. (1975). Useful headway models. *Transpn. Res.* **9,** 371–375.

[6] Cowan, R. (1979). The uncontrolled traffic merge *J. Appl. Prob.* **16,** 384–392.

[7] Cowan, R. (1980). Further results on single-lane flow. *J. Appl. Prob.* **17,** 523–531.

[8] Cowan, R. (1978). An improved model for signalised intersections with vehicle-actuated control *J. Appl. Prob.* **15,** 384–396.

[9] Cowan, R. (1981). An analysis of the fixed-cycle traffic-light problem *J. Appl. Prob.* **18,** 672–683.

[10] Borel, E. (1943). Sur l'emploi du théorème de Bernoulli pour faciliter le calcul d'une infinité de coeffients. Application au problème de l'attente à un guichet. *C. R. Acad. Sci. Paris* **214,** 452–456.

[11] Oliver, R. M. (1961). A traffic counting distribution. *Oper. Res.* **9,** 802–810.

[12] Tanner, J. C. (1953). A problem of interference between two queues. *Biometrika* **40,** 58–69.

[13] Mayne, A. J. (1954). Some further results in the theory of pedestrians and road traffic. *Biometika* **41,** 375–389.

[14] Raff, M. S. (1951). The distribution of blocks in an uncongested stream of automobile traffic *J. Amer. Stat. Soc.* **46,** 114–123.

[15] Oliver, R. M. (1962). Distribution of blocks and gaps in a traffic stream. *Oper. Res.* **10,** 197–217.