Objects Arranged Randomly in Space: an Accessible Theory

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Summary

This expository paper deals with many problems concerning bounded objects arranged randomly in space. The objects are of rather general shapes and sizes, whilst the random mechanisms for positioning and orienting them are also fairly general. There are no restrictions on the dependence between shapes, sizes, orientations and positions of objects. The only substantive assumption is that the objects are distributed in a 'statistically uniform' way throughout the whole of the space. We focus on the statistical properties of features seen in an observation window, itself of general size and shape.

Keywords: Random geometry, integral geometry, geometric statistics, random sets.

Introduction

There has been a resurgence of interest in recent years amongst geometers and probabilists in matters 'random geometric'. Yet, on the topic of this paper, there remains a dearth of expository papers suitable for scientists at large, and even for those mathematicians and statisticians unable to invest time on the blend of integral geometry and probability theory needed. This paper attempts to fill that gap. Although some results in the paper are original, the chief motivation for writing has been the desire to make a large number of useful, elegant formulae accessible. Indeed, most of the paper is 'formula oriented', suitable for immediate use and easy referral by scientists concerned with seemingly random structures in space. Only at the end, do technicalities and proofs dominate. Thus, the paper aims at a different audience than that served by the excellent theoretical book of Stoyan, Kendall and Mecke (1987) [SK&M].

To achieve maximum accessibility, the mathematical treatment is quite informal. In particular the objects considered do not have peculiarities of shape, size, connectedness or surface roughness which are counter-intuitive. Also, the discussion is confined to problems in either 1, 2 or 3 dimensions. Whilst a number of results given here have generalisations in n dimensions we avoid the conceptual overheads and cumbersome notations that a general treatment requires. Using this approach we lose some unity but gain considerable usability.

The paper concentrates on results obtainable with only the meagre assumption of 'statistical uniformity' in the positions and orientations of the objects. We touch only briefly upon the situation where the probability model has additional structure, as for example in the 'Boolean' model (the Poisson process of 'random set theory'). Despite so few structural assumptions used in most of this paper, there is a rich theory to explore.

The paper deals with everyday concepts, such as areas and perimeters of 2-dimensional shapes or volumes and surface areas of 3-dimensional bodies. Two less-familiar entities (the 'connectivity number' known as the Euler characteristic and the 'integral of mean curvature for 3-d bodies') are also essential to the theory. They have a rôle in applications every bit as important as those everyday concepts, so space is devoted to the building of an intuitive framework for these two entities.

The approach taken in the technical discussion (which concludes the paper) should interest specialists in the field, for it is different from that presented in SK&M. In some ways it is more direct, if one accepts the classical and highly intuitive results of Blaschke and Santaló.

Random object process: informal description

Two dimensional space is the most convenient to fix ideas. So we consider firstly a 'random object process' (ROP) in the plane; Figure 1a illustrates such a process with 'filled' objects (the curved boundary plus the region contained by it) whilst Figures 1b and 1c have 'unfilled' objects that are just the curves. Bounded objects (sometimes called 'bodies' or 'grains') of varying shape are located in a *statistically* uniform manner over the plane, with orientations that are also randomly distributed.

Indeed we confine our attention to ROPs which are *statistically homogeneous in mean* (SHIM). We define this concept more completely later, but for the moment we understand the term as follows. If we place an observation window (W say) of given size and shape, the *mean* value of anything one cares to observe within W does not depend upon the position and orientation of W. That is, mean values are invariant under translations and rotations of W. Of course, the statistical variation from region to region implies, in the eyes of the physical scientist, *heterogeneity* of structure not *homogeneity*, hence our use of the term *statistical homogeneity* to convey invariance of mean values.

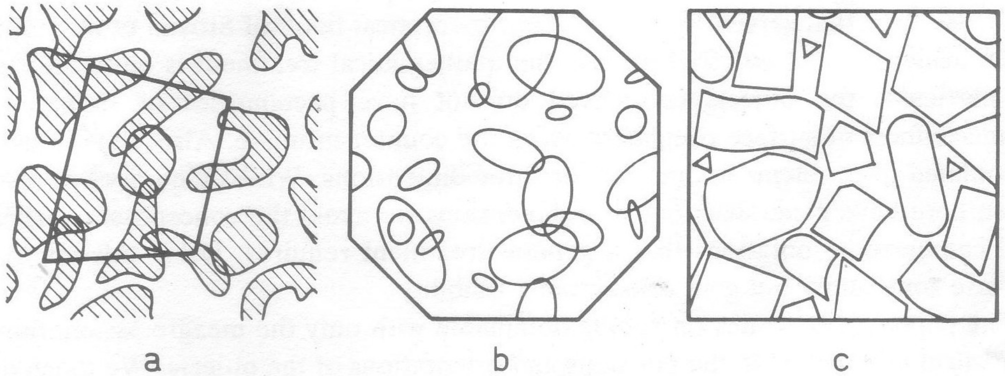

Figure 1. Three examples of ROPs. W is shown in bold outline, in (b) and (c) being all of the domain shown

Under the SHIM assumption, simple formulae exist for the *mean* of numerous features observable in (and near) W. Importantly, these formulae are valid even if the shapes and sizes of objects are statistically *dependent* upon the locations, orientations, shapes and sizes of other objects.

To be specific let us define a space $\mathcal{S}$ as the set of possible bounded objects. $\mathcal{S}$ may be quite specific (i.e. all ellipses) or quite general, but in any case the members of $\mathcal{S}$ will be 'pleasant'. They will be (as sets of $\mathbf{R}^2$) closed, bounded, with finite area A and boundary sufficiently smooth that the perimeter L is finite. Each object in $\mathcal{S}$ has a point which we call the *centre* (any well defined point will suffice). Emanating from the centre is a directional arrow which provides a reference direction when objects are randomly oriented on the plane.

Loosely speaking, we 'construct' the ROP by randomly sampling objects from $\mathcal{S}$ (according to a probability distribution defined on $\mathcal{S}$), and placing the centres of these objects at the points of a *planar point process*. It is not important how this point process arose; it is merely a collection of points having the property that the *mean* number of points in a window W does not depend on the position or orientation of W. The density of points (i.e. the mean number in a window of area 1) is denoted by τ. Pathological collections of points leading to infinite τ are ruled out by the usual regularity conditions applying to point processes (Daley and Vere-Jones, 1972). The orientation of the objects, when placed in the plane, are distributed so that the angle between the reference arrow and the x-axis is uniformly distributed on $[0, 2\pi)$.

One example of a ROP is known as the Boolean scheme [Hadwiger and Giger (1968), Kendall and Moran (1963), Matheron (1975), Serra (1982), Stoyan (1979a), A. M. Kellerer (1983, 1986), H. G. Kellerer (1984)]. In this model, the centre points form a Poisson point process on the plane, all objects are *independently* sampled from $\mathcal{S}$ and located with statistically *independent* orientations. The choice of object from $\mathcal{S}$ and its orientation are mutually independent and independent of the point process of centres.

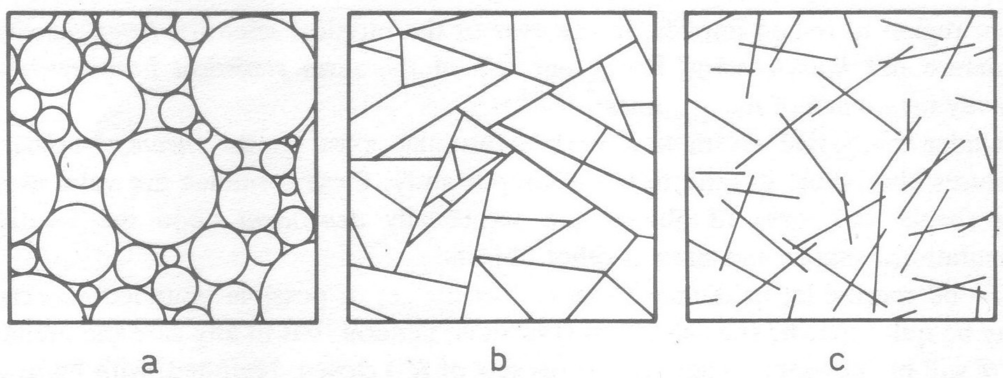

Figure 2. (a) An ROP comprising touching circles; (b) an ROP comprising non-overlapping polygons which cover the plane; (c) a Boolean scheme of line-segments

The Boolean scheme is the most important statistical model for a ROP, and the only one which is manageable for certain types of calculation. But the general SHIM ROP, which is the focus of attention in this paper, has little of the structure of the Boolean scheme because it requires *none* of the assumptions of statistical independence that define the Boolean scheme. (Even the assumption of a Poisson process of centres in the Boolean scheme is a statement of statistical independence; the independence of counts in disjoint windows.)

Thus, in our informal 'construction' of a ROP, any type of statistical dependence is acceptable. Examples which lie within our theory are illustrated in Figures 1 and 2. Figure 1c shows a ROP where the objects must not overlap. Clearly all aspects of this process are inter-dependent. Figure 2a shows circular disks arranged to be touching; Figure 2b illustrates a process of convex polygons constrained to form a mosaic. (See Ambartzumian (1970, 1974a), Cowan (1978, 1980) and Mecke (1980, 1983) for a general theory of homogeneous mosaics and Miles (1964, 1970), Gilbert (1967) and Maillardet (1982) for specially structured mosaics). Alternatively, Figure 2b may be viewed as a line-segment process, the segments being constrained to meet each other at 'junctions'. This is in contrast to the line-segment process in Figure 2c which is a Boolean scheme. (The general homogenous line-segment process is studied in Cowan (1979) whilst Coleman (1972) discusses the Boolean scheme of line segments. Intermediate cases are discussed by Santaló (1977) and Parker and Cowan (1976)).) Of course, the pattern in Figure 2c might also be viewed as a process of *non-crossing* line segments if we think of the original segments cutting each other into sub–segments. Viewed in such a way, the sub-segment process is *not* a Boolean scheme.

Although our theory envisages an uncountably infinite collection of general shapes and sizes in $\mathcal{S}$, some authors have specialised $\mathcal{S}$ by allowing it to contain just one ele-

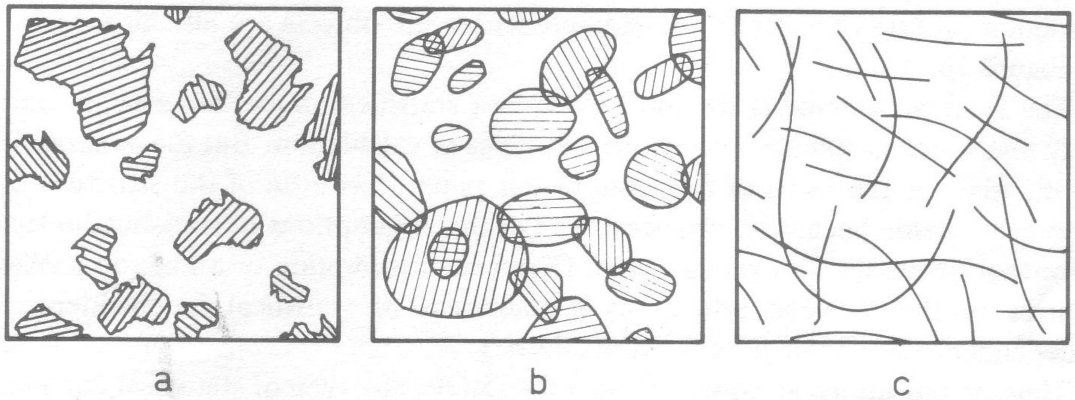

Figure 3. (a) Objects with size variation only; (b) convex objects; (c) curved fibres

ment, or more generally, to comprise a *finite* number of objects of fairly general shape (Berman, 1977). Another approach, used by Fava and Santaló (1978), allows $\mathcal{S}$ to comprise one general shaped object, s say, together with all contractions and dilations of s; size variation without shape variation (Figure 3a). Hadwiger and Giger (1968) consider $\mathcal{S}$ as the class of all bounded *convex* sets (see Figure 3b) whilst Ambartzumian (1974b, 1977), Mecke and Stoyan (1980), Stoyan (1981), Stoyan, Mecke and Pohlmann (1980) and Mecke (1981a) have specialised $\mathcal{S}$ to contain only *fibres* (Figure 3c). Matheron (1975), when discussing collections of objects, deals mainly with bounded convex shapes. In many of these studies a Boolean scheme is also assumed.

Objects of 'infinite extent' have been considered in a number of studies, in particular full-length lines and thickened lines [Miles (1964, 1973), Davidson (1974), Solomon (1978)]. In these studies, however, the mathematical framework differs considerably from ours. A difficulty exists if one tries to use objects of 'infinite extent' using our approach, because if such objects were 'centred' at the points of a stationary point process of density τ, every observation window would be crossed by an infinite number of objects. Our assumption of *bounded* objects eliminates these cases from the discussion.

For the most part it is natural to think of objects as 'connected' in a topological sense, an object made of many disconnected parts being viewed as multiple objects. But there can be cases where there is a defined nexus between the parts, and then it can be useful to consider just a single, multi-part, disconnected object within $\mathcal{S}$. Figure 4a shows a ROP comprised of two-part objects, namely pairs of circles. Dotted lines, themselves not part of the object, indicate the pairing.

Whilst it seems unnecessary to consider multi-part objects, there is a good reason for doing so. Consider Figure 1a, made up of one-piece objects. One object has an intersection with the window W made up of two disconnected parts. Thus, even with one-piece objects, there arises a need for some 'disconnectedness concept' to describe the

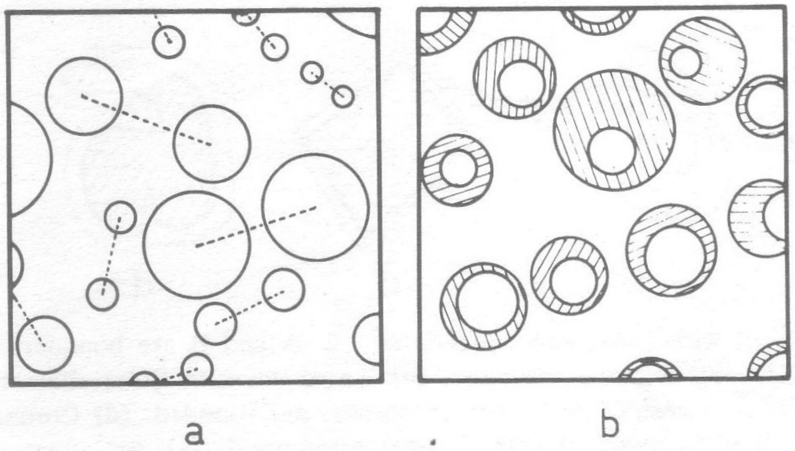

Figure 4. (a) An ROP with objects made up of paired circles; (b) annuli shaped objects

patterns seen in a window (unless the objects and the window are all convex). In setting up the simple topological concept to describe pattern disconnectedness, one finds that it is economical to allow the objects themselves to be multi-part and disconnected.

The Euler characteristic, χ

The Euler characteristic (sometimes called Euler-Poincaré characteristic) is an integer which summarises the topological character of a set. It is a kind of 'connectivity number'. For a two dimensional set (at least for an extremely rich class of 2-d sets sufficient for our purposes), χ is simply defined as the number of disconnected pieces of the set *minus* the number of holes (see Figures 5a and 5b).

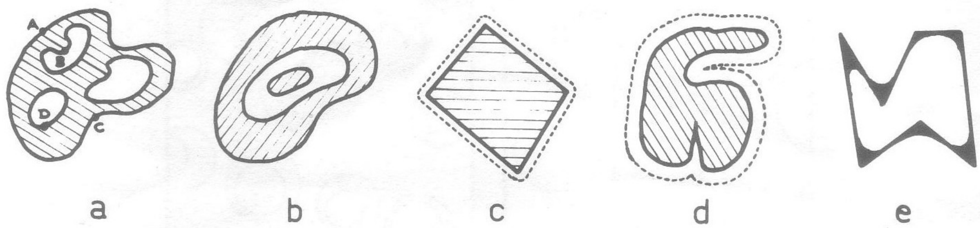

Figure 5. (a) An object with Euler characteristic of −2. A and B are boundary points with positive curvature, D and C having negative curvature. (b) An object with Euler characteristic of +1. (c) A convex object and its δ-extension. Note that all corners are rounded. (d) Corner rounding does not necessarily occur with non-convex objects if δ-extension used. (e) An alternative corner-rounding scheme, ensuring that curvature is defined at all boundary points

Clearly, χ quantifies the disconnectedness of objects in $\mathcal{S}$, but it also provides a topological parameter for sets with holes. If χ proves to be the appropriate topological parameter for our objects [and it does, Hadwiger (1957)] then we can allow the objects in

$\mathcal{S}$ to have holes. Is there any value in this? Firstly, the generalisation allows one to study a process like that shown in Figure 4b, where the objects are circular annuli. Of course, if the annuli are made arbitrarily thin we have a process of circular boundaries. Figure 2a could thus be treated as a ROP of circles rather than 'filled' disks. Indeed all our examples could be viewed as processes of closed curves, rather than of 'filled' objects. They would have area zero, Euler characteristic zero and a distribution of perimeter lengths. (Note the perimeter of a curve is defined, as we see later, to be *twice* the length.) Secondly, the mechanism for generating the ROP sometimes leads naturally to objects having holes. Consider a random continuous surface, for example, the realisation of a statistically homogeneous Gaussian process (that is, a Gaussian distributed random variable defined at each point in the plane with imposed continuity and homogeneity). This might model the height of land in an undulating countryside. If we flood all the land below a certain altitude, we form a ROP comprising land islands with internal sea-level lakes (though for some unsmooth Gaussian processes, the islands will have boundaries so rough that perimeters are infinite). Adler (1981) gives the average Euler characteristic of these 'islands' as part of his study on random Gaussian surfaces. Thirdly, when we discuss later the 3-dimensional ROP, it is of interest to question the nature of pattern on a planar section. (That is, consider the 3-dimensional viewing window as a rectangular prism which can be made arbitrarily thin and of arbitrarily large lateral extent.) Clearly the pattern on the section is a 2-d ROP, one where objects may contain holes even if the 3-d objects are themselves connected and without holes. Only a convexity assumption on the 3-d objects guarantees 2-d objects without holes.

So there is sufficient motivation for the study of objects with holes. Also, in the patterns that result from a ROP, we see 'clumps' (of objects), many clumps having holes. The patterns of wet circles seen when raindrops begin to fall on a dry path illustrate these 'clumps' with their dry 'holes'. (In the context of Boolean schemes (though not raindrops), A. M. Kellerer (1983) derives the Euler characteristic of the 'wet' region within a window W.)

Window formulae

Suppose the objects selected from $\mathcal{S}$ have mean area μ_A, mean perimeter μ_L and mean Euler characteristic μ_χ. Denote the area, perimeter and Euler characteristic of the observation window W by A, L and χ. Let $N(W)$ be the number of objects which intersect W. For each such object s, one can measure certain quantities (such as area of overlap with W) and sum the measurements over the $N(W)$ objects. For example, using the notation that ∂s and ∂W represent the boundaries of s and W respectively, let

$$A(W) \;=\; \sum(\text{area of } s \cap W)$$

$$
\begin{aligned}
L_i(W) &= \sum(\text{length of } \partial s \text{ inside } W) \\
L_\partial(W) &= \sum(\text{length of } \partial W \text{ covered by } s) \qquad (1)\\
L(W) &= \sum(\text{perimeter of } s \cap W) = L_i(W) + L_\partial(W) \\
\chi(W) &= \sum(\text{Euler characteristic of } s \cap W).
\end{aligned}
$$

Provided neither the window W nor the objects sampled from $\mathcal{S}$ are fibres or have fibrous components, we have the following simple formulae for mean values.

$$
\begin{aligned}
\mathbf{E}A(W) &= \tau A \mu_A, &(2)\\
\mathbf{E}L_i(W) &= \tau A \mu_L, &(3)\\
\mathbf{E}L_\partial(W) &= \tau L \mu_A, &(4)\\
\mathbf{E}L(W) &= \tau(A\mu_L + L\mu_A), &(5)\\
\mathbf{E}\,\chi(W) &= \tau[A\mu_\chi + \chi\mu_A + \frac{1}{2\pi}L\mu_L], &(6)
\end{aligned}
$$

where τ is the density of the point process of centres. In particular, if each object is connected and without holes, with W likewise (implying $\chi = \mu_\chi = 1$), then (6) simplifies. In the special case of convex W and convex shaped objects, $\chi(W) = N(W)$. So under these convexity conditions

$$
\mathbf{E}N(W) = \tau[A + \mu_A + \frac{1}{2\pi}L\mu_L]. \qquad (7)
$$

It is useful to note that A, L and χ denote characteristics of 'dimension' 2, 1 and 0 respectively, as do μ_A, μ_L and μ_χ. We see that the multiple of τ in (6) contains dimensionally consistent terms. Formulae (2)–(5) also have appropriate dimensions.

As an example, consider a random process of filled rectangles observed within the circular field (radius r) of a microscope. Formulae (2), (3), (4), and (7) apply with $A = \pi r^2$, $L = 2\pi r$ and $\mu_L = 2(\mu_a + \mu_b)$, where μ_a and μ_b are *mean* side lengths (longer and shorter respectively) of the rectangles. Using obvious notations for standard deviations and correlation coefficients, $\mu_A = \rho_{ab}\sigma_a\sigma_b + \mu_a\mu_b$. Though it is not my intention to dwell on issues of statistical estimation, it is obvious that observed values of $N(W)$, $A(W)$, $L_i(W)$ and $L_\partial(W)$ in such an example provide some information about τ, μ_A and μ_L if these need to be estimated.

Of course if W is a grid of k distinct points, then (6) helps show that $\mathbf{E}\,\chi(W) = k\tau\mu_A$ for the general ROP, since $A = L = 0$ and $\chi = k$.

Fibres and fibrous components

The formulae above need some qualification if either W or the objects are fibres (i.e. curves of zero thickness) or have fibrous components. This includes the case where they comprise a closed curve without the region it encloses. The easiest way to use the basic formulae correctly in such cases is by giving the fibre a small thickness Δ, applying the formula and then taking $\Delta \to 0$. Figure 6 shows some objects of this type and the Δ–modified form.

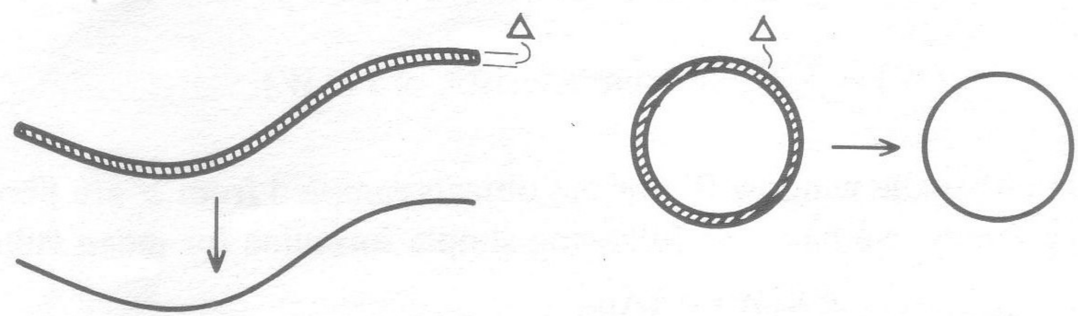

Figure 6. Illustrations of Δ-thickening

This device forces us to use a notion of 'perimeter of a fibre' which is a little counter-intuitive. The perimeter of a thickened fibre of length ℓ is approximately $2(\ell + \Delta)$, ignoring minor curvature effects themselves of order Δ, so the unthickened fibre has perimeter equal to $\lim_{\Delta \to 0} 2(\ell + \Delta)$ or 2ℓ, *twice* its length. Thus, if the line-segments of Figure 2c have mean lengths μ_ℓ, then $\mu_L = 2\mu_\ell$, $\mu_\chi = 1$ and $\mu_A = \lim_{\Delta \to 0} \Delta\mu_\ell = 0$. Thus (6) implies

$$\mathbf{E}\ \chi(W) = \tau[A + L\mu_\ell/\pi] \tag{8}$$

and, if W is convex, this is also the formula for $\mathbf{E}N(W)$. Formula (8) also applies to the curved fibres of Figure 3c, but since these are not convex, $\mathbf{E}N(W) \neq \mathbf{E}\ \chi(W)$. Of course, if W is fibrous too, with length ℓ, (8) becomes $\mathbf{E}\ \chi(W) = 2\tau\ell\mu_\ell/\pi$, where χ is now the number of intersection points. As another example, consider Figures 1, 2a and 3b as processes of closed curves of mean length μ_ℓ. Now $\mu_A = 0$, $\mu_L = 2\mu_\ell$ and $\mu_\chi = 0$. So $\mathbf{E}\ \chi(W) = \tau L\mu_\ell/\pi$. Of course, whilst the shapes in 3b are convex, the closed boundary curves are not, so here $\mathbf{E}N(W) \neq \mathbf{E}\ \chi(W)$.

Clumps

Note that it is necessary to be able to distinguish overlapping objects in order to determine the geometric entities such as $A(W)$ and $L(W)$. If one cannot do this (for example, if all the objects in a ROP were a common colour), one must be content with measurements (such as areas or perimeters) on 'clumps' of objects. Thus, **in cases where**

mean clump size is finite[1], we envisage a ROP of non-overlapping 'clumps' having its own parameters, τ^*, μ_A^*, μ_L^* and μ_χ^* say, as statistical descriptors of the clumps. If these $*$-parameters are used in (2)–(6), we obtain expectations of total clump area, perimeter and Euler characteristic within W.

An obvious question is the relationship between (τ^*, μ_A^*, μ_L^*, μ_χ^*) and (τ, μ_A, μ_L, μ_χ). Without further assumptions concerning the dependence structure of shapes and positions, nothing can be said about the relationship. Even in the case of Boolean schemes, there are unsolved problems concerning this relationship though, with a rearrangement of some results of A. M. Kellerer (1983), one can show that $\tau^*\mu_A^* = 1 - \exp(-\tau\mu_A)$, $\tau^*\mu_L^* = \tau\mu_L \exp(-\tau\mu_A)$ and $\tau^*\mu_\chi = \tau A \exp(-\tau\mu_A)[\mu_\chi - \tau\mu_L^2/4\pi]$. Clearly another relation, for τ^* say, is needed, but none is available. It is obvious that $\tau = \bar{n}\tau^*$, where $\bar{n}$ is the mean number of objects per clump, but there are no examples where $\bar{n}$ is known.

Additional window formulae

Here we present more window formulae. These involve the *boundaries* of either W, the objects s or both. Initially we look at the case where neither W nor the objects are fibrous.

Let

$$
\begin{aligned}
\chi(\partial W) &= \sum(\text{Euler characteristic of } s \cap \partial W), && (9)\\
\chi_\partial(W) &= \sum(\text{Euler characteristic of } \partial s \cap W), && (10)\\
\chi_\partial(\partial W) &= \sum(\text{Euler characteristic of } \partial s \cap \partial W), && (11)
\end{aligned}
$$

where the summation is over all objects which intersect W. Note that $\chi_\partial(W) = \chi(\partial W) = \frac{1}{2}\chi_\partial(\partial W)$. To find $\mathbf{E}\ \chi(\partial W)$, consider ∂W as another observation window, called W^* say. As such it has $A^* = 0$, $L^* = 2L$ and $\chi^* = 0$. Thus from (6), $\mathbf{E}\ \chi(W^*) = \tau L\mu_L/\pi$. Therefore

$$
\mathbf{E}\ \chi(\partial W) = \mathbf{E}\chi_\partial(W) = \frac{1}{2}\,\mathbf{E}\ \chi_\partial(\partial W) = \tau L\mu_L/\pi. \qquad (12)
$$

If the s in (9) or the W in (10) are themselves fibrous of length ℓ, then the L in (12) equals 2ℓ. But the W in (9) or (11) and the s in (10) or (11) are never taken to be fibrous, for it is confusing to talk of 'boundaries of fibres'.

[1]This clause in bold font was overlooked in the original paper, but it is a necessary addition to render the discussion of *clumps* correct.

One-dimensional processes

A random object process on the line is defined in an analogous manner. The objects, which are simply intervals or collections of disconnected intervals are 'centred' on points which are arranged in a statistically homogeneous manner on the line. In one dimension, we denote the density of points by λ. The 'content' of an object is the sum of lengths for its disconnected pieces, whilst its Euler characteristic (now denoted by η) is simply the number of pieces (there being no concept of holes). The mean content of objects is μ_C and mean Euler characteristic μ_η, whilst the (possibly disconnected) window W has content C and Euler characteristic η. Let $C(W) = \sum$(content of s within W) and define $\eta(W)$, $\eta(\partial W)$ and $\eta_\partial(W)$ by analogy with (1), (9) and (10) respectively. The appropriate window formulae are

$$\begin{aligned} \mathbf{E}\eta(W) &= \lambda(C\mu_\eta + \eta\mu_C) \qquad (13)\\ \mathbf{E}C(W) &= \lambda C\mu_C \\ \mathbf{E}\eta(\partial W) &= 2\lambda\eta\mu_C \\ \mathbf{E}\eta_\partial(W) &= 2\lambda C\mu_\eta. \end{aligned}$$

The 'clumping' relationships are $\lambda^*\mu^*_\eta = \lambda\mu_\eta\exp(-\lambda\mu_C)$ and $\lambda^*\mu^*_C = 1 - \exp(-\lambda\mu_C)$. If one specifies that $\mu_\eta = 1$, then $\mu^*_\eta = 1$ and a complete set of relationships can be found [including $\bar{n}$, the mean number of objects per clump equalling $\exp(\lambda\mu_C)$].

Line transects of 2-dimensional processes

A two-dimensional ROP observed along a line transect yields a one-dimensional ROP. To find the properties of the transect ROP, consider W as an $\ell \times \Delta$ rectangle in the plane. Then as $\Delta \to 0$, $\mathbf{E}\chi(W) \to \tau(\mu_A + \ell\mu_L/\pi)$ using the two-dimensional formula, (6). But (13) shows that if W is an interval of length ℓ, $\mathbf{E}\eta(W) = \lambda(\ell\mu_\eta + \mu_C)$, in terms of one-dimensional features. Thus $\lambda\mu_\eta = \tau\mu_L/\pi$ and $\lambda\mu_C = \tau\mu_A$. If the objects in 2-d are convex, these formulae simplify to $\lambda = \tau\mu_L/\pi$ and

$$\mu_C = \pi\mu_A/\mu_L. \qquad (14)$$

Thus, in this case, the transect process comprises *intervals* placed at centres whose mean density λ is proportional to the mean perimeter of the planar objects. The intervals have mean length μ_C, proportional to the ratio of mean areas to mean perimeters. In the non-convex case, the objects on the line transect may be multi-part, and one can show that the parts are intervals of mean length $\pi\mu_A/\mu_L$.

Extended window formulae: 2 dimensions

In this section some new formulae for the statistical features of objects which intersect W are presented. Let $I(W)$ be the number of objects whose centres lie within W and $I(W,a,\ell)$ be the number of these objects with area $\leq a$ and perimeter $\leq \ell$. It is the full area and perimeter of such an object which concerns us here, even though the object may extend beyond the window W. ¿From the definitions of the ROP, we have that $\mathbf{E}I(W) = \tau A$ and

$$\mathbf{E}I(W,a,\ell) = \tau A G_{AL}(a,\ell) \tag{15}$$

where G_{AL} is the 'joint distribution function' of area and perimeter for the objects in $\mathcal{S}$.

Thus by observing the totality of all objects with centres in W we obtain a sample of areas and lengths which is unbiased. On the other hand, bias occurs if we observe the full areas and perimeters of the $N(W)$ objects which intersect W (the 'hitting objects'). This is not unexpected, since 'large' objects are more likely than 'small' objects to hit W. We see however that the aspect of largeness which influences bias depends on the context. The nature of bias can be examined for convex W and convex objects. Let $N(W,a,\ell)$ be the number of hitting objects with area $\leq a$ and perimeter $\leq \ell$. One can show (with convexity assumptions) that

$$\mathbf{E}N(W,a,\ell) = \tau[AG_{AL}(a,\ell) + \int_o^a \int_0^\ell (x + yL/2\pi) dG_{AL}(x,y)]. \tag{16}$$

Let $H_{AL}(a,\ell) = \mathbf{E}N(W,a,\ell)/\mathbf{E}N(W)$. From (7) and (16), H_{AL} has properties of a joint distribution function. In a certain sense (discussed later), H_{AL} is the joint distribution of areas and perimeters for hitting objects. We have

$$H_{AL}(a,\ell) = \frac{AG_{AL}(a,\ell) + \int_0^a \int_0^\ell (x + yL/2\pi) dG_{AL}(x,y)}{A + \mu_A + L\mu_L/2\pi} \ . \tag{17}$$

When areas and perimeters have a joint *density*, g_{AL} say, the joint probability density of areas and perimeters of hitting objects is

$$h_{AL}(a,\ell) = \frac{(A + a + \ell L/2\pi) g_{AL}(a,\ell)}{A + \mu_A + L\mu_L/2\pi} \ . \tag{18}$$

The marginal probability densities are

$$h_A(a) \ = \ \frac{[2\pi(A+a) + L\mu_{L|a}]g_A(a)}{2\pi(A+\mu_A) + L\mu_L} \tag{19}$$

$$h_L(\ell) \ = \ \frac{[2\pi(A+\mu_{A|\ell}) + \ell L]g_L(\ell)}{2\pi(A+\mu_A) + L\mu_L} \tag{20}$$

where $\mu_{L|a}$ and $\mu_{A|\ell}$ denote conditional expectations. The means of these distributions are extremely interesting. The mean area of the hitting objects is

$$\mu_A + \frac{(2\pi\sigma_A + L\rho_{AL}\sigma_L)\sigma_A}{2\pi(A+\mu_A) + L\mu_L} \tag{21}$$

whilst the mean perimeter is

$$\mu_L + \frac{(L\sigma_L + 2\pi\rho_{AL}\sigma_A)\sigma_L}{2\pi(A+\mu_A) + L\mu_L} \ , \tag{22}$$

where σ_A^2 and σ_L^2 are variances and ρ_{AL} is the correlation coefficient between areas and perimeters. The bias is clearly evident when there is variability of object sizes.

It is easy to show, using (18), that if W is a disk of radius r then $\lim_{r\to\infty} h_{AL} = g_{AL}$. As expected, bias disappears for large 'rotund' windows. But large elongated windows behave differently. Let W be a $\Delta \times x$ rectangle. Then as $x \to \infty$,

$$\begin{aligned}
h_{AL}(a,\ell) &\to \frac{\pi\Delta + \ell}{\pi\Delta + \mu_L} g_{AL}(a,\ell) \\
h_A(a) &\to \frac{\pi\Delta + \mu_{L|a}}{\pi\Delta + \mu_L} g_A(a) \\
h_L(\ell) &\to \frac{\pi\Delta + \ell}{\pi\Delta + \mu_L} g_L(\ell) \ .
\end{aligned}$$

If $\Delta \to 0$, we recover results for line transect sampling. Thus the objects 'hit' by a line transect are 'perimeter biassed' with mean perimeter $\mu_L + \sigma_L^2/\mu_L$, mean area $\mu_A + \rho_{AL}\sigma_A\sigma_L/\mu_L$ and

$$\begin{aligned}
h_{AL}(a,\ell) &= \ell g_{AL}(a,\ell)/\mu_L, \\
h_A(a) &= \mu_{L|a} g_A(a)/\mu_L, \\
h_L(\ell) &= \ell g_L(\ell)/\mu_L.
\end{aligned} \tag{23}$$

Cauchy's classical formula states that the mean projection of a 2-d convex body onto a randomly oriented line is $1/\pi$ times its perimeter. This explains the bias in favour of objects with large perimeter, because an object's chance of being hit is proportional to its projected length on a line *orthogonal* to the randomly-oriented transect. Perimeter biassing, together with another geometric result from the 19th century provide an alternative justification for (14). Crofton showed that the mean length of a random chord of a convex set is π times the ratio of area to perimeter. Thus μ_C of (14) equals $\int\int(\pi a/\ell)h_{AL}(a,\ell)da\, d\ell$, which equals $\pi\mu_A/\mu_L$ from (23).

If W collapses to a single point, (18)–(22) show that the sampling is now 'area' biassed, a well known phenomenon. Areas and perimeters have means $\mu_A + \sigma_A^2/\mu_A$ and $\mu_L + \rho_{AL}\sigma_A\sigma_L/\mu_A$ respectively with joint density $a g_{AL}(a,\ell)/\mu_A$.

We now consider a different type of 'extended window formula'. Let $Y_A(W)$ be the sum of areas for all objects hitting W and $Y_L(w)$ be the sum of perimeters. Under convexity assumptions, one can show that

$$\mathbf{E}Y_A(W) \ = \ \tau[A\mu_A + \sigma_A^2 + \mu_A^2 + \frac{L}{2\pi}(\rho_{AL}\sigma_A\sigma_L + \mu_A\mu_L)]$$

$$\mathbf{E}Y_L(W) \quad = \quad \tau[A\mu_L + \rho_{AL}\sigma_A\sigma_L + \mu_A\mu_L + \frac{L}{2\pi}(\sigma_L^2 + \mu_L^2)].$$

These formulae contrast with (2) and (3), where the sums use only areas and perimeters *interior* to W.

Extended window formulae: 1 dimension

Using obvious notation, the analogous results are $\mathbf{E}I(W) = \lambda C, \mathbf{E}I(W, c) = \lambda C G_C(c)$ in general, whilst under convexity assumptions

$$\begin{aligned} H_C(c) &= \frac{CG_C(c) + \int_0^c x dG_C(x)}{C + \mu_C}, \\ h_C(c) &= \frac{(C + c)g_C(c)}{C + \mu_C} \end{aligned}$$

with the mean content of hitting objects being $\mu_C + \sigma_C^2/(C + \mu_C)$. When W is a point and hence $C = 0$, the well-known 'length' biassing is demonstrated. Also

$$\mathbf{E}Y_C(W) = \lambda[C\mu_C + \sigma_C^2 + \mu_C^2].$$

Some basic geometry and topology

As a preliminary to the discussion of 3-dimensional ROPs, certain notions about the geometry and topology of 3-dimensional bodies must be mentioned. Our texts for this are Santaló (1976), Matheron (1975), A.D. Aleksandrov (1963) and Hadwiger (1957).

We have seen that in two dimensions the important features of a body are area A, perimeter L and Euler characteristic χ. It is perhaps surprising that the curvature κ at points around the boundary does not enter the argument. In fact it does, but only in an integrated form. For bodies, such as those in Figures 5a and 5b where curvature is uniquely defined at all boundary points, the integral of curvature over the whole boundary equals $2\pi\chi$. (For this purpose, adopt the convention that curvature is positive at points like A and B in Figure 5a and negative at points like C and D.) For a convex body with corners as in Figure 5c, the result is still $2\pi\chi$ provided that the integral is first taken over the 'δ-extended body' with δ then taken to zero. The 'δ-extended body' is defined as the union of all circular disks of radius δ, the centres of which are points of the original body (Figure 5c). This commonly used technique 'rounds' corners in a way suitable for calculation of the integral. With non-convex bodies, δ-extension will not 'round' corners that are re-entrant and may introduce new corners (Figure 5d). Then, other 'corner-rounding' devices must be used (e.g. Figure 5e). (Of course, the result, which is essentially the Gauss-Bonnet theorem in the plane, is usually stated in terms of the angles at the corners, but corner rounding is a notion easily extended to higher dimensions.)

So in two dimensions the integral of curvature plays a role as the topological invariant $2\pi\chi$. In three dimensions there are two important integrals of curvature; one is proportional to a topological invariant (which is the three dimensional Euler characteristic called φ), the other an important parameter denoted by M. A point Q on the surface ∂D of a 3-dimensional body D is deemed 'regular' if it has a unique tangent plane $\mathcal{T}$. A plane $\mathcal{H}$ orthogonal to $\mathcal{T}$ cuts ∂D, forming a plane curve with well defined curvature κ at Q. Each such $\mathcal{H}$ yields a curvature κ. The maximum and minimum values of κ are known as the 'principal curvatures' and denoted by κ_1 and κ_2. It turns out that their planes, $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, are always mutually orthogonal and that for a plane $\mathcal{H}$ at angle α to $\mathcal{H}_1$ the corresponding κ is $\kappa_1 \cos^2\alpha + \kappa_2 \sin^2\alpha$. Thus the 'mean curvature at Q' over all planes $\mathcal{H}$ is $(\kappa_1 + \kappa_2)/2$. The 'integral of mean curvature' over all points Q on the surface ∂D is denoted by M. It is a quantity as fundamentally important for the body D as its volume V and surface area S.

For bodies with edges and corners (i.e. not all points Q regular) the integral is first taken over a version of D with rounded edges and corners. Some values of M for common bodies are given in Table 1, others can be found in Santaló (1976, p. 229). The method of δ-extension adequately copes with edges and corners for bodies[2] in Table 1.

Body	M	S	V
Ball, radius r	$4\pi r$	$4\pi r^2$	$4\pi r^3/3$
Cylinder,height h, radius r	$\pi(h+\pi r)$	$2\pi r(h+r)$	$\pi r^2 h$
Rectangular prism, sides a, b, c	$\pi(a+b+c)$	$2(ab+bc+ca)$	abc
Cone, height h, radius r	$\pi^2 r + \pi h - \pi h \tan^{-1}(h/r)$	$\pi r^2 + \pi r(h^2+r^2)^{\frac{1}{2}}$	$\pi r^3 h/3$
Hemi-ball, radius r	$2\pi r(1+\pi/4)$	$3\pi r^2$	$2\pi r^3/3$
Cylinder with hemi-ball caps	$\pi(h+4r)$	$2\pi r(h+2r)$	$\pi r^2(h+4r/3)$
Torus, circle radius a revolved about axis at distance b from its centre	$2\pi^2 b$	$4\pi^2 ab$	$2\pi^2 a^2 b$

Table 1: Integral of mean curvature, M, surface area S and volume V for some common 3-dimensional solid bodies.

The integral of $\kappa_1\kappa_2$, the product of principal curvatures at Q, over ∂D (with due

[2]In this Table and in other parts of the original paper, I used the words 'solid sphere', or just 'sphere' with 'solid' implied, instead of the better word 'ball'. In this revised version, 'ball' has been used in some places.

allowance for edges and corners using 'corner rounding') yields a multiple φ say, of 4π. φ equals 1 for the ball and its topological equivalents and k for a multi-part object made up of k such bodies. The structure of 'holes' within D influences φ. Holes are made up of 'cavities' and 'tunnels'.

We speak of holes like those inside a good Swiss cheese as *simple* cavities. A connected body with x simple cavities has $\varphi = 1 + x$. Alternatively, if a hole is drilled *through* a ball, we speak of a *simple* tunnel. A ball with x simple non-intersecting tunnels has $\varphi = 1 - x$. Thus a torus is like a ball with one simple tunnel and has $\varphi = 0$, whilst a solid figure-of-eight is like a ball with two simple tunnels and has $\varphi = -1$. Complicated tunnel systems can be made by the sequential addition of tunnels, each of which must have two ends. The tunnel ends may both be on the surface (as with a simple tunnel), but in general the ends may be another tunnel, a cavity or the surface. Each *additional* tunnel decreases φ by 1. A labyrinth of tunnels can be analysed by counting the number of sequentially added tunnels. Cavities can also be quite complicated, but there is a neat rule. View the cavity in 'inverted mode'; treat void space (and the interface surface) as solid and solid as void. Calculate the Euler characteristic, φ^* say, for the inverted mode object. For the original body the effect on φ due to a cavity is the addition of φ^*. For example, a ball with x torus-shaped cavities and y figure-of-eight cavities has $\varphi = 1 - y$.

There remains the contingency that the tunnels (which are the elements of tunnel networks) may have a complicated cross-section, for example, an annulus. It is usually not difficult to establish φ for these cases, but the methods are *ad hoc*. Finally we note that φ for the surface ∂D is twice that of the body D, since ∂D is topologically equivalent to D with a D-shaped cavity.

Classical Statistical Geometry

We now have the full complement of geometric and topological features for objects: η and C in 1 dimension; χ, L and A in 2; φ, M, S and V in 3. In general, if the space has dimension k, there are $k+1$ fundamental numbers describing entities of dimension $0, 1, 2, ...k$. This section shows the interplay between these 9 quantities in numerous problems of statistical geometry, in a manner which underpins the results already presented for 1 and 2 dimension and those yet to come in 3 dimensions. They deal with projections of objects (Cauchy-Kubota formulae), sections of objects (Crofton formulae), δ-extensions (Steiner formulae) and the links between the Euler characteristics in different dimensions. We give a comprehensive account of these, partly because they are essential for our central theme, partly to emphasise that the less-familiar entities such as χ, φ and M play a vital rôle in random geometry, and partly to present the 'projection and section' formulae in a novel way (using the language of statistical sampling instead of concepts from 'invariant measure theory').

A *random diameter* of a 'ball' (circular filled disk in $\mathbf{R}^2$, solid sphere in $\mathbf{R}^3$) is found by sampling a point P uniformly distributed on the ball's boundary and drawing a line through the centre O. For a solid sphere in $\mathbf{R}^3$, a random equatorial plane is found by first sampling a random diameter and taking the plane orthogonal to it through O. In $\mathbf{R}^2$, a random 'transect' of the disk is found by sampling a point Q uniformly distributed on a random diameter and taking the line through Q orthogonal to OQ. In $\mathbf{R}^3$, a random 'line probe' of the solid sphere is defined similarly, except that Q is uniformly distributed on a random equatorial plane. A random 'section' of the solid sphere is defined as the plane orthogonal to OQ when Q is uniformly distributed on a random diameter.

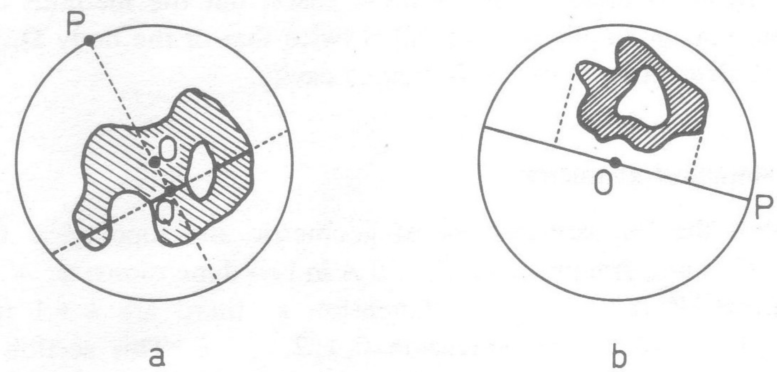

Figure 7. (a) Random diameter through O defined by endpoint P uniformly distributed on circle chosen to enclose shaded object. Random transect of object sampled by line orthogonal to OQ at Q, uniformly distributed on diameter. (b) Projection on to random diameter

For a general body D, random transects, line probes and sections are defined by enclosing D within a ball (Figure 7a), sampling a random transect (etc.) of the ball and, in the event $\mathcal{E}$ that it cuts D, taking it as D's random transect (etc). Pairs (P, Q) are sampled until $\mathcal{E}$ occurs.

Also defined for general D is the projection onto a random diameter (Figure 7b) or, for D in $\mathbf{R}^3$, a random equatorial plane. It is here that the Cauchy-Kubota formulae apply. For connected D in $\mathbf{R}^2$, the projection onto a random diameter has mean content

$$\begin{aligned} \mathbf{E}(C) &= L_h/\pi \\ &= L/\pi \quad (D \ \ convex) \end{aligned}$$

where L and L_h are the perimeters of D and its convex hull respectively. Projections of connected D in $\mathbf{R}^3$ onto a random diameter yield

$$\begin{aligned} \mathbf{E}(C) &= M_h/2\pi \\ &= M/2\pi \quad (D \ \ convex) \end{aligned} \tag{24}$$

whilst projections onto a random equatorial plane yield[3]

$$\begin{aligned}\mathbf{E}(A) &= S/4 \quad (D \ \ convex) \\ \mathbf{E}(L) &= M/2. \quad (D \ \ convex)\end{aligned}$$

For a convex D, (24) shows that $\mathbf{E}(C)$, often called the 'mean caliper diameter of D', is proportional to M.

We now deal with random transects, line probes and sections of D, in particular the nature of their intersection with D. If r is the radius of the enclosing ball and D is a connected body in the appropriate dimension, then the probability of $\mathcal{E}$ is $L_h/2\pi r$ (transects), $S/4\pi r^2$ (probes of convex D) and $M_h/4\pi r$ (sections). For a random transect of a connected D in $\mathbf{R}^2$, $\mathbf{E}(\eta|\mathcal{E}) = L/L_h$ and $\mathbf{E}(C|\mathcal{E}) = \pi A/L_h$. For a random probe of a convex D in $\mathbf{R}^3$, $\mathbf{E}(C|\mathcal{E}) = 4V/S$, whilst for a random section of a connected D, $\mathbf{E}(\chi|\mathcal{E}) = M/M_h$, $\mathbf{E}(A|\mathcal{E}) = 2\pi V/M_h$ and $\mathbf{E}(L|\mathcal{E}) = \pi^2 S/(2M_h)$. We refer to as these as 'Crofton formulae', though Crofton's attention was restricted to transects of convex domains in $\mathbf{R}^2$.

A general class of geometric results concerning δ-extensions are referred to as Steiner formulae. For convex D [and certain types of more general shapes, Hadwiger (1957)], these formulae relate the properties $C_\delta, A_\delta, L_\delta, V_\delta, S_\delta, M_\delta$ for the δ-extension to those of D itself. Briefly (where dimension is obvious) $C_\delta = C + 2\delta$, $A_\delta = A + L\delta + \pi\delta^2$, $L_\delta = L + 2\pi\delta$, $V_\delta = V + S\delta + M\delta^2 + 4\pi\delta^3/3$, $S_\delta = S + 2M\delta + 4\pi\delta^2$ and $M_\delta = M + 4\pi\delta$.

Lastly, we consider the links, established by Hadwiger (1957), between φ, χ and η. As the line AB in Figure 8a moves up the page remaining parallel with the position shown, it 'sweeps' across the 2-d domain D. As it does so, the η value for the 1-d transects changes when the sweep passes over various boundary points (marked $\bullet$). At those marked E the sweep is 'entering' D. Dots marked L signify the sweep 'leaving' D. χ is simply the sum of η-changes at the E points, namely $(1-0) + (1-2)$ in Figure 8a. Changes in the orientation of the sweep leave χ unchanged. The link between φ and χ is based on a sweeping *plane* XY (seen edge view in Figure 8b). φ is the sum of χ-changes at the E points, namely $(1-0) + (1-2) + (1-0)$.

So we have seen that there is a rich assortment of geometric, topological and statistical relationships between the nine variables $\eta, C, \chi, L, A, \varphi, M, S$ and V. We are now ready to consider their role in three-dimensional ROPs.

[3] In these formulae and those in the next paragraph, I made some mistakes concerning three-dimensional convex-hulls; these have now been corrected.

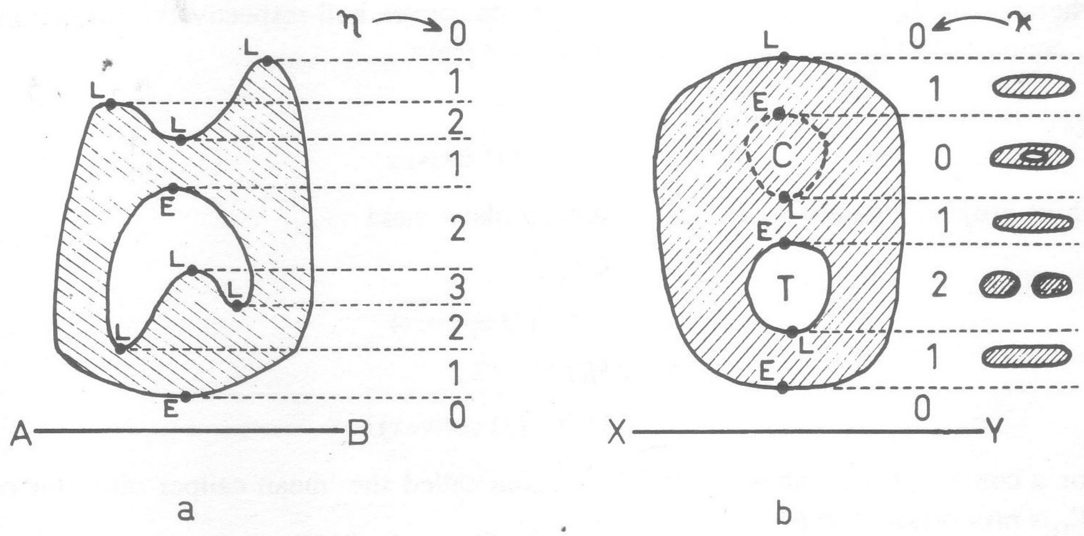

Figure 8. Illustration of Hadwiger's algorithm for linking φ, χ and η (see text)

Three-dimensional random object processes

In $\mathbf{R}^3$, the centroids of objects have density θ, whilst objects have mean volume μ_V, mean surface area μ_S, mean 'integral of mean curvature' μ_M and mean Euler characteristic μ_φ. As before, assumptions of 'statistical homogeneity in mean' apply.

Window formulae in R^3

Let W be a window with features V, S, M and φ, initially considered as non-fibrous and non-lamina. With this 'full-bodied' assumption applied to the objects too, we define

$$
\begin{aligned}
V(W) &= \sum(\text{volume of } s \cap W) \\
S_i(W) &= \sum(\text{area of } \partial s \text{inside } W) \\
S_\partial(W) &= \sum(\text{area of } \partial W \text{ covered by } s) \\
S(W) &= \sum(\text{surface area of } s \cap W) = S_i(W) + S_\partial(W) \\
M_i(W) &= \sum(\text{integral of m.c. for } \partial s \text{ inside } W) \\
M_\partial(W) &= \sum(\text{integral of m.c. for } \partial W \text{ covered by } s) \\
M(W) &= \sum(\text{integral of m.c. for } s \cap W) \\
\varphi(W) &= \sum(\text{Euler characteristic of } s \cap W),
\end{aligned}
$$

where the summation is over all objects which intersect W. We note that $M(W) \neq M_i(W) + M_\partial(W)$ since there is a contribution from the edge $\partial s \cap \partial W$. We have

$$\mathbf{E}V(W) = \theta V \mu_V \tag{25}$$

$$
\begin{aligned}
\mathbf{E}S_i(W) &= \theta V\mu_S \\
\mathbf{E}S_\partial(W) &= \theta S\mu_V \\
\mathbf{E}S(W) &= \theta(V\mu_S + S\mu_V) && (26) \\
\mathbf{E}M_i(W) &= \theta V\mu_M \\
\mathbf{E}M_\partial(W) &= \theta M\mu_V \\
\mathbf{E}M(W) &= \theta(V\mu_M + M\mu_V + \pi^2\, S\mu_S/16) && (27) \\
\mathbf{E}\varphi(W) &= \theta[V\mu_\varphi + \varphi\mu_V + (S\mu_M + M\mu_S)/4\pi]. && (28)
\end{aligned}
$$

Note the matching dimensionality of each term in equations like (26), (27) or (28). For convex objects and convex W, $\varphi(W) = N(W)$, the number of objects hitting W, and then $\mathbf{E}N(W) = \theta[V + \mu_V + (S\mu_M + M\mu_S)/4\pi]$.

In a fashion analogous to (9)–(11) we define

$$
\begin{aligned}
\varphi(\partial W) &= \sum(\text{Euler characteristic of } s \cap \partial W) \\
\varphi_\partial(W) &= \sum(\text{Euler characteristic of } \partial s \cap W) \\
\varphi_\partial(\partial W) &= \sum(\text{Euler characteristic of } \partial s \cap \partial W)
\end{aligned}
$$

and note that $\varphi_\partial(\partial W) = 0$. Also, from (28) and arguments similar to those used in establishing (12), $\mathbf{E}\varphi(\partial W) = \theta(4\pi\varphi\mu_V + S\mu_M)/2\pi$ and $\mathbf{E}\varphi_\partial(W) = \theta(4\pi V\mu_\varphi + M\mu_S)/2\pi$. If the boundaries ∂s and ∂W intersect, $\partial s \cap \partial W$ takes the form of one (or more) closed space curves. One can show that the sum of lengths of the space curves for all bodies intersecting W has expectation $\pi\theta S\mu_S/4\pi$.

Adaptations for Curved Plates and Fibres

A window that is lamina (eg a curved 'plate' or '2-d manifold') in $\mathbf{R}^3$ can be viewed as the limit, as $\Delta \to 0$, of its Δ-thickened version. Thus it has contributions to S and M from both its sides. The contributions are additive for S, so S is twice the 'nominal' area, f say, of the plate. The contributions cancel in the calculation of M, but there is a contribution of $\pi\ell/2$ from the plate's *rim*, assumed to be one (or more) space curve(s) with total length ℓ. Of course, V is zero and φ is determined in the usual way by the number of parts and the number of holes, all holes being of the 'tunnel' type. (Hole boundaries contribute to the rim). Similar considerations apply for *objects* which are plates: $\mu_V = 0, \quad \mu_S = 2\mu_f, \quad \mu_M = \pi\mu_\ell/2$ with some μ_φ.

Subject to these considerations formulae (25) - (28) hold. For example, if W is a plate and the objects full-bodied, $\mathbf{E}V(W) = 0, \quad \mathbf{E}S(W) = 2\theta f\mu_V, \quad \mathbf{E}M(W) = \pi\theta(4\ell\mu_V + \pi f\mu_S)/8$ and

$$\mathbf{E}\varphi(W) = \theta(8\pi\varphi\mu_V + 4f\mu_M + \pi\ell\mu_S)/8\pi. \tag{29}$$

Due care is needed in interpretations: for example, $\mathbf{E}S(W)$ involves double counting of the 'one-sided areas' of the individual $s \cap W$ plates. Other window formulae are not applicable if they involve the concept of 'boundary of a plate' (as distinct from its rim).

Similar considerations apply if W or the objects are fibres in $\mathbf{R}^3$. If W is a fibre of length ℓ, it has $V = S = 0, \;\; M = \pi\ell$, with φ determined by its number of parts. Likewise if the objects are fibres, $\mu_V = \mu_S = 0, \;\; \mu_M = \pi\mu_\ell$ with some μ_φ. As an example of the window formula, we see that W fibrous of length ℓ implies $\mathbf{E}\varphi(W) = \theta(\varphi\mu_V + \ell\mu_S/4)$.

Sections and Probes of 3-d Processes

A section by a reference plane yields a ROP in $\mathbf{R}^2$. A probe by a line yields a ROP in $\mathbf{R}$. To investigate sectioning consider W as a connected *flat* plate of 'nominal' area f, rim length ℓ and $\varphi = 1$. Matching (29) with (6) we have $\tau\mu_\chi = \theta\mu_M/2\pi$, $\tau\mu_L = \theta\pi\mu_S/4$ and $\tau\mu_A = \theta\mu_V$. If the 3-d objects are convex, we recover the 'stereological' formulae $\tau = \theta\mu_M/2\pi$, $\mu_L = \pi^2\mu_S/2\mu_M$ and $\mu_A = 2\pi\mu_V/\mu_M$. Alternatively consider W as a line-segment of length ℓ. Matching (29) and (13) yields $\lambda\mu_\eta = \theta\mu_S/4$ and $\lambda\mu_C = \theta\mu_V$, which under convexity assumptions on the 3-d objects, yield $\lambda = \theta\ \mu_S/4$ and $\mu_C = 4\mu_V/\mu_S$.

Extended Window Formulae: 3 Dimensions

We give now a range of extended window formulae for convex objects and convex W. The notation is an obvious extension of the 2-d case. In general $\mathbf{E}I(W) = \theta V$ and $\mathbf{E}I(W, v, s, m) = \theta V G_{VSM}(v, s, m)$, whilst under the convexity conditions (assumed for the rest of this section)

$$H_{VSM}(v,s,m) = \frac{VG_{VSM}(v,s,m) + \int_0^v \int_0^s \int_0^m \; [x + (zS + yM)/4\pi] \; dG_{VSM}(x,y,z)}{V + \mu_V + (S\mu_M + M\mu_S)/4\pi},$$

$$h_{VSM}(v,s,m) = \frac{[V + v + (mS + sM)/4\pi] g_{VSM}(v,s,m)}{V + \mu_V + (S\mu_M + M\mu_S)/4\pi},$$

$$h_V(v) = \frac{[4\pi(V+v) + S\mu_{M|v} + M\mu_{s|v}] g_V(v)}{4\pi(V + \mu_V) + S\mu_M + M\mu_S}$$

$$h_S(s) = \frac{[4\pi(V + \mu_{V|s}) + S\mu_{M|s} + sM] g_S(s)}{4\pi(V + \mu_V) + S\mu_M + M\mu_S}$$

$$h_M(m) = \frac{[4\pi(V + \mu_{V|m}) + mS + M\mu_{S|m}] \; g_M(m)}{4\pi(V + \mu_V) + S\mu_M + M\mu_S}.$$

Formulae for h-type densities depend on the existence of g-type densities. The mean values for volume, surface area and integral of mean curvature for the objects which 'hit'

W are respectively

$$\begin{aligned}
\mu_V &+ \frac{(4\pi\sigma_V + S\rho_{VM}\sigma_M + M\rho_{VS}\sigma_S)\ \sigma_V}{4\pi(V+\mu_V) + S\mu_M + M\mu_S}, \\
\mu_S &+ \frac{(4\pi\rho_{VS}\sigma_V + S\rho_{SM}\sigma_M + M\sigma_S)\ \sigma_S}{4\pi(V+\mu_V) + S\mu_M + M\mu_S}, \quad \text{and} \\
\mu_M &+ \frac{(4\pi\rho_{VM}\sigma_V + S\sigma_M + M\rho_{SM}\sigma_S)\ \sigma_M}{4\pi(V+\mu_V) + S\mu_M + M\mu_S}.
\end{aligned}$$

Let W be a cylinder of radius r and height Δ. As $r \to \infty$ and $\Delta \to 0$ we recover formulae for the objects which 'hit' a planar section.

$$\begin{aligned}
h_{VSM}(v,s,m) &\overset{r\to\infty}{\longrightarrow} \frac{(2\pi\Delta + m)\ g_{VSM}(v,s,m)}{2\Delta + \mu_M} \\
&\overset{\Delta\to 0}{\longrightarrow} m g_{VSM}(v,s,m)/\mu_M. \qquad (30)
\end{aligned}$$

Thus, objects hit by a section are biassed to be those with large integrals of mean curvature. It is clear from (24) that this should be so, for the chance that a body is cut by a plane section is proportional to its projected length on a line orthogonal to the section. From (30), we see that the hitting objects have mean i.m.c. of $\mu_M + \sigma_M^2/\mu_M$, mean surface area $\mu_S + \rho_{SM}\sigma_S\sigma_M/\mu_M$ and mean volume $\mu_V + \rho_{VM}\sigma_V\sigma_M/\mu_M$.

By taking W as a cylinder of radius Δ and height h and letting $h \to \infty$ and $\Delta \to 0$, we recover results for probe sampling. Now $h_{VSM}(v,s,m) \to s\ g_{VSM}(v,s,m)/\mu_S$, demonstrating that such objects are 'surface area biassed'. Likewise if W is a point, the results are seen to be 'volume biassed'.

To conclude this section we note the analogous 'Y-formulae'.

$$\begin{aligned}
\mathbf{E}Y_V(W) &= \frac{\theta}{4\pi}[4\pi(V\mu_V + \sigma_V^2 + \mu_V^2) + S(\rho_{VM}\sigma_V\sigma_M + \mu_V\mu_M) + M(\rho_{VS}\sigma_V\sigma_S + \mu_V\mu_S)] \\
\mathbf{E}Y_S(W) &= \frac{\theta}{4\pi}[4\pi(V\mu_S + \rho_{VS}\sigma_V\sigma_S + \mu_V\mu_S) + S(\rho_{SM}\sigma_S\sigma_M + \mu_S\mu_M) + M(\sigma_S^2 + \mu_S^2)] \\
\mathbf{E}Y_M(W) &= \frac{\theta}{4\pi}[4\pi(V\mu_M + \rho_{VM}\sigma_V\sigma_M + \mu_V\mu_M) + S(\sigma_M^2 + \mu_M^2) + M(\rho_{SM}\sigma_S\sigma_M + \mu_S\mu_M)]
\end{aligned}$$

The Extended Formulae Qualified

The distribution functions H, examples of which commence with (17), are not exact results in the context of the general ROP. If the window W is large, however, the H-formulae can be used with confidence. The functions H are defined by ratios of expectations, when the true distributions are expectations of ratios, given the event $\mathcal{E}$ that at least one object intersects W. For example in a 2-d ROP, the probability that an object randomly sampled from those that hit W has area $\leq a$ and perimeter $\leq \ell$ given $\mathcal{E}$ is $\mathbf{E}[N(W,a,\ell)/N(W) \mid \mathcal{E}]$ which is approximately equal to $H_{AL}(a,\ell)$ when W

is large enough. An ergodic assumption is needed to formalise the 'large W' approximation (Miles, 1961; Cowan, 1978). When the process is Boolean it can be shown (see Cowan, 1979, Lemma 2) that this probability equals $H_{AL}(a, \ell)$ exactly for all W. The H-formulae like (17) are therefore presented in the knowledge that they are exact for Boolean schemes and asymptotically appropriate in general ROPs. They are like the demographic formulae from the theory of age-dependent branching processes, where ratios of expectations are also used successfully as approximations.

Technical Discussion

We conclude with a brief outline of methods for proof of these formulae and further discussion of the literature. Each object is defined by the position $x \in \mathbf{R}^n$ of its centre, its orientation t defined by a point on the surface of the unit ball B_n in $\mathbf{R}^n$ and its 'shape and size' $s \in \mathcal{S}$. Thus the *random object process* is defined as a random point process on the space $Z = \mathbf{R}^n \times \partial B_n \times \mathcal{S}$. This 'associated' point process (APP) has the usual regularity properties of 'almost-sure orderliness' and 'finite expected count in bounded subsets'. Let $\mathcal{B}(Z)$ be the class of Borel sets of Z (any sensible topology on $\mathcal{S}$, and hence on Z, will suffice without restricting our theory in any practical way). For each $U \in \mathcal{B}(Z)$, define $K(U)$ as the number of points of the APP within U. Define $\mathcal{T}$ as the group of translations in $\mathbf{R}^n$ and let $\mathcal{R}$ be the group of rotations in $\mathbf{R}^n$ expressed as motions on the surface ∂B_n. Any $T \in \mathcal{T}$ and $R \in \mathcal{R}$ can be defined on Z; if $z \in Z$ is the point (x, t, s) then $Tz = (Tx, t, s)$ and $Rz = (x, Rt, s)$. Thus TU and RU are defined for $U \in \mathcal{B}(Z)$. A random object process is deemed *statistically homogeneous in mean* (SHIM) if $\mathbf{E}K(U) = \mathbf{E}K(TU) = \mathbf{E}K(RU)$ for all motions $T \in \mathcal{T}$ and $R \in \mathcal{R}$.

It is clear that the representation

$$\mathbf{E}K(U) = \int_U \mathbf{E}K(dz) \tag{31}$$

holds in general. Noteworthy is the fact that under the SHIM assumption, $\mathbf{E}K(.)$ is a product measure on $(Z, \mathcal{B}(Z))$. Normally we associate product measures with assumptions of *independence* but no such assumption is needed here. Invariance of $\mathbf{E}K(\cdot)$ for motions in the $\mathbf{R}^n \times \partial B_n$ subspace of Z with s held fixed is sufficient.

The proof is based on a very simple lemma proved earlier (Cowan, 1979), but because the 'factorisation' of the measure $\mathbf{E}K(\cdot)$ establishes the framework for all the formulae of this paper, we restate the lemma.

Consider a space (such as our Z) which can be represented as a product $C \times D$ with C being a locally compact group and D being a topological space. Let $\mathcal{F}$ be the class of functions mapping $C \to C$ defined for each $a \in C$ by $f_a(c) = a^{-1}c; c \in C$. These functions can also be defined as maps $Z \to Z$ via $f_a(c, d) = (a^{-1}c, d); a, c \in C, d \in D$.

Lemma: *Suppose μ is a measure on $(Z, \mathcal{B}(Z))$ invariant under the action of functions in $\mathcal{F}$; that is for $U \in \mathcal{B}(Z), a \in C, \mu(U) = \mu(f_a U)$. Then, μ is a product measure, being defined on product sets $U_1 \times U_2$ by $\mu(U_1 \times U_2) = \rho(U_1)\nu(U_2); U_1 \in \mathcal{B}(C), U_2 \in \mathcal{B}(D)$, where $\mathcal{B}(\cdot)$ denotes Borel sets of said space. Here ν is some measure on $(D, \mathcal{B}(D))$ and ρ is a Haar measure on $(C, \mathcal{B}(C))$.*

Thus, when the Lemma is applied in the current context, $C = \mathbf{R}^n \times \partial B_n$ with ρ being a multiple of Lebesgue measure, whilst $D = \mathcal{S}$ with ν being the measure which determines the sampling of shapes and sizes. ν induces the various distribution functions $G_A, G_{AL}, \ldots$ and must be sufficiently regular to give finite $\mu_A, \mu_L, \sigma_A, \ldots$. In one, two and three dimensions, ρ is respectively $\lambda/2, \tau/2\pi$ and $\theta/4\pi$ times Lebesgue measure.

Concentrating on $\mathbf{R}^2$, (31) becomes

$$\mathbf{E}K(U) = \int\int\int_{(x,t,s)\in U} (\tau/2\pi)\ dtdx\ \nu(ds), \tag{32}$$

an easy form to use in applications. For example, to prove (15), let U be the set of points $z = (x, t, s) \in Z$ such that an object s, placed with centre at x and orientation t, has centre inside a window $W \subset \mathbf{R}^2$, area $\leq a$ and perimeter $\leq \ell$. Thus $K(U)$ equals the count of such z in the APP and hence equals $I(W, a, \ell)$. So

$$\mathbf{E}I(W, a, \ell) = \int_X \int_W \int_0^{2\pi} (\tau/2\pi)\ dtdx\ \nu(ds) \tag{33}$$

where $X \in \mathcal{B}(\mathcal{S})$ is the subset of shapes and sizes which have areas $\leq a$ and perimeters $\leq \ell$. Since $\nu(X) = G_{AL}(a, \ell)$, (15) follows from (33).

More substantial calculations rely upon the power of integral geometry for the evaluation of the two inner-most integrals in representations like (32),in particular, upon the *fundamental kinematic formulae* of Blaschke (1936), Santaló (1953, 1976) and Chern (1952). We give three examples in detail.

Let U be the set of $z = (x, t, s)$ such that the object s placed with centre at x and orientation t intersects a window W. Thus $K(U) = N(W)$. Also entities like $\mathbf{E}A(W), \mathbf{E}L(W)$ and $\mathbf{E}\chi(W)$ in (2), (5) and (6) have integral representations of the form[4]

$$\int\int\int_{(x,t,s)\in U} (\tau/2\pi)\ f(x, t, s)\ \ dtdx\ \nu(ds)$$

where f represents (respectively) area, perimeter and Euler characteristic of $s \cap W$. The kinematic formulae tell us that for *fixed s*

[4]Note that the original paper had a typing error in the formula, using ρ instead of ν.

$$\begin{aligned}\int\int\ f(x,t,s)\ dtdx &= 2\pi AA_s, && (f = \text{area})\\ &= 2\pi(AL_s + LA_s), && (f = \text{perimeter})\\ &= 2\pi(A\chi_s + \chi A_s) + LL_s, && (f = \text{Euler characteristic})\end{aligned}$$

where the integration is over all positions x and orientations t such that $s \cap W \neq \phi$ and where s has features A_s, L_s and χ_s. Thus (2), (5) and (6) follow.

All of the window formulae are derived in this manner, as are 'extended' window formulae like (16) and those 'Y-type' formulae. (Indeed versions of the 'Y-type' formulae involving sums of higher moments of the 'hitting objects' are available by the same methods).

There remains the issue of generality of shapes for the valid application of the kinematic formulae, and indeed other geometric aspects of this paper. There appear to be two distinct classes of sets for which the kinematic formulae hold. Both are extremely rich for practical purposes. One is the 'convex ring' (Hadwiger, 1957), the other the sets of 'positive reach' (Federer, 1959). Further extensions to the class of 'Hausdorff rectifiable sets' have been established by Zähle (1982). See Weil (1983) for a good account from the specialist geometer's view. Of course, our exposition also imposes conditions (like piecewise twice differentiable boundaries) when discussing perimeters, surface areas and curvatures.

Much of the probabilistic foundation for the random process of objects comes from the notion of a random set in $\mathbf{R}^n$ established by Matheron (1975) and Kendall (1974), though their approach and emphasis differs from ours. They address basic questions of measurability and point-set topology and establish the 'minimal' class of events on which a probability measure need be defined for its extension to events of wider interest. This is an outgrowth from the theory of point processes [Daley & Vere-Jones (1972), Matthes, Kerstan and Mecke (1978), Ripley (1981), Diggle (1983)] and a prelude to the practice of image processing (Serra, 1982).

It is noteworthy that we do not study here any features involving the interaction of two (or more) random objects within W. Naturally, assumptions concerning the second (or higher) order structure of the ROP are needed in this case. Readers can consult Streit (1970), Santaló (1976) and Kellerer (1983, 1986), where problems of this type are discussed in the context of Boolean schemes.

Much of the work on general processes of geometric objects has come from the East German school, notably the works of Mecke and Stoyan (Mecke, 1981a, 1981b; Stoyan, 1979b, 1981, 1982; Pohlmann, Mecke & Stoyan, 1981; Stoyan & Mecke, 1983; Zähle, 1982 and the book SK&M) and is based on the machinery of 'marked point processes', 'Campbell's theorem' and 'Palm measures' (Matthes, Kerstan & Mecke, 1978; Mecke, 1967), with less emphasis on the highly intuitive results from integral geometry. Some of the formulae in this paper can be found in their work, and most could be derived using their approach. They also consider extensions to higher dimensions, the consequences of dropping the isotropic component of the SHIM assumption and statistical estimation issues.

This paper is based however on my own approach incorporating the 'lemma on product measures' (Cowan, 1979) with the kinematic formulae and point process theory. (The general two-dimensional theory of this paper was given in three lectures to an Australian statistical conference in 1979.) It is a natural approach, one used also by Fava & Santaló (1978, 1979) in their study of plates and line segments in $\mathbf{R}^3$ and general manifolds in $\mathbf{R}^n$ (though their work uses unnecessary assumptions concerning independence of shapes, sizes, positions, etc.). The parallels between the Mecke/Stoyan methods and mine are not obvious, though with any two approaches that start with the same premises and reach results of similar content, analogies will be found.

This paper touches on issues 'stereological', the inference of structure from lower dimensional sections or probes. This is now a large field of study, reviewed by Jensen, Baddeley, Gundersen and Sundberg (1985). Important recent developments, following the seminal paper of Davy and Miles (1977), have been in its sampling theory without statistical models for the arrangement of objects throughout space. Rather the emphasis is on the choice of sampling technique to best reveal the structure of a given opaque specimen. Impetus for this work comes largely from the extensive work of R. E. Miles (e.g. 1977, 1985) but other important developments can be found in Davy (1978), Cruz-Orive (1980), Gundersen (1986), Baddeley, Gundersen & Cruz-Orive (1986) and Voss (1982).

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