Kähler-Ricci Flow and Complex Monge-Ampère Equation

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Plan for the series of talks

- Part I: Introduction of Kähler geometry
- Part II: Complex Monge-Ampère equation and Ricci flow
- Part III: Set-up of Kähler-Ricci flow
- Part IV: Topics

I. Introduction of Kähler geometry

A few references:
"Differential Analysis on Complex Manifolds" (Wells);
"Principles of Algebraic Geometry" (Griffiths-Harris);
"Canonical Metrics in Kähler Geometry" (Tian).

- Smooth manifold and Riemannian structure

1. Smooth manifold: a topological space M with local coordinate systems (i.e. $\mathbb{R}^n$ structure, $\dim_{\mathbb{R}} M = n$) of smooth coordinate transition.

Example: $S^2 = \mathbb{CP}^1$, or even just S^1.

2. Smooth vector bundle (over a smooth manifold M): a smooth manifold V with a smooth map $F : V \to M$, with smoothly-varying real Euclidean structure on each fibre (i.e. pre-image of each point of M).

Note: Smoothly-varying means existence of local smooth trivialization.

Typical example: tangent and cotangent bundle of smooth manifold, TM and T^*M.

Intuitively, TM is the collection of directions (of curves) in M. Expression in local coordinates $\{x_1, \cdots, x_n\}$.

Another example: the universal bundle of $\mathbb{CP}^1$. (No need to always think of matrix-valued transition functions with compatibility conditions.)

Curvature form: for a smooth vector bundle with connection ∇ (i.e. covariant derivative, some explanation here), it is ∇^2, where the second time of ∇ treats the T^*M part as (form) coefficient.

3. Riemannian structure (over M): a smoothly-varying (symmetric) positive definite form over fibres of TM, denoted by g.

Infinitesimal length of curves in M, i.e. "speed".

Example: Euclidean space and induced ones over submanifolds.

Levi-Civita connection: a canonical connection on tensor fields (coming from that on TM),

$\nabla : \Gamma(M, TM) \to \Gamma(M, T^*M \otimes TM)$ such that

$\nabla g = 0$ and $\nabla_Y Z - \nabla_Z Y = [Y, Z]$ (Lie bracket) for $Y, Z \in \Gamma(M, TM)$.

Riemannian curvature (as the classic definition) is nothing but the above curvature of tangent bundle as a vector bundle with Levi-Civita connection.

- Complex manifold and Kähler structure

M: a smooth manifold of $\dim_{\mathbb{R}} M = 2n$.

1. Complex structure: an "integrable" almost complex structure".

Almost complex structure: a "bundle map" $J : TM \to TM$ "covering" $Id_M : M \to M$, i.e. mapping fibre to the same fibre, such that $J^2 = -Id_{TM} : TM \to TM$ with $-$ using vector bundle structure.

Integrable: the existence of local complex coordinate systems (i.e. $\mathbb{C}^n$ structure) with holomorphic coordinate transition.

In local holomorphic coordinates: $\{z_1, \cdots, z_n\}$, $z_j = x_i + \sqrt{-1}y_j$,

$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}$, $J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}$.

Notations like dz_j, $\frac{\partial}{\partial z_j}$, $d\bar{z}_j$ and $\frac{\partial}{\partial \bar{z}_j}$ are for convenience.

Holomorphic coordinate transition: consider transition matrix between "smooth coordinates",

$\{z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n\}$.

Note: meaningful special structure should survive coordinate transition. Locally, it doesn't mean anything.

Equivalent conditions for integrability:

a) $N(J) = [J, J] - J[J,] - J[, J] - [,] = 0$, tensorial.

b) "$[T^{1,0}, T^{1,0}] \in T^{1,0}$", not tensorial, flavor of Frobenius Theorem.

$(TM)^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$ using eigen-decomposition of J with explicit expression, and $\overline{T^{1,0}M} = T^{0,1}M$. $T^{1,0}M$ is isomorphic to TM as real bundles. This only needs almost complex structure. ("Type" of forms.)

$T^{1,0}M$ is a "holomorphic vector bundle" if J is integrable. By taking J as $\sqrt{-1}$ for TM to see TM can also be seen as a holomorphic bundle.

c) $d = \partial + \bar{\partial}$ (for higher differential), type consideration. (What is ∂ and $\bar{\partial}$? The composition of d and the projection to proper type of forms.)

2. Symplectic structure: a real closed two form ω with $\omega^n \neq 0$.

Fact: there is always an almost complex structure J such that $J^*\omega = \omega$ and $g(Y, Z) = \omega(Y, JZ)$ is a Riemannian structure ("compatible conditions").

ω is of type $(1, 1)$. (Why? Exercise.)

3. Kähler structure: starting from symplectic structure, the almost complex structure J is integrable.

Equivalently, starting from a Riemannian manifold, (M, g) has an almost complex structure J, $J^*g = g$ and $\nabla J = 0$ (i.e. Levi-Civita preserving type.).

ω is the Kähler form, representing a class in $H^{1,1}(M; \mathbb{C})$ (Dolbeault Theory) or $H^2(M; \mathbb{R})$ (de Rham Theory). Kähler class is thus defined.

"Holomorphic vector bundle": similar definition as "smooth vector bundle".

Smooth hermitian metric: a smoothly-varying hermitian metric on fibre, h. (Only smooth! Flat otherwise. Consider transition to understand.)

Holomorphic connection: the unique connection ∇ such that $\nabla h = 0$ and $\nabla^{0,1} = \bar{\partial}$ (well defined using holomorphic structure).

Fact: Levi-Civita connection gives the "holomorphic connection" of $T^{1,0}M$ with the hermitian metric h induced from g, i.e. for $Y, Z \in T^{1,0}M \subset (TM)^{\mathbb{C}}$, $h(Y, Z) = g_{\mathbb{C}}(Y, \bar{Z})$ with $g_{\mathbb{C}}$ being linear complexification of g.

Note: in order to see the relation between Riemannian curvature tensor, Ricci tensor and their expressions in Kähler case, one needs to use the First Bianchi Identity a lot.

Alternative ways of understanding (or defining) Kähler structure:

a) $d\omega = 0$;

b) local Eucildean expression of the metric with second order error (very handy in computation for Kähler Identities);

c) existence of the complex counterpart of geodesic coordinates .

- De Rham (or Dolbeault) Hodge decomposition and $\partial\bar{\partial}$-Lemma

  M: a smooth "closed" (compact without boundary) manifold.

  Note: de Rham and Dolbeault cohomology theory are available for non-compact case, but one needs elliptic theory for Hodge decomposition.

  1. Hodge decomposition.

  Elliptic differential operators (for forms):

  $\Delta_d = dd^* + d^*d$,

  $\Delta_\partial = \partial\partial^* + \partial^*\partial$,

  $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$,

  with corresponding "inverse" (Green's operators) G_d, G_∂, and $G_{\bar{\partial}}$ to give Hodge decomposition for any smooth tensor field, Γ:

  $$\Gamma = Proj_{_{Ker(\Delta)}}(\Gamma) + \Delta G(\Gamma).$$

  The upper $*$ indicates the corresponding dual using the Hermitian product integrated over the manifold. Of course, for de Rham d operator, one can reduce to real category and use the Riemannian product. Hodge star operator is involved here which involves a complex conjugation (for the Hermitian metric). Each of d^*, ∂^* and $\bar{\partial}^*$ would involve twice of Hodge star operation. It's more necessary to make this clear when twisting with a general holomorphic bundle.

  In Kähler case: $\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta_d$, and so the decompositions are compatible to give decomposition of De

Rham cohomology by Dolbeault cohomology. (What are these cohomology spaces? Quotient spaces of closed forms by exact forms, simply speaking.)

Hodge Diamond: Dolbeault cohomology and symmetries from Hodge star and complex conjugation. One thing is left ...... Mirror Symmetry.

Remark: Kähler classes form an open cone in the finitely dimension space $H^{1,1}(M;\mathbb{C}) \cap H^2(M;\mathbb{R})$.

Note: for $F \in C^\infty(M)$,
$\Delta_{\bar{\partial}} F = g^{i\bar{j}} F_{i\bar{j}} = \langle \omega, \sqrt{-1}\partial\bar{\partial} F \rangle$. So $\sqrt{-1}\partial\bar{\partial} F = 0$ means F is a constant over a closed Kähler manifold, using integration by parts for "$F\sqrt{-1}\partial\bar{\partial} F$" (or "$F\Delta_{\bar{\partial}} F$") to see this.

2. $\partial\bar{\partial}$-Lemma: for a global closed (p,q)-form α trivial in cohomology, there is a global $(p-1,q-1)$-form η such that $\alpha = \sqrt{-1}\partial\bar{\partial}\eta$.

Prove by going through identities from Hodge decompositions and using compatibility ($\alpha = \Delta G \Delta G \alpha$ and use the versions for ∂ and $\bar{\partial}$ respectively).

- Curvature form of holomorphic line bundle and the first Chern class

  1. Holomorphic line bundle.

  Begin with a rank 2 real vector bundle.

  Rank 1 complex vector bundle, i.e. complex line bundle: transition matrix preserving complex structure over fibre, i.e. "multiplication by a complex number".

  Holomorphic line bundle: holomorphic transition function (i.e. holomorphically-varying over base manifold).

  2. The first Chern class.

  For a complex line bundle L with any ∇, $\frac{\sqrt{-1}}{2\pi}[\nabla^2] = c_1(L) \in H^2(M;\mathbb{Z})$ using the sheaf exact sequence

$$0 \to \mathbb{Z} \to \mathcal{A} \to \mathcal{A}^* \to 0,$$

  which gives the long exact sequence of sheaf cohomology, and the identification between de Rham and sheaf cohomology spaces (noticing that as sheaf, closed and exact are the same). This is only the torsion-free component of the more topologically defined first Chern class.

  The $\mathbb{C}$-valued smooth function sheaf $\mathcal{A}$ is a fine sheaf, and so with vanishing sheaf cohomology spaces of positive degrees. Hence from the long exact sequence, $H^2(M,\mathbb{Z})$ is isomorphic to $H^1(M,\mathcal{A}^*)$ where the later one corresponds to the space of smooth complex line bundles over M (direct from definitons).

So the first Chern class decides the *smooth* structure of the complex line bundle.

In the case of holomorphic line bundle L with the hermitian metric h (over a complex manifold M), we take the corresponding holomorphic connection ∇ and by direct computation,

$$\sqrt{-1}\nabla^2 = -\sqrt{-1}\partial\bar{\partial}\log(|\sigma|_h^2),$$

where σ is a local holomorphic base vector field. (Hint: compute the action of ∇^2 on σ, using the definition of holomorphic connection.)

$\frac{\sqrt{-1}}{2\pi}[\nabla^2] = c_1(L) \in H^{1,1}(M;\mathbb{C}) \cap H^2(M;\mathbb{Z})$ using

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

in a similar way as before. Here the curvature form is clearly a $(1,1)$-form from the formula above.

Note: in future, the universal constant $\frac{1}{2\pi}$ (or $\frac{\sqrt{-1}}{2\pi}$) would constantly be ignored with no affect. Only the notation $H^{1,1}(M,\mathbb{C}) \cap H^2(M,\mathbb{Z})$ makes use of M being closed and Kähler.

The holomorphic function sheaf is not fine. (No partition of unit by holomprhic function, right?) The first Chern class is NOT going to decide the *holomorphic* line bundle.

This provides one way to understand Calabi's Conjecture.

II. Complex Monge-Ampère equation and Ricci flow

II-1. Complex Monge-Ampère equation

- Computation for Ricci curvature in Kähler setting.

  Recall:

  1) The Levi-Civita connection ∇ for (M, g) induces the holomorphic connection for $T^{1,0}M$.

  2) Ricci curvature is a "trace" of Riemannian curvature, and so Ricci form, $\mathrm{Ric}(Y, JZ) = Ricci(Y, Z)$, is the curvature form for $\wedge^n T^{1,0}M$ with the hermitian metric induced from g. (Explanation of $R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}}$ as a "trace": the cancelation when computing curvature form of the highest degree wedge bundle, even easier if using complex geodesic coordinates).

  So we have recovered the following classic computation in Kähler geometry:

$$\begin{aligned}\mathrm{Ric} &= -\sqrt{-1}\partial\bar{\partial}\log \left|\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n}\right|_g^2 \\ &= -\sqrt{-1}\partial\bar{\partial}\log det(g_{i\bar{j}}),\end{aligned}$$

  where $g_{i\bar{j}} = g_{\mathbb{C}}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right)$. Hence the study of Ric (Ricci form) is reduced to analyzing the volume form. This also reduces the order of derivative.

  In cohomology, we have

$$[\mathrm{Ric}] = c_1(\wedge^n T^{1,0}M) = c_1(T^{1,0}M) =: c_1(M),$$

  the first Chern class of (M, J).

Summary: given any Kähler metric ω, its Ricci form $\mathrm{Ric}(\omega)$ represents the first Chern class of (M, J) (in the cohomology space $H^2(M;\mathbb{Z})$).

If M is closed, then the cohomology space could be

$$H^{1,1}(M;\mathbb{C}) \cap H^2(M;\mathbb{Z}).$$

Remark: clearly, any deformation of the (almost) complex sturcture J will not change the class in $H^2(M,\mathbb{Z})$, and so, for example, there is a well-defined first Chern class for a symplectic manifold by considering only those compatible almost complex structures. The construction makes use of the compatible metric coming from the symplectic form and the almost complex structure.

Consider M being closed for a while.

In the other direction of **Summary**, we have the following important problem and its solution.

- Calabi's Conjecture and Calabi-Yau Theorem.

1. Calabi's Conjecture: over a closed manifold M, in any Kähler class α, for any real closed $(1,1)$-form T representing $c_1(M)$, there exists a unique Kähler metric ω with $[\omega] = \alpha$ and $\mathrm{Ric}(\omega) = T$.

Another way to understand: suppose M is simply connected. For the holomorphic line bundle $\wedge^n T^{1,0}M$, being trivial smoothly implies being trivial holomorphically.

2. Reduction to a complex Monge-Ampère equation.

Take a Kähler metric ω_0 in α.
$[\mathrm{Ric}(\omega_0)] = c_1(M) = [T]$, and so $[\mathrm{Ric}(\omega_0) - T] = 0$.
By $\partial\bar{\partial}$-Lemma, we have $\mathrm{Ric}(\omega_0) - T = \sqrt{-1}\partial\bar{\partial}F$ for $F \in C^\infty(M)$ unique up to an addtive constant.
The desired ω has to be in the form of $\omega_0 + \sqrt{-1}\partial\bar{\partial}u$ by $\partial\bar{\partial}$-Lemma, and we need $\mathrm{Ric}(\omega) = T$.
Hence $\mathrm{Ric}(\omega_0) - \mathrm{Ric}(\omega) = \sqrt{-1}\partial\bar{\partial}F$, which is

$$-\sqrt{-1}\partial\bar{\partial}\log\frac{\omega_0^n}{V_E} + \sqrt{-1}\partial\bar{\partial}\log\frac{\omega^n}{V_E} = \sqrt{-1}\partial\bar{\partial}F.$$

So $\sqrt{-1}\partial\bar{\partial}\log\frac{\omega^n}{\omega_0^n} = \sqrt{-1}\partial\bar{\partial}F$. Although $\frac{\omega_0^n}{V_E}$ and $\frac{\omega^n}{V_E}$ are locally defined, their quotient $\frac{\omega^n}{\omega_0^n}$ is a smooth function over M.

So M being closed gives $\log\frac{\omega^n}{\omega_0^n} = F + C$, and that is

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}u)^n = e^{F+C}\omega_0^n.$$

The constant C is clearly fixed by considering integrals over M and can indeed be absorbed by a proper choice of F. Obviously the above computation can be reversed.

Conclusion: the statement of Calabi's Conjecture is equivalent to the uniqueness and existence of solution (up to additive constants) for the following complex Monge-Ampère equation,

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}u)^n = e^F\omega_0^n$$

for "any" $F \in C^\infty(X)/\{\text{additive } \mathbb{R}\}$.

Uniqueness (easy exercise): assume two solutions u, v, and use
$0 = \int_X (u-v)\left((\omega_0 + \sqrt{-1}\partial\bar{\partial}u)^n - (\omega_0 + \sqrt{-1}\partial\bar{\partial}v)^n\right)$.

3. Calabi-Yau Theorem: existence is true.

C^0 estimate from Moser Iteration, Laplacian estimate and higher order derivative estimates.

4. Other Kähler-Einstein equations.

Different signs: $(\omega_0 + \sqrt{-1}\partial\bar{\partial}u)^n = e^{\pm u+F}\omega_0^n$.
Measure equation: $(\omega_0 + \sqrt{-1}\partial\bar{\partial}u)^n = f\omega_0^n$.
Degenerate case: $(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = f\Omega$ for $\omega \geqslant 0$.
More precisely, $[\omega]$ is NOT a Kähler class.

- Pluripotential Theory for Monge-Ampère operator and Kolodziej's breakthrough.

  1. The other part of the story for complex Monge-Ampère equation $\longrightarrow$ low regularity category.

  Problem: how to make sense of the complex Monge-Ampère equation when $\sqrt{-1}\partial\bar{\partial}u$ doesn't even exist pointwise (i.e. "measure equation" before).

  Answer: in the weak sense (measure and distribution).

  Positive current (Lelong).

  Plurisubharmonic function (Lelong, Oka).

  Monge-Ampère operator (Bedford-Taylor).

  Relative capacity (Bedford-Taylor).

  References: books by Lelong, Kolodziej and Demailly (online).

  Search for existence, uniqueness and regularity of the (weak) solution.

  2. Kolodziej's breakthrough (for ω_0 being Kähler).

  L^∞ estimate from measure "$L^{p>1}$" condition on the right hand side.

  Continuity.

  Hölder continuity.

II-2. Ricci flow

References: books by Chow and others.

- Set-up, uniqueness and short time existence.

$$\frac{\partial g(t)}{\partial t} = -2Ricci\left(g(t)\right), \quad g(0) = g_0.$$

"*Weak parabolicity from diffeomorphism invariance*" (Hamilton): the static equation of a parabolic equation would be elliptic, and so would have a finite number of solutions (M being closed). But this is not true considering action by diffeomorphism.

Weakly parabolic flow in the (positive) symmetric 2-tensor space (Hamilton).

Linearization: $\frac{\partial g}{\partial s} = v$ and $V = Tr_g(v)$, then the evolution term is linearized to be

$$\frac{\partial(-2Ricci)}{\partial s} = \Delta_L v + Symm(\nabla X),$$

where $X = \frac{\nabla V}{2} - div(v)$. There are second order derivative terms from ∇X other than the parabolic term $\Delta_L v$.

The symbol is weakly parabolic, i.e. with some eigenvalues being 0. Hamilton's computation makes use of the contracted Second Bianchi Identity, which has a lot to do with the term X above.

DeTurck's Trick: reduce to a parabolic flow by a family of diffeomorphisms, i.e. "fixing gauge".

Ricci-DeTurck flow:

$$\frac{\partial g}{\partial t} = -2Ricci + Symm(\nabla W), \quad g(0) = g_0,$$

where the 1-form $W =$ "$Tr(\nabla - \bar{\nabla})$" with $\bar{\nabla}$ be the Levi-Civita connection for a fixed metric.

Key feature of such W: the linearization is $-X$ up to lower order terms in $v = \frac{\partial g}{\partial s}$. So Ricci-DeTurck flow is parabolic which guarantees uniqueness and short time existence.

The dual of W, W^* is a time-depending vector field, and so solving $\frac{\partial \Phi}{\partial t} = -W^*$ with initial data $\Phi(0) = id$ gives a one-parameter family of diffeomorphisms.

$\Phi^* g$ solves Ricci flow with the initial g_0 by noticing $L_{W^*} g = Symm(\nabla W)$.

This gives *short time existence* (i.e. local existence).

How about uniqueness?

A solution of Ricci flow $g(t)$, gives a one-parameter family of diffeomorpisms $\Psi(t)$ as the solution of harmonic map heat flow (parabolic) with $\Psi(0) = id$.

The pullback $(\Psi^{-1})^* g$ is the solution for Ricci-Deturck flow with initial data $g(0)$. (*unique*)

By rewriting the harmonic map heat flow, one realizes that Ψ is nothing but the Φ above. (*unique from a solution of Ricci-Deturck flow*)

So one can get back to the solution of Ricci flow and achieve uniqueness.

Key of the argument:

from Ricci flow to Ricci-Deturck flow, the one-parameter family of diffeomorphisms is obtained by solving harmonic map heat flow;

from Ricci-DeTurck flow to Ricci flow, the one-parameter family of diffeomorphisms is by integrating a time-depending vector field.

Remark: backward uniqueness. (Brett Kotschwar)

- Finite time singularity.

  PDE point of view: blow-up of the solution (metric coefficients) or the derivatives (covariant derivative).

  Hamilton: blow-up of $|\mathrm{Rm}\,(g(t))\,|_{g(t)}$ (in sight of Shi's estimates).

  Perelman: local noncollapsing result.

  Original form, *κ-noncollapsed below the scale ρ*: $\forall r \in (0, \rho)$, in $B(x, r)$, $|\mathrm{Rm}| \leqslant r^{-2}$, then $\mathrm{Vol}(B) \geqslant \kappa r^n$.

  (Two proofs: W-functional, stronger but only for closed manifold; reduced distance, weaker but also for complete manifold.)

  There are improved versions for local noncollapsing results by Sesum-Tian-Wang and others. The version by Topping which changes $|Rm|$ to $|R|$ is the best so far.

  <u>It is very useful in the study of Ricci flow.</u>

  a) uniform injectivity radius lower bound for dilation sequence (and hence smooth Cheeger-Gromov-Hamilton convergence for the dilations at finite time in general), resulting in singularity models which can be classified in low dimensions.

  Cheeger-Gromov-Taylor: volume noncollapsed (i.e. ratio lower bound) implies injectivity radius lower bound.

  Explain: enemies for injectivity radius are conjugate point (exp being degenerate, controlled by curvature) and geodesic loop, 2 *minimal geodesics. Dilaten* *to have flat space with volume of Euclidean growth, which is Euclidean space, and a small loop. Contradiction!*

b) rule out some "nice" solitons as singularity models for the sake of arranging surgery.

Examples: (steady solitons)

i) cigar soliton $\frac{dx^2+dy^2}{1+x^2+y^2} = \frac{dr^2}{1+r^2} + \frac{r^2}{1+r^2} d\theta^2$ over $\mathbb{R}^2$, exponentially asymptotically flat but cylindrical. So local noncollapsing result rules out the product of cigar with $\mathbb{R}^1$ as a singularity model.

ii) 3-D Bryant soliton ($\sim dr^2 + r g_{S^2}$ with $|\mathrm{Rm}| \sim O(\frac{1}{r})$), is not ruled out. It is asymptotically necklike at infinity.

After Perelman's:

Sesum: blow-up of $|\mathrm{Ric}\,(g(t))\,|_{g(t)}$. In real dimension 3, blow-up of scalar curvature. (David Glickenstein's result is applied. Perelman's local non-collapsing result can be avoid.)

Wang and others: blow-up of the space-time integral of power of curvature norm. ("Ricci lower bound assumption")

Enders-Müller-Topping (and others): blow-up of scalar curvature for Type I singularity (compact or complete). (Perelman's pseudolocality result using reduced distance and Naber's result to obtain the soliton as dilation limit.)

Kähler-Ricci flow over closed manifold (Z.): scalar curvature blows up. (Argument of very different flavour.)

[Blow-up of scalar curvature is conjectured for finite time singularity of Ricci flow in general.]

- Geometric implications.

  This is by no means a complete list.

  [Uhlenbeck's trick: time-depending frame for a neat expression on Riemannian curvature evolution.]

  Hamilton: $\dim_{\mathbb{R}} M = 3$ and $\mathrm{Ric}\,(g_0) > 0 \Longrightarrow M$ is a quotient of S^3.

  Notion: invariant set for Ricci flow curvature ODE.

  Chow and Hamilton: flows over closed Riemann surfaces (equivalent to Kähler-Ricci flow over closed M with $dim_{\mathbb{C}} M = 1$) being completely settled.

  Metric evolves in a fixed conformal class and scalar curvature is "everything".

  Tools: Harnack and entropy are introduced by Hamilton, assuming $R > 0$ in the sphere case, when bounding R (from above). This additional assumption is removed by Chow using $R + C$. The adjustment of Harnack part is quite direct. The entropy part requires different consideration.

  Perelman followed by several groups of people: Poincaré Conjecture and Geometrization Program.

  Böhm and Wilking: $\mathrm{Rm}\,(g_0) > 0 \Longrightarrow M$ is quotient of S^n.

  Notion: curvature pinching set for Ricci flow curvature ODE. [pinching family as the general setting of algebra structure and generalized pinching set for application to flow.]

Schoen-Brendle: curvature (strictly and pointwise) $\frac{1}{4}$-pinch indicates smooth geometry of space form.

Main difficulty:

Nonnegative isotopic curvature: weak but preserved by Ricci flow ($\dim_{\mathbb{R}} = 4$ is done by Hamilton). Meanwhile, it can not prevent premature finite time singularities and so one needs to do surgeries (Chen-Zhu for 4-fold).

$\frac{1}{4}$-pinch curvature: strong but not preserved by Ricci flow.

Somewhere in between lies the proper curvature condition preserved by Ricci flow and pinching to constant curvature.

III. Set-up of Kähler-Ricci flow

- Set-up, cohomology information and scalar potential flow.

  This is the special case of Kähler class being fixed.

  1. Some history: Huaidong Cao and others.
     - Alternative proof of Yau's Theorem (not just "Ricci-flat case"):
       $\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}\,(\widetilde{\omega}_t) + T, \quad \widetilde{\omega}_0 = \omega_0$
       for a Kähler metric ω_0 and T representing $c_1(X)$.
       Convergence.
     - $c_1(X) < 0$ case:
       $\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}\,(\widetilde{\omega}_t) - \widetilde{\omega}_t, \quad \widetilde{\omega}_0 = \omega_0$
       with $[\omega_0] = -c_1(X)$.
       Convergence.
     - Fano (or $c_1(X) > 0$) case:
       $\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}\,(\widetilde{\omega}_t) + \widetilde{\omega}_t, \quad \widetilde{\omega}_0 = \omega_0$
       with $[\omega_0] = c_1(X)$.
       Stability and convergence.
  2. Special features of Kähler-Ricci flow.
     - Equivalence of metric (form) flow and scalar (metric potential) flow (as explained in detail in Chau's work).
       Scalar flow (discussed in detail later) to metric flow: take $\sqrt{-1}\partial\bar{\partial}$.
       Metric flow to scalar flow: time ODE for each space point and uniqueness.

– **Kähler-Ricci flow is Ricci flow with initial metric being Kähler**: uniqueness of Ricci flow and existence of Kähler-Ricci flow. (This settles the logic.)

– Easier uniqueness and short time existence for Kähler-Ricci flow (in the space of Kähler metrics).

  It is a parabolic flow! Ways to see this:

  (a) Scalar (metric potential) flow: linearized to have the leading term being Laplacian.

  (b) Metric form flow: in the space of real, smooth, **closed** $(1,1)$-forms, the leading term is Laplacian (in sight of $(\phi_{i\bar{j}})_k = (\phi_{k\bar{j}})_i$).

- Flow with evolving class and more general setting.

  1. $\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}\,(\widetilde{\omega}_t) - \widetilde{\omega}_t, \quad \widetilde{\omega}_0 = \omega_0$ with ω_0 being ANY Kähler metric.

  Cohomology.

  Scalar flow.

  2. $\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}\,(\widetilde{\omega}_t) - \widetilde{\omega}_t + \mathrm{Ric}(\Omega) - L, \quad \widetilde{\omega}_0 = \omega_0$ with ω_0, Ω and L being any Kähler metric, a smooth volume form and a real smooth closed $(1,1)$-form.

  Cohomology.

  Scalar flow.

- Optimal existence result and cases of singularities.

  1. Tian-Z. (a weaker version by Cascini-La Nave):

  Kähler-Ricci flow exists as long as the class remains to be Kähler (from formal consideration).

  Idea of proof: choice of background form, estimates and equivalent metric flow.

  2. Cases from picture in Kähler cone: infinite and finite time.

- Convergence in non-degenerate case.

  Generalization of H. D. Cao's result in $c_1(X) < 0$ case by removing the cohomology restriction on the initial class.

- Relation with other versions of Kähler-Ricci flow.

  1. Classic Ricci flow ("Ricci-flat"):
  $\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}\left(\widetilde{\omega}_t\right), \quad \widetilde{\omega}_0 = \omega_0.$
  Rescaling of time and metric: the same infinite and finite time.

  Evolution of class.

  2. "Fano":
  $\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}\left(\widetilde{\omega}_t\right) + \widetilde{\omega}_t, \quad \widetilde{\omega}_0 = \omega_0.$
  Rescaling of time and metric: infinite time becomes finite time.

  Evolution of class.

  3. The implication of Sesum-Tian's result (following Perelman's idea): the recent work by Song on finite time extinction of classic Kähler-Ricci flow.

IV. Some recent topics on Kähler-Ricci flow

Focus on the flow $\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}\,(\widetilde{\omega}_t) - \widetilde{\omega}_t, \quad \widetilde{\omega}_0 = \omega_0.$

- Minimal manifold of general type.

  The canonical class $K_X = -c_1(X)$ is nef. and big.

  Translation for differential geometer: there exists $P : X \to \overline{X} \subset \mathbb{CP}^N$ with $dim_{\mathbb{C}}\overline{X} = dim_{\mathbb{C}}X$ and $P^*(H) = mK_X$ for a positive integer m.

  Semi-ample: quite essentially involved in the argument up to this point.

  Theorem (Tian-Z.):

  1) Smooth local convergence (of $\widetilde{\omega}_t$).

  2) Weak global convergence.

  3) Uniqueness of the limit (singular Kähler-Einstein metric).

  4) Continuity of the limiting potential.

  5) Uniform control of scalar curvature.

  Idea of proof:

  1) and 2): Tsuji's trick, $K_X - \epsilon E > 0$.

  3): an observation.

  4): pluripotential theory.

  5): Schwarz Lemma computation, gradient and Laplacian estimates.

- Minimal manifold of lower Kodaira dimension.

Song-Tian: the limit should be collapsed to the base manifold with the fibration structure coming in as Weil-Peterson metric (moduli information).

More recently, Fong-Z.: metric collapsing for regular fibration. [One could probably work out a degenerate version for general fibration by combining with the argument in Song-Tian.]

Tosatti and others: using elliptic setting for Calabi-Yau case.

- Generalization of Kolodziej's results.

  Under assumption similar to minimal manifold of general type,

  L^∞-estimate:

  Eyssidieux-Guedj-Zeriahi

  Demailly-Pali

  Z.

  Continuity:

  Z. by applying Fornaess-Narasimhan's extension result for weak PSH function and Kolodziej's original argument.

  Collapsing case:

  Eyssidieux-Guedj-Zeriahi and Demailly-Pali can still be applied to achieve L^∞-estimate.

- Manifold of general type.

  Finite time volume non-collapsing case: similar convergence.

[L^∞-estimate from simple flow argument for finite time singularity case, requiring only semi-ample limiting class.]

- What to expect for finite time singularities?

  Tian's Program.

  Recent justification for examples about the pictures of contraction by Song-Weinkove and flip by Song-Yuan.

  In order to get a canonical limit, we need to *continue* it to the time infinity. That leads us to next topic.

- Weak flow.

  Chen-Tian-Z.: for singular initial metric with bounded potential, one can define a (unique) weak flow which becomes smooth instantly.

  Application in general type surface case.

  Song-Tian and Z.: more general pictures.

- Further singularity analysis.

  Z.: scalar curvature behavior.

  Song-Tian: partial metric information from Schwarz Lemma.

  Z.: Ricci lower bound. [related to the examples by Knopf and Maximo on the sign of Ricci tensor along Ricci flow]

- Flows in complete non-compact setting.

  Shi; Chau, Tam; Chen-Zhu; Lott-Z.; Rochon-Z.