

Schur and the birth of projective representations

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Abstract. We provide a commentary on Schur's paper [19] on the representations of a group via fractional linear transformations, which was published in *Crelle* in 1904. We give some historical context for Schur's work, and discuss the results and methods of the paper in detail. Our aim is to make this beautiful paper more accessible to the modern reader.

1. Introduction

Schur's paper [19], see also the translation [48], on the representations of groups via fractional linear transformations is remarkable in several aspects. It emerged only a few years after Frobenius, Molien, Burnside and Schur began studying the representation theory of finite groups. The concepts of central extension and (Schur) multiplier which it introduces have had a profound impact on group theory. One sees one of the first theorems proved via group presentations, and the first glimmer of cohomological methods. Both ideas would go on to deeply influence 20th century algebra and topology. It is also interesting to study a paper written at the point where the group concept was clearly articulated and understood, but before abstract axiomatic mathematics had become mainstream.

We attempt to provide a motivated historical introduction to Schur's work. We also include a modern discussion of all results of the paper, closely following Schur's line of argument. We warn the reader that the author is not a historian of mathematics, and is far from an expert on this period of early 20th century algebra. On the other hand, even as an amateur the reading of Schur's and other papers of the time is a meaningful and rewarding activity. Our goal is to help the modern mathematician read and achieve a greater appreciation of Schur's work.

This paper is structured as follows: we first discuss the evolution of the group concept through the works of Lagrange, Galois, Cayley, von Dyck and Frobenius. We then discuss the emergence of the theory of group representations in the work of Frobenius and Molien. We also discuss the theory of associative algebras (then called hypercomplex number systems) and the comparatively small impact this had on Frobenius and Schur. We discuss his results in detail, as a companion to the English translation of the paper [50], and point out some historical curiosities along the way. We do not discuss Schur's life; for this the reader is referred to [30, 40, 41].

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2. Historical context

Studying the history of the development of a subject has a fractal nature: the more one examines it, the more complexity and interconnectedness it reveals. Our aim here is to provide some context for Schur's paper, whet the reader's appetite, and to point out the excellent existing literature. Our principal sources when preparing this section were the books of Curtis [30] and Wussing [52] and the articles of Hawkins [33–35].

2.1. The concept of a group. The group concept is surely one of the most useful in mathematics. The three axioms (associativity, unit, and existence of inverses) are close to universally accepted, and it is difficult to anticipate the incredible richness that emerges from these three axioms.¹⁾ ²⁾ One can view the axiomatization of groups as one of the first successes of the formal approach to mathematics, advocated by Hilbert and others. However it took great effort (and most of the 19th century) for this idea to crystallize.

In [52], Wussing undertakes a detailed study of the emergence of the group concept. His conclusion is that the widespread notion that the group concept arose via abstraction of the notion of a permutation group is simplistic. In fact, group theory had three distinct and equally important historical roots in the 19th century: the theory of algebraic equations, number theory and geometry.

For most of the 19th century, the modern group concept was not explicitly articulated, which makes the tracing of its historical roots all the more difficult. For example, in Galois' work, the concept of a "group" is fluid, becoming more and more precise as his work evolves [52, p. 112]. Throughout the 19th century, it seems that most authors had in mind the object that Burnside defines in his first English-language textbook on group theory [1, Chapter 2], which was published in 1897:

Let $A, B, C \dots$ denote a set of operations, which can be performed on the same object, or set of objects. Suppose this set of operations has the following characteristics...

Burnside then goes on to what nowadays appears a rather convoluted discussion of associativity and the existences of inverses. The key point is that a group is a concrete collection of operations, and the modern idea of "abstract group" is entirely absent.

As an example of the impact of a missing definition, the classification theorem for finite abelian groups is implicit in Gauß's *Disquisitiones Arithmeticae* (see [26, p. 184]) and explicit in Kronecker's 1870 work [14], although without reference to the group concept. In 1879, Frobenius and Stickelberger reprove the same theorem [6], this time with the group concept

¹⁾ Mathematicians were already aware of this richness at the time of Schur's work. As Hawkins [34, §4] points out, Pierpont [48] wrote in 1904 that "the group concept, hardly noticeable at the beginning of the century, has at its close become one of the fundamental and most fruitful notions in the whole range of our science."

²⁾ Other mathematical concepts like "space" and "function" are arguably more universal, but there is no widely agreed upon single set of axioms describing these concepts [28, 32].

in mind. It is clear from their introduction that they are beginning to be aware of the group concept as a unifying framework.³⁾

2.2. Emergence of the axioms of a group. In a remarkable paper in 1854, Cayley [2] implicitly defines what we would today call an abstract group. He describes the multiplication table and classifies groups of prime order and of order up to 6. Cayley's point of view seems to have been ignored for over two decades, before obtaining general currency. Cayley returned to this theme in 1878 [3], where he articulated his famous dictum: "A group is defined by means of the laws of combination of its symbols."

A turning point in the development of the group concept is provided by Dyck's papers [4, 5]. He defines what we would today call a finitely-presented group. As is apparent when reading the introductions to these papers, Dyck is motivated by an attempt to unify the various appearances of the group concept in number theory, algebra and geometry. Dyck's paper plays a crucial role in Schur's work: in order to construct the Schur covering group, Schur first considers an infinite finitely-generated group and then constructs the covering group as a quotient of this group, which he argues is finite. We thus encounter a compelling early example where infinite groups are used crucially in a question concerning only finite groups.

By the 1880s it is clear that the group concept has reached something close to its modern conceptual form. However, interestingly the axioms used by mathematicians in the 1880s and 1890s still differed from the modern axioms. Here is Frobenius' definition from his famous Crelle paper of 1884 on the Sylow theorems [7]:

- (i) Each two elements A and B determine a *unique* third element, that will be denoted by AB .
- (ii) From the equation $AC = BC$ or $CA = CB$ follows $A = B$.
- (iii) For the operation, which out of A and B produces AB , the associative rule holds $(AB)C = A(BC)$. (However, the commutative law $AB = BA$ does not necessarily hold.)
- (iv) The number of elements is finite.

Frobenius then goes on to *prove* the existence and unicity of the unit element, and establish the existence of inverses.

Remark 2.1. It is interesting to trace the emergence of the inverse axiom. All early sources studying finite groups prove the existence of inverses, rather than asserting their existence. Klein states that in his investigations with Lie in the 1870s "Lie found it necessary to require that for each A the group should contain its inverse A^{-1} " [13, Kapitel 8]. Dyck's work alluded to above is notable in that the existence of an inverse is axiomatized (see [4, p. 74]). However, even around 1904 at the time of Schur's article [19] it was still not standard to include the existence of the inverse as part of the definition of a group (see e.g. [12]). In [19], Schur checks the cancellative law (i.e. axiom (ii) above), rather than demonstrate inverses.

The modern mathematician is now used to axioms, and usually doesn't think twice about the axioms of a group. However, it is interesting to reflect on what might be lost in our modern

³⁾ In [19], Schur uses the classification of *finitely-generated* abelian groups in a crucial way. Amusingly, he cites [6] which only proves the theorem for *finite* abelian groups.

approach. In this direction, the following opinion of Klein [13] from a lecture delivered around 1915 is thought-provoking. After introducing the modern group axioms, he writes:

... there is a complete loss of the appeal to the imagination. On the other hand, the logical skeleton is carefully prepared. ... This abstract formulation is excellent for the working out of proofs but it does not help one find new ideas and methods. Rather, it represents the conclusion of an advance. It therefore seems to ease the learning process, for it helps one prove known theorems in a flawless and simple manner. On the other hand, it makes matters far more difficult for the mind of the student, for he confronts something closed, does not know how one arrives at these definitions, and can imagine absolutely nothing. In general, the disadvantage of the method is that it fails to encourage thought. All one must beware of is that one does not violate the four commandments [i.e. the group axioms].⁴⁾

2.3. Representations and the character table. Following a detailed analysis of the 19th century literature, Hawkins [34, §5] traces the roots of character theory and representation theory to at least four places: 1) the structure theory of finite-dimensional algebras (Molien, Cartan); 2) the theory of Lie groups and algebras (Cartan, Burnside); 3) Klein's form problem (Maschke); 4) number theory and group determinants (Dedekind, Frobenius). For Schur's paper, it is this last motivation that exerts the strongest influence, and we discuss this first. For a detailed historical treatment the reader is referred to the papers of Hawkins [33, 35] and for a more mathematically driven discussion to [30, 46].

The notion of a character of an *abelian* group emerged naturally at the beginning of the 19th century in Gauß's work on quadratic forms, and on Dirichlet's work on primes in arithmetic progressions. Dedekind played a substantial role in advancing it as a unifying concept [36, §2].

Around 1880, motivated by number-theoretic questions, Dedekind began studying the *group determinant*, which he defined as follows. For a finite group G , consider commuting variables x_g indexed by elements of G , and consider the matrix

$$(2.1) \quad (x_{gh^{-1}})_{g,h \in G}$$

for some fixed ordering $g_1, g_2, g_3, \dots$ of its elements. (Thus, this matrix is, up to reordering of the columns, simply an encoding of the multiplication table of G .) The group determinant is its determinant:

$$\Phi_G = \det((x_{gh^{-1}})_{g,h \in G}) \in \mathbb{C}[x_g \mid g \in G].$$

Dedekind was able to prove that if G is abelian then Φ_G factors into linear factors, given by the characters of G :

$$\Phi_G = \prod_{\chi \in \hat{G}} \left(\sum_{g \in G} \chi(g) x_g \right).$$

In two letters to Frobenius in 1896, Dedekind defined the group determinant, provided the factorization for several small non-abelian groups, and stated some conjectures.

Motivated by Dedekind's questions, Frobenius was able to completely settle the matter in two papers [8, 9] which appeared at the end of 1896.⁵⁾ He proved that for any finite group

⁴⁾ Translated by Wussing [52, p. 228].

⁵⁾ Frobenius' struggle towards these theorems in the second half of 1896 is vividly recounted, based on letters exchanged between Dedekind and Frobenius, in [33, 35].

one has a factorization

$$(2.2) \quad \Phi_G = \prod_{\chi \in \hat{G}} (\Phi_\chi)^{d_\chi}$$

indexed by the characters $\hat{G}$ of G , where Φ_χ is an irreducible polynomial of degree d_χ . He also proved that if one sets $x_g = x_{g'}$ whenever g and g' are conjugate in G , then each irreducible polynomial may be factored into linear factors

$$(2.3) \quad \Phi_\chi = \left(\sum_{C \in \text{cl}(G)} \frac{|C|}{d_\chi} \chi(C) x_C \right)^{d_\chi},$$

where $\text{cl}(G)$ denotes the set of conjugacy classes in G , $\chi(C)$ denotes the value of χ on such a conjugacy class, and $x_C = x_g$ for any $g \in C$.

Here we are being somewhat ahistorical: we assume the reader knows what a character is, and use them to factor the group determinant. In Frobenius' work in 1896 the theory proceeded the other way around—the characters of a finite group G are defined so as to enable the factorization (2.3). Thus the group determinant is central to early treatments. The following year Frobenius established [10] that his characters were indeed the traces of elements of g in the irreducible representations of G , recovering the modern definition.

As Frobenius' student, Schur was deeply familiar with the group determinant, and the connection between the characters and the irreducible representations of G . The group determinant plays an important role in [19]. Frobenius' arguments are very complicated, and in [20] Schur achieved a major simplification in 1902 which became a standard reference. It is this paper that introduces Schur's lemma [20, Satz I].

2.4. Hypercomplex number systems. The modern reader should have no trouble translating both Dedekind's question and Frobenius' answer into modern language. Indeed, if $\mathbb{C}G$ denotes the group algebra, then (2.1) is simply the matrix of the left action of a “general element”

$$\gamma = \sum_{g \in G} x_g g$$

in the regular representation. Basic representation theory provides the decomposition

$$\mathbb{C}[G] = \bigoplus_{\chi \in \hat{G}} V_\chi^{\oplus d_\chi}$$

and (2.2) is immediate.⁶⁾ Moreover, for any conjugacy class C in G the element $\sum_{g \in C} g$ is central, and hence acts as a scalar c_χ in any irreducible representation, by Schur's lemma. Taking traces yields the relation

$$c_\chi = \frac{|C|}{d_\chi} \chi(C),$$

which implies Frobenius' linear factorization (2.3).

⁶⁾ With the exception of the fact that each Φ_χ is irreducible, which follows with a little more thought. For a modern proof see [46].

It is the absence of the group algebra that is most striking to the modern reader of Frobenius and Schur. At the time, associative algebras were known as hypercomplex number systems, and there was considerable interest in these structures from several mathematicians [35, §1]. It is noteworthy that in his letters, Dedekind repeatedly asked Frobenius whether his theory can be interpreted in terms of hypercomplex number systems [33].

Remarkably, at almost the same time as Frobenius was proving his theorems on the factorization of the group determinant, Molien established many of the same theorems using the theory of hypercomplex number systems developed in his thesis. Molien appears to have been deeply influenced by Killing's work on Lie algebras, as well as Klein's work. He realized that finite groups provide a rich class of hypercomplex number systems via their group algebras. He was able to use a result from his thesis [15] to deduce that group algebras were isomorphic to a direct product of matrix rings, and deduce the fundamental theorems of group representation theory [16, 17]: that the number of irreducible representations agrees with the number of conjugacy classes, the fundamental orthogonality relations for characters, and that the character degrees divide the order of the group [34, §4].

Throughout the early development of the subject, Frobenius was aware of the theory of hypercomplex number systems and this had some impact on his work [35, § 6]. In a letter to Dedekind in 1896 he wrote:

Your ideas involve thoughts about which I presently have a very obscure notion. The theory of groups is concerned with only one operation which obeys certain rules. In this respect, it reminds one of tightrope-walking, where the feet have only one dimension at their disposal. Many times I have thought about how, to the multiplication of elements, one could adjoin another operation (addition).⁷⁾

When Frobenius became aware of Molien's work he was quick to acknowledge its importance (see [12]). The group algebra and module point of view seems never to have been appealing to Schur [42].⁸⁾ We owe its prominence today largely to the work of E. Noether [53].

3. Discussion of the paper

We now turn to Schur's paper in detail. We discuss its contents, using modern notation. Our aim is to help the reader follow what Schur is doing as efficiently as possible.

3.1. On the introduction to Schur's paper. Throughout the second half of the 19th century the classification of permutation groups on n letters for small values of n was a well-known problem. It appears that the corresponding problem for matrix groups (i.e. the determination of finite subgroups of $GL_n(\mathbb{C})$ for small values of n) received much less attention. Famously, Burnside wrote in his introduction to [1]:

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily

⁷⁾ See [35, § 6], translation due to Hawkins.

⁸⁾ See [41, § 5] for an interesting anecdote.

by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.⁹⁾

In contrast, Klein emphasized the importance of this problem in the 1880s and 1890s [34, §4]. A well-known result of Jordan stated that this number is finite.

Schur opens his paper with a discussion of the question of the finite subgroups of $GL_n(\mathbb{C})$, and its apparent difficulty. He points out that representation theory furnishes a sort of converse to this problem—namely, given a finite group G , its representations provide all ways in which G can be realized as a matrix group. He notes that the theory of Frobenius and Molien provides a solution to this problem. (Interestingly, solving this problem seems to have been a primary motivation for Molien [34, §4], whereas for Frobenius this consequence appears as a happy accident.) Schur then goes on to address the problem studied in this work, namely the classification of representations of a fixed finite group via fractional linear transformations. (Thus, rather than classifying realizations inside $GL_n(\mathbb{C})$, Schur aims at classifying realizations inside the projective general linear group $PGL_n(\mathbb{C})$.)

He then observes that a representation by fractional linear transformations of a finite group G is the same thing as an assignment $x \mapsto A_x$ of matrices to each element of G so that for all $x, y \in G$ the relation $A_x A_y = r_{x,y} A_{xy}$ holds, for some complex numbers $r_{x,y}$. In the body of the paper the numbers $r_{x,y}$ play a central role. At this point Schur simply points out that representations correspond to such systems of numbers, and that if all $r_{x,y} = 1$ then one recovers the theory of representations via linear transformations. He introduces crucial basic notions like degree [Grad], equivalence [Äquivalenz] and irreducibility [which Schur calls Primitivität, following Frobenius], exactly as in the case of “ordinary” representations.

He also points out the crucial role played by central extensions. Namely, consider a short exact sequence

$$A \hookrightarrow G \twoheadrightarrow H$$

where A is a central subgroup of G . By Schur’s lemma, A acts by scalars on any irreducible representation of G , and hence furnishes an irreducible projective representation of H . This tight connection between central extensions and projective representations is a major theme of the paper.

Schur then summarizes the main results of the paper:

- (i) For any finite group H , there exists a central extension as above such that all projective representations of H lift to ordinary representations of G . Schur calls such a group sufficiently augmented [hinreichend ergänzt]. Such a group of minimal order is called a representation group [Darstellungsgruppe]. Nowadays, such groups are called *Schur covering groups*.
- (ii) In general, for each H there are several non-isomorphic covering groups G , however they all have the same order and the kernel of the surjection $G \twoheadrightarrow H$ provides a finite abelian group, which depends only on H . Schur terms this group the multiplier [Multiplikator]. Nowadays, this is called the *Schur multiplier* of H and may be identified with the second homology group $H_2(H, \mathbb{Z})$.

⁹⁾ Burnside writes in the preface to the second edition (published in 1911) that “... the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good. In fact, it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions.”

Schur points the reader to a number of results at the end of the paper which often allow the efficient determination of the multiplier of a finite group. He promises further theory and computations for specific groups in future works, which include [21, 22].

Remark 3.1. The reader may notice that several of Schur's early papers (including this one) are signed "J. Schur". An explanation is given in [43, pp. i-li] quoting Alfred Brauer:¹⁰⁾ "In the first of these papers Schur's first name had been abbreviated wrongly and Schur considered it proper not to change this for the subsequent papers in the journal. But in fact he always attached great importance to being quoted as I. Schur. Nevertheless, even today his first name is sometimes wrongly abbreviated."

3.2. Interlude concerning group cohomology. In the following discussion, it will be useful to have the modern terminology of group cohomology at our disposal. Given a discrete group G its cohomology $H^*(G, \mathbb{C}^*)$ is defined to be the cohomology of the complex

$$C^m(G) = \{f: G^m \rightarrow \mathbb{C}^*\}$$

(where $C^0(G)$ is understood to be maps $f: \{1\} \rightarrow \mathbb{C}^*$) and differential

$$d^m f(g_1, \dots, g_{n+1}) = \prod_{i=0}^{n+1} f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1})^{(-1)^i}$$

where, by convention, the 0^{th} and final terms are $f(g_2, \dots, g_{n+1})$ and $f(g_1, \dots, g_n)^{(-1)^{n+1}}$.

One checks easily that d^0 is trivial, and that a 1-cocycle is a map $f: G \rightarrow \mathbb{C}^*$ satisfying

$$1 = f(g_1)f(g_1 g_2)^{-1}f(g_2) \quad \text{for all } g_1, g_2 \in G$$

or in other words

$$f(g_1 g_2) = f(g_1)f(g_2).$$

Thus $H^0(G, \mathbb{C}^*) = \mathbb{C}^*$ and $H^1(G, \mathbb{C}^*) = G^\vee$ is the character group of G . Now, 2-cocycles are maps satisfying

$$1 = f(g_2, g_3)f(g_1 g_2, g_3)^{-1}f(g_1, g_2 g_3)f(g_1, g_2)^{-1} \quad \text{for all } g_1, g_2, g_3 \in G,$$

or in other words,

$$f(g_1, g_2)f(g_1 g_2, g_3) = f(g_2, g_3)f(g_1, g_2 g_3) \quad \text{for all } g_1, g_2, g_3 \in G.$$

Such a two cocycle is cohomologous to zero if it is of the form $\frac{f(g_1)f(g_2)}{f(g_1 g_2)}$ for some $f: G \rightarrow \mathbb{C}^*$.

In a similar way one defines homology groups $H_*(G, \mathbb{Z})$ and one has

$$H^i(G, \mathbb{C}^*) = H_i(G, \mathbb{Z})^\vee$$

where $(-)^\vee$ denotes the dual abelian group.

Remark 3.2. Group cohomology is a big subject with connections to many areas of mathematics. For detailed treatments, see [31, Chapter 15] and [24, 25, 27]. For an inspiring history of the development of group cohomology see [47] (see also Remark 3.6 below).

¹⁰⁾ Alfred Brauer served as Schur's assistant for many years in Berlin, and is the brother of the famous group theorist Richard Brauer.

3.3. Schur's Section 1. Schur begins with an assignment $x \mapsto A_x$ of matrices to the elements $x \in H$, “belonging to the number system $r_{x,y}$ ”, i.e.

$$(3.1) \quad A_x A_y = r_{x,y} A_{xy} \quad \text{for all } x, y \in H.$$

By expanding the associative law, he deduces that $r_{x,y}$ is a 2-cocycle.

He then proceeds to the first theorem of the paper, which states that there exist projective representations associated to any 2-cocycle. He proves this by producing explicit $|H| \times |H|$ -matrices, satisfying (3.1). Let us give an alternative explanation using the group algebra $\mathbb{C}H = \bigoplus_{z \in H} \mathbb{C}z$ which is close to Schur's argument. Consider the operator of left multiplication by x , which in the basis of group elements is given by

$$L_x: z \mapsto xz.$$

(When written in a basis, we get a permutation matrix of size $|H|$.) We can twist this operator using our scalars $r_{x,y}$:

$$L'_x: z \mapsto r_{x,z} xz.$$

(When written in a basis, we get a matrix with one non-zero entry per row and column, with entries determined by our 2-cocycle $r_{x,z}$.) Now we easily calculate, using the 2-cocycle condition

$$L'_x L'_y(z) = L'_x(r_{y,z} \cdot yz) = r_{x,yz} r_{y,z} \cdot xyz = r_{x,y} r_{xy,z} \cdot xyz = r_{x,y} \cdot L'_{xy}(z).$$

Or in other words, the operators L'_x satisfy the relations

$$(3.2) \quad L'_x L'_y = r_{x,y} L'_{xy} \quad \text{for all } x, y \in H.$$

These are Schur's matrices satisfying (3.1).

Remark 3.3. One can consider an algebra $\mathbb{C}^r H$ with generators corresponding to the elements of the group, and multiplication determined by $g \cdot h = r_{g,h} gh$. This object is usually called the *twisted group algebra*, and representations of $\mathbb{C}^r H$ are exactly projective representations of H associated to the 2-cocycle r . For more on twisted group algebras and their relations to projective representations, see [29, 39, 44, 45].

Schur then points out that if we replace our matrices A_x above by $A'_x = c_x A_x$ for some scalars $c_x \in \mathbb{C}^\times$ then our 2-cocycle above is transformed into a new 2-cocycle

$$r'_{x,y} = \frac{c_x c_y}{c_{xy}} r_{x,y}.$$

Schur calls such 2-cocycles associated [assoziert]. We saw in the previous section that this is precisely the condition for $r_{x,y}$ and $r'_{x,y}$ be cohomologous, i.e. represent the same element in $H^2(H, \mathbb{C}^*)$.

He then proceeds to argue that $H^2(H, \mathbb{C}^*)$ is finite. To this end, let d_x denote the determinant of L'_x . Taking determinants on both sides of (3.2) yields

$$r'_{x,y}^{|H|} = \frac{d_x d_y}{d_{xy}}.$$

Thus, if we replace $r_{x,y}$ by the cohomologous 2-cocycle

$$s_{x,y} = \frac{\delta_x^{-1} \delta_y^{-1}}{\delta_{xy}^{-1}} r_{x,y} \quad \text{for all } x, y \in H$$

where δ_x are fixed $|H|$ -roots of d_x , then

$$s_{x,y}^{|H|} = 1 \quad \text{for all } x, y \in H.$$

In other words, any cocycle is cohomologous to one in which all values are $|H|^{th}$ roots of unity. Finiteness of $H^2(H, \mathbb{C}^*)$ follows. It also follows that $H^2(H, \mathbb{C}^*)$ is of exponent dividing $|H|$.

Having already defined the (Schur) multiplier in the introduction, Schur then redefines the multiplier to be the group of 2-cocycles modulo the equivalence relation of being associated, i.e. $H^2(H, \mathbb{C}^*)$. Nowadays, authors typically reserve the term multiplier for the group $H_2(H, \mathbb{C}^*)$ dual to $H^2(H, \mathbb{C}^*)$. This agrees with Schur's definition in the introduction (see e.g. [49, § 11.4]). Schur does not appear to distinguish a finite abelian group from its dual, possibly because he is only concerned with these groups up to isomorphism.

Remark 3.4. Interestingly, it is only at the end of this section that Schur shows that cycles up to equivalence form an abelian group. Rather than demonstrate inverses, he shows the “cancellative property”: that an identity $K_\alpha K_\beta = K_\alpha K_\gamma$ implies that $K_\beta = K_\gamma$. Thus he seems to have the axioms stated by Frobenius (discussed in §2.2) in mind.

3.4. Schur's Section 2. In this section, Schur studies central extensions and the projective representations that they give rise to. Consider a central extension

$$(3.3) \quad A \hookrightarrow G \xrightarrow{p} H.$$

Schur chooses a set-theoretic section of p

$$s: H \rightarrow G: x \mapsto s_x$$

(i.e. $p \circ s = id$). Because (3.3) is exact, we can find unique $a_{x,y} \in A$ such that

$$(3.4) \quad s_x s_y = a_{x,y} s_{xy} \quad \text{for all } x, y \in H.$$

Schur observes that in any irreducible representation D of G on a complex vector space, any element $a \in A$ necessarily acts by a scalar $\psi(a) \in \mathbb{C}^*$, and we obtain in this way a character ψ of A . (Indeed, A is central and hence $\rho(A)$ is scalar by Schur's lemma.) In both Schur's and modern language, we say that D affords the central character ψ . On the other hand, for any character ψ of A there exists a representation of G affording ψ . (Schur deduces this result from Frobenius [11, §7]. The modern reader can convince themselves of this fact by observing that all characters occur when the regular representation of G is decomposed according to characters of A .) In any such representation, we obtain a 2-cocycle via

$$r_{x,y} = \psi(a_{x,y}).$$

As Schur points out in a footnote, one can expand (3.4) to deduce directly that $\psi(a_{x,y})$ yields a 2-cocycle. In particular, the 2-cocycle does not depend on the irreducible representation D affording ψ .

Thus, we obtain a homomorphism

$$m: A^\vee \rightarrow H^2(H, \mathbb{C}^*),$$

where $A^\vee$ denotes the character group of A . (Recall that $H^2(H, \mathbb{C}^*)$ is the finite abelian group dual to the Schur multiplier.)

Lemma 3.1. The kernel of m consists of those characters of A which extend to characters of G .

Proof. The image of ψ in $H^2(H, \mathbb{C}^*)$ is trivial if and only if there exists $\varphi: H \rightarrow \mathbb{C}^*$ such that

$$\psi(r_{x,y}) = \frac{\varphi(x)\varphi(y)}{\varphi(xy)},$$

which is easily seen to be equivalent to the fact that

$$\psi: G \rightarrow \mathbb{C}^*: as_x \mapsto \psi(a)\varphi(x) \quad \text{for } a \in A, x \in H$$

is a one-dimensional character of G .¹¹⁾ □

Now, one-dimensional characters of G are the same thing as characters of $G/[G, G]$, and thus a character of A extends to a character of G if and only if it factors over $A \rightarrow G/[G, G]$ if and only if it is trivial on $A \cap [G, G]$. Hence we obtain an injection

$$m': (A \cap [G, G])^\vee \hookrightarrow H^2(H, \mathbb{C}^*),$$

which is Schur's Theorem 2. Schur also points out that if $A \cap [G, G]$ is of cardinality equal to that of $H^2(H, \mathbb{C}^*)$, then m' must be an isomorphism. He is able to deduce that in this case every projective representation lifts to G . Thus, in this case G is a sufficiently augmented [hinreichend ergänzte] group.

Remark 3.5. A few remarks of a historical nature concerning this section:

- (i) The fact that one-dimensional characters are precisely those that factor over $G/[G, G]$ plays an important role, and appeared early in the theory, as such characters account for the linear factors of the group determinant. (This was conjectured to hold in Dedekind's second letter to Frobenius.)
- (ii) The modern distinction between an abelian group and its dual is not made by Schur. He points out in [19, § 1(6)] that one can always choose an isomorphism between an abelian group A and group of characters, and uses it when needed. This lack of distinction between A and $A^\vee$ makes some of the proofs (particularly of Schur's version of Lemma 3.1 above, which occurs just before his proof of Satz II) difficult to read.

¹¹⁾ One-dimensional characters are often called "linear characters" in the literature. However this terminology can lead to confusion, and we avoid it here.

3.5. An overview of Schur’s Section 3. From a modern point of view this is probably the most surprising section of the paper. We first give a high-level overview before discussing the details.

Given a finite group H , Schur’s goal is to establish the existence of a “Schur cover”. In other words a central extension

$$A \hookrightarrow G \xrightarrow{p} H,$$

such that (a) every projective representation of H lifts to an ordinary representation of G and (b) the order of G is minimal with this property.

Why is it surprising that Schur is able to construct G ? For one, there were limited means at the time of constructing new groups from old. Secondly, Schur was aware that in general G is not unique, and this ambiguity must arise somehow in the construction of G .

Schur proceeds as follows. He considers a free group F on generators corresponding to the elements of H , and considers the corresponding presentation of H

$$R \hookrightarrow F \twoheadrightarrow H.$$

Note that R is finitely generated. He then forces each element of R to be central, by quotienting by the commutator subgroup $[F, R] \subset F$. The result is an extension

$$N' \hookrightarrow K' \twoheadrightarrow H$$

with N' finitely generated and central (in particular abelian). Schur then *chooses* a free subgroup $Z \subset N'$ such that N'/Z is isomorphic to the torsion subgroup of N' . The resulting quotient

$$N \hookrightarrow K \twoheadrightarrow H$$

is the sought-after Schur cover of H . Thus the two challenges discussed above are addressed as follows:

- (i) Schur uses *infinite* groups and presentations to construct the *finite* group G . He relies crucially on the influential treatment of von Dyck [4,5] from 1882, which first introduced the concept of a finitely-presented group.
- (ii) Given a finitely-generated abelian group A the torsion subgroup is canonical but in general there are many possible free subgroups $A' \subset A$ such that A/A' is isomorphic to the torsion subgroup. The non-uniqueness of the Schur cover arises from different possible choices of such an A' .

Remark 3.6. Hopf’s paper [37] from 1942 is widely regarded as ushering in the study of group cohomology. For any group H , Hopf defines its second homology to be

$$N' = R \cap [F, F]/[F, R] = N' \cap [K', K'],$$

which—as we will see below—is exactly Schur’s construction! This formula for the second homology is often called “Hopf’s formula” (e.g. [49, § 11.4.15], [27, II, § 5]). As Mac Lane [47, §2] points out, “Hopf, while a student in Berlin, had been an assistant to Schur, but his 1942 paper does not mention the connection with the multiplier.”

3.6. Nuts and Bolts of Schur's Section 3. We now explain how Schur goes about proving that the construction outlined in the previous section furnishes the Schur cover of H . Schur's proof is beautiful, and has some advantages over modern treatments (e.g. [38, §23]), and so we give some detail.

Consider the central extension

$$N' \hookrightarrow K' \twoheadrightarrow H$$

considered in the previous section. All three groups are finitely generated and N' is central. Let

$$N'_{\text{tors}} \hookrightarrow N' \twoheadrightarrow N'_{\text{free}}$$

denote the torsion subgroup and free quotient of N' . The crucial statements are:

$$(3.5) \quad N'_{\text{free}} \text{ is of rank } |H|,$$

$$(3.6) \quad (N'_{\text{tors}})^\vee \cong H^2(H, \mathbb{C}^*),$$

$$(3.7) \quad [K', K'] \cap N' = N'_{\text{tors}}.$$

(Note that (3.7) implies the non-trivial fact that the commutator subgroup $[K', K']$ is finite.)

Indeed, if we know (3.6), then any choice of free subgroup $Z \subset N'$ such that N'/Z is isomorphic to the torsion subgroup will provide (by quotienting) a central extension

$$N \hookrightarrow K \twoheadrightarrow H$$

such that $N \subset [K, K]$ and $|N| = |H^2(H, \mathbb{C}^*)|$ which then must be a Schur covering group, by facts established in § 2.¹²⁾

Instead of working with the abelian group N' in question directly, Schur works with “arbitrary assignments of values satisfying the relations”. (For an example, see the statement of the classification of finitely-generated abelian groups, part way into § 3.) This makes parts very difficult to follow, however this difficulty is only a question of language. For the modern reader we can interpret what Schur is saying as follows: We begin with a finitely-generated abelian group M and consider group homomorphisms

$$X_M = \text{Hom}(M, \mathbb{C}^*).$$

This is an algebraic group in a natural way (see e.g. [51, § 3.2]). Schur's arguments are easily interpreted as algebro-geometric statements about groups of the form X_M . It will be important below to note that elements of M naturally provide functions on X_M . We will use the language of algebraic groups in what follows, as it makes Schur's arguments rather transparent.

Recall that F denotes a free group on generators corresponding to H . Let us write q_h for $h \in H$ for these generators. Then Schur's group K' is generated by q_h for $h \in H$ subject only to the condition that the $|H|^2$ elements

$$(3.8) \quad j_{x,y} := q_x q_y q_{xy}^{-1} \quad \text{for } x, y \in H,$$

which generate the kernel of the canonical map $K' \rightarrow H$, are central. The associativity relation implies that the elements $j_{g,h}$ satisfy the 2-cocycle condition

$$(3.9) \quad j_{x,y} j_{xy,z} = j_{x,yz} j_{y,z} \quad \text{for all } x, y, z \in H.$$

¹²⁾ Actually, Schur proceeds slightly differently. He establishes (3.5) and (3.6), and then uses the natural isomorphism in (3.6) to conclude that K is a Schur covering group, before moving on to (3.7).

Schur argues that (3.9) is in fact a presentation of N' . It follows that $X_{N'}$ is nothing other than the group of 2-cocycles of H . Let us define

$$Y_H = \{r: H \rightarrow \mathbb{C}^*\}$$

to be all maps of sets from H to $\mathbb{C}^*$, which is an algebraic group in a natural way (i.e. $Y_H \cong (\mathbb{C}^*)^H$). We have a natural map

$$d: Y_H \rightarrow X_{N'}$$

that associates to each 1-cocycle its coboundary:

$$j_{x,y} = \frac{r(x)r(y)}{r(xy)}.$$

As we saw above, $X_{K'}$ is the set of all 2-cocycles, and $d(X_H)$ consists of all coboundaries. Thus

$$X_{N'}/d(Y_H) = H^2(H, \mathbb{C}^*).$$

Finally, Schur argues that Y_H and $X_{N'}$ have the same dimension $|H|$.

In summary, we have a long exact sequence, in which the first two terms both have dimension $|H|$:¹³⁾

$$(3.10) \quad Y_H \xrightarrow{d} X_{N'} \longrightarrow H^2(H, \mathbb{C}^*) \longrightarrow \{1\}.$$

Dualizing (3.10) (i.e. passing back to finitely-generated abelian groups) yields a short exact sequence of finitely-generated abelian groups

$$0 \rightarrow H_2(H, \mathbb{C}^*) \rightarrow N' \rightarrow \mathbb{Z}^H.$$

Thus $N'/H_2(H, \mathbb{C}^*)$ embeds inside the free module $\mathbb{Z}^H$, from which it follows that $H_2(H, \mathbb{C}^*)$ is isomorphic to the torsion subgroup of N' . It also follows that N' and $\mathbb{Z}^H$ have the same rank $|H|$. This establishes (3.5) and (3.6).

To establish (3.7), Schur notes that $g \in K'$ belongs to $[K', K']$ if and only if $\chi(g) = 1$ for any one-dimensional character of K' . By construction, one-dimensional characters of K' are simply arbitrary assignments of values to each q_h . In other words, we have a canonical identification

$$Y_H = \text{one-dimensional characters of } K'.$$

Moreover, given such a one-dimensional character $\chi: q_h \mapsto c_h$, its value on $j_{x,y}$ is, by (3.8),

$$\chi(j_{x,y}) \mapsto \frac{c_x c_y}{c_{xy}}.$$

In other words, the map from one-dimensional characters of K' to characters of N' is nothing other than the differential d above. Thus (3.7) follows by following a sequence of beautiful tautologies!

Schur then points out that although the central extension

$$N \hookrightarrow K \twoheadrightarrow H$$

¹³⁾ Of course, the kernel of the first map in (3.10) consists of the one-dimensional characters of H , also known as $H^1(H, \mathbb{C}^*)$.

depends on choices, the induced central extension

$$N \hookrightarrow [K, K] \twoheadrightarrow [H, H]$$

does not. He concludes that the commutator subgroups of any two Schur covering groups agree, and that the Schur cover is unique up to isomorphism if H is perfect (i.e. $H = [H, H]$).

Remark 3.7. A well-known result of Schur states that if the centre of an infinite group G is of finite index, then the derived subgroup $[G, G]$ is finite. Moreover, it is common to cite [19], although this statement is not explicit in the paper. Most modern proofs use the transfer map (see e.g. [49, § 10.1.4] or [38, Satz 2.3]). However, one may derive a different proof from this section. Consider the central extension (with G potentially infinite)

$$Z(G) \hookrightarrow G \twoheadrightarrow H = G/Z(G)$$

and consider the subgroup G' of G generated by a fixed choice $g_0, g_1, \dots, g_{h-1}$ of preimages of elements of $G/Z(G)$. As Schur points out in this section, every commutator in G is a commutator of these elements, and hence $[G, G] \subset G'$. Now, G' is a homomorphic image of K' discussed above, where we have already seen that there the commutator subgroup is finite. It is clear (in particular from the line that cites von Dyck and Burnside) that Schur is aware of the possibility of reasoning along these lines.

3.7. Schur's Section 4. In this section, Schur studies the question of determining all irreducible projective representations associated to a fixed 2-cocycle $r_{x,y}$. If $A \hookrightarrow G \twoheadrightarrow H$ denotes a Schur cover, then Schur first points out that irreducible projective representations of H corresponding to $r_{x,y}$ can be obtained via the group determinant of G , using results of Frobenius. (In modern language, one may see this easily as a consequence of the fact that the twisted group algebra $\mathbb{C}^r H$ of H can be realized as a direct summand inside the group algebra of G . This also explains the connection which Schur makes with the matrices considered in Schur's § 1.)

Schur then goes on to count the number of irreducible projective representations corresponding to r in terms of the conjugacy classes of H satisfying the condition that $r_{x,y} = r_{y,x}$ whenever x belongs to the centralizer of y . The argument here is somewhat involved; modern readers can convince themselves of this theorem by computing the centre of the twisted group algebra (see [29, §1, Remark 4]). Schur finishes this section by proving that the dimension of any irreducible projective representation divides the order of $|H|$. The argument is similar to Frobenius' classic argument for ordinary representations, which is still used today.

3.8. Schur's Section 5. In this section, Schur obtains several bounds on the cardinality m of the multiplier of H . Indeed, as Schur points out, one often has a candidate for the Schur multiplier, and knowing good bounds can sometimes allow one to deduce equality. The techniques of this section are standard, and a modern reader should have little difficulty in following the original paper. It is perhaps worth noting that the proof of Theorem 10 exploits the transfer map, which was introduced by Schur in [18], was rediscovered by Artin in 1929 (see [23, pp. 159–164]) and went on to play an important role in number theory and topology.

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