The University of Sydney
Math1003 Integral Calculus and Modelling

Semester 2 **Exercises for Week 12** 2014

Assumed Knowledge: Finding the roots of quadratic equations. Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

Objectives:

(11a) To be able to write down the auxiliary (or characteristic) equation associated with a second-order differential equation with constant coefficients.

(11b) To be able to construct the solutions to such differential equations in terms of exponential and trigonometric functions.

Preparatory questions:

1. Write down the auxiliary or characteristic equation for the following second-order differential equations and find its roots:

(i) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - 8y = 0$

(ii) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - 4y = 0$

(iii) $\dfrac{d^2y}{dt^2} - 9y = 0$

(iv) $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + 5y = 0$

(v) $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + x = 0$

Practice Questions:

2. Find the particular solution of Preparatory Question 1 (i) which satisfies $y(0) = 0$ and $y'(0) = 3$.

3. Find the general solution of Preparatory Question 1 (ii).

4. Find the particular solution of Preparatory Question 1 (iii) which satisfies the initial conditions $y = 3$ and $\dfrac{dy}{dt} = 3$ when $t = 0$.

5. Find the general solution of Preparatory Question 1 (iv), expressing your answer in terms of real functions.
What is the particular solution satisfying $y(0) = 1$ and $y(\pi/4) = 2$?

6. Find the particular solution of Preparatory Question 1 (v) which satisfies $x(0) = 1$ and $x'(0) = 2$.

More Exercises

7. Find the general solutions of these second-order homogeneous equations. In each case give your answer in terms of real functions.

(i) $2\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 3y = 0.$

(ii) $\dfrac{d^2y}{dx^2} + 3y = 0.$

(iii) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0.$

8. Find the general solution of each of these second-order linear homogeneous equations. Hence find the particular solution for the given conditions. In each case give your answer in terms of real functions.

(i) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 20y = 0, \quad y(0) = y'(0) = 1$

(ii) $\dfrac{d^2y}{dx^2} + 9y = 0, \quad y(0) = 1, \quad y(\pi/6) = 3.$

(iii) $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 7y = 0, \quad y(0) = 0, \quad \dfrac{dy}{dt} = 3$ when $t = 0$.

(iv) $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 4x = 0, \quad x(0) = 1, \quad x(1) = 3e^2.$

Answers to Preparatory Questions

1. (i) The auxiliary equation is $\lambda^2 + 2\lambda - 8 = 0$ so that $\lambda = 2$ or $\lambda = -4$.

(ii) The auxiliary equation is $\lambda^2 + 2\lambda - 4 = 0$ so that $\lambda = -1 + \sqrt{5}$ or $\lambda = -1 - \sqrt{5}$.

(iii) The auxiliary equation is $\lambda^2 - 9 = 0$ so that $\lambda = 3$ or $\lambda = -3$.

(iv) The auxiliary equation is $\lambda^2 - 2\lambda + 5 = 0$.

The roots of this equation are $\lambda = \dfrac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$.

(v) The auxiliary equation is $\lambda^2 + 2\lambda + 1 = 0$ and this has two equal roots $\lambda = -1$.