Semester 2 **Exercises for Week 13** 2014

Assumed Knowledge:

Objectives:

(12a) To be able to rewrite two coupled first-order differential equations as a single second-order differential equation.

(12b) To be able to sketch the solutions of second-order differential equations with constant coefficients.

Preparatory questions:

1. Two species, struggling to compete against each other in the same environment, have populations at time t of $x(t)$ and $y(t)$, satisfying the equations

$$x'(t) = 3x(t) - 4y(t)\,, \qquad y'(t) = -2x(t) + y(t)\,.$$

Find the second-order differential equation satisfied by $x(t)$.

Practice Questions:

2. Find $x(t)$ and $y(t)$ in Preparatory Question 1.

3. Two species are in a predator-prey relationship. One species, which numbers Y individuals, eats the other, which numbers X individuals. Historically the numbers of these species have been constant with $X = 3000$ and $Y = 1500$. After a severe environmental disturbance the populations cease to be constant and start to change with time.

Let $x(t)$ and $y(t)$ be the difference between the historically constant population numbers and the new, changing population numbers $X(t)$ and $Y(t)$.
Then $x(t) = X(t) - 3000$ and $y(t) = Y(t) - 1500$ are the sizes of the perturbations from the historically steady states.
The sizes of the perturbations are described by the following:

$$x'(t) = 3x(t) - 2y(t)$$
$$y'(t) = 4x(t) - y(t).$$

(i) Show that $x''(t) - 2x'(t) + 5x(t) = 0$.

(ii) Find $x(t)$ if $x(0) = 100$ and $x'(0) = 100$. (Take $t = 0$ to be the time at which monitoring of the population sizes begins.)

(iii) Hence find $y(t)$.

(iv) Sketch $x(t)$ and $y(t)$ and then $X(t)$ and $Y(t)$ as a function of t. What will happen to the original populations?

More Exercises

4. Find the general solution of the following system of equations:

$$\frac{dx}{dt} = x + y, \qquad \frac{dy}{dt} = 3x - y\,.$$

5. Find the general solution of the pair of differential equations

$$\frac{dx}{dt} = 5x - 3y\,, \qquad \frac{dy}{dt} = 2y\,,$$

by first solving the second equation and then substituting into the first. (There are two equations, so you should have *two* arbitrary constants of integration at the end.) Find the particular solution satisfying the initial conditions $x = 1$, $y = 2$ when $t = 0$.

6. (i) For each of the following systems of differential equations for $x(t)$ and $y(t)$, find an equivalent second-order differential equation. (You do not need to find any solutions.)

(a) $x' = y + 4x, \qquad y' = 6x.$

(b) $x' = y, \qquad y' = x + 5y.$

(ii) Eliminate $y(t)$ from the following system to obtain a nonlinear second-order equation for $x(t)$. (You do not need to find any solutions.)

$$x' = y, \qquad y' = x - xy + 3t$$

7. Find whether each of the following first-order equations is separable, linear or neither. If the equation is separable or linear find its general solution.

(i) $\dfrac{dy}{dx} = \dfrac{x^2 - y}{x}$ (ii) $\dfrac{dy}{dx} = \dfrac{y - 1}{x(1 + x)}$

(iii) $\dfrac{dy}{dx} = \dfrac{\sin x - y}{x - \cos y}$ (iv) $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{2(1 + x)y}$

Answers to Preparatory Questions

1. Differentiating $x' = 3x - 4y$ gives $x'' = 3x' - 4y'$. But $y' = -2x + y$, so

$$\begin{aligned} x'' &= 3x' - 4(-2x + y) \\ &= 3x' + 8x - 4y \\ &= 3x' + 8x + (x' - 3x) \quad (\text{since } -4y = x' - 3x) \\ &= 4x' + 5x\,. \end{aligned}$$

That is, $x'' - 4x' - 5x = 0$.