THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

# Solutions to Week 5 Exercises and Objectives

MATH1003: Integral Calculus and Modelling — Semester 2, 2015

Web Page: www.sydney.edu.au/science/maths/u/UG/JM/MATH1003/
Lecturers: B. Crossman, R. Marangell, M. Myerscough, S. Stephen

**Assumed Knowledge:** Simple properties of the functions $\ln x$ and $e^x$, including their derivatives.

**Objectives:**

**(5a)** To know and be able to use the properties of the ln function.

**(5b)** To know and be able to use the properties of the exp function.

**(5c)** To know and be able to use the properties of the generalised exponential function $a^x$.

**(5d)** To be able to perform a logarithmic differentiation.

**Exercises:**

**1.** Simplify each of the following expressions:

(a) $e^{\ln 6}$

***Solution:*** $e^{\ln 6} = 6$.

(b) $\ln \sqrt{e}$

***Solution:*** $\ln \sqrt{e} = \ln\left(e^{1/2}\right) = \frac{1}{2}\ln e = \frac{1}{2}$.

**2.** Find $dy/dx$ for each of the following:

(a) $y = 2^x$

***Solution:*** $2^x \ln 2$.

(b) $y = \log_{10} x$

***Solution:*** $\dfrac{1}{x \ln 10}$.

(c) $y = \log_{10} \sqrt{x}$

***Solution:*** $\log_{10} \sqrt{x} = \dfrac{1}{2}\log_{10}(x) = \dfrac{\ln x}{2\ln 10}$, so $\dfrac{dy}{dx} = \dfrac{1}{2x\ln 10}$.

(d) $y = (\sin x)^x$

***Solution:*** First write $(\sin x)^x$ as $e^{x\ln(\sin x)}$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left((\sin x)^x\right) = \frac{d}{dx}\left(e^{x\ln(\sin x)}\right) \\ &= e^{x\ln(\sin x)}\left(x \times \frac{1}{\sin x} \times \cos x + \ln(\sin x)\right) \\ &= (\sin x)^x(x\cot x + \ln(\sin x)).\end{aligned}$$

**3.** The hyperbolic sine and hyperbolic cosine functions, $\sinh x$ and $\cosh x$, are defined as $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$.
Using properties of the exponential function, show that:

(a) $\frac{d}{dx} \sinh x = \cosh x$

***Solution:*** $\frac{d}{dx} \sinh x = \frac{1}{2}\frac{d}{dx}\left(e^x - e^{-x}\right) = \frac{1}{2}\left(e^x + e^{-x}\right) = \cosh x.$

(b) $\frac{d}{dx} \cosh x = \sinh x$

***Solution:*** $\frac{d}{dx} \cosh x = \frac{1}{2}\frac{d}{dx}\left(e^x + e^{-x}\right) = \frac{1}{2}\left(e^x - e^{-x}\right) = \sinh x.$

(c) $\cosh A \cosh B + \sinh A \sinh B = \cosh(A + B)$

***Solution:***

$$\begin{aligned}
&\cosh A \cosh B + \sinh A \sinh B \\
&\quad = \frac{1}{4}(e^A + e^{-A})(e^B + e^{-B}) + \frac{1}{4}(e^A - e^{-A})(e^B - e^{-B}) \\
&\quad = \frac{1}{4}(e^{A+B} + e^{A-B} + e^{-A+B} + e^{-(A+B)} + e^{A+B} - e^{A-B} - e^{-A+B} + e^{-(A+B)}) \\
&\quad = \frac{1}{2}(e^{A+B} + e^{-(A+B)}) = \cosh(A + B)\,.
\end{aligned}$$

(d) $2(\cosh A)^2 - 1 = \cosh(2A)$

***Solution:***

$$\begin{aligned}
2\left(\frac{e^A + e^{-A}}{2}\right)^2 - 1 &= \frac{1}{2}(e^{2A} + 2e^A e^{-A} + e^{-2A}) - 1 \\
&= \frac{1}{2}(e^{2A} + e^{-2A}) = \cosh(2A).
\end{aligned}$$

**4.** Consider the function $f(x) = \frac{\ln x}{x}$, which is defined for all $x > 0$. Show that this function is strictly increasing on the interval $(0, e)$ and strictly decreasing on the interval $(e, \infty)$, and thus has a global maximum at $x = e$. Hence show that $f(x) \le \frac{1}{e}$ for all $x > 0$. Use this result with $x = \pi$ to show that $\pi^e < e^\pi$.

***Solution:*** The derivative $f' = \frac{1}{x}\left(\frac{1}{x}\right) - \frac{1}{x^2}\ln x = \frac{1 - \ln x}{x^2}$. Since $x^2 > 0$ for $x > 0$, the sign of $f'(x)$ is determined by the sign of $1 - \ln x$. This is positive when $1 - \ln x > 0$, i.e. $\ln x < 1$, and negative when $1 - \ln x < 0$, i.e. $\ln x > 1$. Now $\ln x = 1$ when $e^{\ln x} = e^1$, i.e. when $x = e$. Therefore, when $0 < x < e$, $f'(x) > 0$ so the function is increasing, and when $x > e$, $f'(x) < 0$ so the function is decreasing. Hence, $f(x)$ has a maximum when $x = e$.
If $f(x)$ has a maximum at $x = e$, then $f(x) \le \frac{\ln e}{e} = \frac{1}{e}$ since $\ln e = 1$.
Now $\pi^e = e^{\ln \pi^e} = e^{e \ln \pi}$. But if we put $x = \pi$ in the previous result, $\ln \pi / \pi < 1/e$ or $e \ln \pi < \pi$ (an inequality holds because the maximum occurs at $x = e$, not $x = \pi$). Since $e^x$ is an increasing function of $x$, $e^{e \ln \pi} < e^\pi$, i.e. $\pi^e < e^\pi$.