The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial for Week 12

MATH1903: Integral Calculus and Modelling (Advanced) — Semester 2, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1903/
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Material covered

- ☐ Homogeneous linear second order differential equations with constant coefficients.
- ☐ Inhomogeneous linear second order differential equations with constant coefficients.

Outcomes

After completing this tutorial you should

- ☐ be confident in solving homogeneous second order homogeneous and inhomogeneous differential equations in various contexts.

Summary of essential material

Homogeneous linear second order equations with constant coefficients. Consider a differential equation of the form

$$ay'' + by' + c' = 0$$

with $a, b, c \in \mathbb{R}$ constants and $a \neq 0$. To find the general write down the *auxiliary equation*

$$a\lambda^2 + b\lambda + c = 0$$

and find its roots (real or complex). Depending on the nature of the roots apply the relevant case:

Case 1: The auxiliary equation has *two distinct real roots* $\lambda_1 \neq \lambda_2$. Then the general solution is

$$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

Case 2: The auxiliary equation has one (real) *double root* λ. Then the general solution is

$$y(t) = (A + Bt)e^{\lambda t}$$

Case 3: The auxiliary equation has a *pair of complex conjugate roots* $\lambda = \mu \pm i\omega$. Then the real form of the general solution is

$$y(t) = e^{\mu t}\bigl(A\cos(\omega t) + B\sin(\omega t)\bigr)$$

Inhomogeneous linear second order equations with constant coefficients. Consider a differential equations of the form

$$ay'' + by' + c' = f(t)$$

with $a, b, c \in \mathbb{R}$ constants and $a \neq 0$. The function f is called the *inhomogeneity*. The general solution is of the form

$$y(t) = y_h(t) + y_p(t),$$

where y_h is the general solution of the homogeneous problem $ay'' + by' + c' = 0$ and y_p an arbitrary solution of the inhomogeneous problem we call a *particular solution*. To find a particular solution we often find a solution that has a similar form to the inhomogeneity f. The idea is to determine the unknown parameters by subsitituion into the differential equations.

Inhomogeneity $f(t)$	Form of particular solution $y_p(t)$	($C, D, E, \dots$ to be determined)
$Ae^{\mu t}$	$Ce^{\mu t}$	
$A\cos(\omega t)$ or $B\cos(\omega t)$	$C\cos(\omega t) + D\sin(\omega t)$	(both terms unless there is symmetry)
At	$C + Dt$	
At^2	$Ct^2 + Dt + E$	(all terms unless there is symmetry)
polynomial of degree n	polynomial of degree n	(all terms, unless there is symmetry)
$f(t)$ solves the homogeneous equation	$Ctf(t)$	

Questions to do before the tutorial

1. Find the general solution of each of the following.

(a) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 5y = 0.$

Solution: The auxiliary equation $\lambda^2 + 4\lambda - 5 = 0$ has roots $\lambda = -5, 1$, and so the general solution is $y = Ae^{-5x} + Be^{x}$.

(b) $\dfrac{d^2y}{dt^2} + 9y = 0.$

Solution: The auxiliary equation $\lambda^2 + 9 = 0$ has complex roots $\lambda = \pm 3i$, and so the general solution is $y = C\cos 3t + D\sin 3t$.

2. Consider the second-order non-homogeneous differential equation $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2.$

(a) Find the general solution of the above differential equation.

Solution: The auxiliary equation $\lambda^2 - 2\lambda + 1 = 0$ has a double root $\lambda = 1$, and so the general solution of the homogeneous equation (also called the complementary equation) is $y_h = Ae^x + Bxe^x$. For a particular solution, try $y_p = ax^2 + bx + c$. Substituting this into the differential equation gives

$$2a - 2(2ax + b) + (ax^2 + bx + c) = x^2.$$

Comparing coefficients of like powers gives $a = 1$, $b - 4a = 0$ and $2a - 2b + c = 0$, and hence $a = 1$, $b = 4$ and $c = 6$. So a particular solution is $y_p = x^2 + 4x + 6$, and the general solution is

$$y = (A + Bx)e^x + x^2 + 4x + 6.$$

(b) Find the particular solution of the above differential equation satisfying the initial conditions $y(0) = y'(0) = 4$.

Solution: The solution above gives $y(0) = A + 6$ and $y'(0) = A + B + 4$. So $y(0) = 4$ and $y'(0) = 4$ imply that $A = -2$ and $B = 2$, and so the required particular solution is $y = 2(x - 1)e^x + x^2 + 4x + 6$.

Questions to complete during the tutorial

3. Find the general solution of each of the following differential equations.

(a) $\dfrac{d^2x}{dt^2} - 6\dfrac{dx}{dt} + 9x = 0.$

Solution: The auxiliary equation $\lambda^2 - 6\lambda + 9 = 0$ has repeated roots $\lambda = 3, 3$, and so the general solution is $x = Ae^{3t} + Bte^{3t}$.

(b) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 25y = 0.$

Solution: The auxiliary equation $\lambda^2 - 6\lambda + 25 = 0$ has complex roots $\lambda = 3 \pm 4i$, and so the general solution is $y = e^{3x}(C\cos 4x + D\sin 4x)$.

4. Solve the following equations, giving the general solution and then the particular solution $y(x)$ satisfying the given boundary or initial conditions.

(a) $y'' + 4y' + 5y = 0, \quad y(0) = 2,\ y'(0) = 4$

Solution: The auxiliary equation $\lambda^2 + 4\lambda + 5 = 0$ has roots $-2 \pm i$, and so the general solution is $y(x) = e^{-2x}(C\cos x + D\sin x)$, which gives $y'(x) = e^{-2x}\{(D - 2C)\cos x - (C + 2D)\sin x\}$. Hence $y(0) = C$ and $y'(0) = D - 2C$, so the initial conditions imply $C = 2$ and $D = 8$, and the particular solution is $y(x) = 2e^{-2x}(\cos x + 4\sin x)$.

(b) $y'' - 2y' + y = 0, \quad y(2) = 0,\, y'(2) = 1$

Solution: The auxiliary equation $\lambda^2 - 2\lambda + 1 = 0$ has one double root $\lambda = 1$, and so the general solution is $y(x) = (A + Bx)e^x$, which gives $y'(x) = (A + B + Bx)e^x$. Hence $y(2) = (A + 2B)e^2$ and $y'(2) = (A + 3B)e^2$, so the initial conditions imply $A = -2e^{-2}$ and $B = e^{-2}$, and the particular solution is $y(x) = (x - 2)e^{x-2}$.

5. We considered the case of a second order differential equation where the auxiliary equation has a double root, say λ_0. Here we provide an argument why $te^{\lambda_0 t}$ is expected to be a solution. The differential equation in that case is

$$y'' - 2\lambda_0 y' + \lambda_0^2 y = 0.$$

The idea is to look at a perturbed equation that has two distinct real roots, then obtain the solution $te^{\lambda_0 t}$ as a limit of solutions of the perturbed equation.

(a) Check that $e^{\lambda_0 t}$ and $e^{(\lambda_0+h)t}$ are solutions to $y'' - (2\lambda_0 + h)y' + \lambda_0(\lambda_0 + h)y = 0$. Briefly explain why

$$\frac{e^{(\lambda_0+h)t} - e^{\lambda_0 t}}{h}$$

is a solution of the same perturbed equation.

Solution: The auxiliary equation of the given differential equation is

$$0 = \lambda^2 - (2\lambda_0 + h)\lambda + \lambda_0(\lambda_0 + h) = (\lambda - \lambda_0)\big(\lambda - (\lambda_0 + h)\big).$$

Hence the roots are λ_0 and $\lambda_0 + h$ and thus $e^{\lambda_0 t}$ and $e^{(\lambda_0+h)t}$ are solutions as required. According to the superposition principle, also

$$\frac{1}{h}e^{(\lambda_0+h)t} - \frac{1}{h}e^{\lambda_0 t} = \frac{e^{(\lambda_0+h)t} - e^{\lambda_0 t}}{h}$$

is a solution as well.

(b) Let $h \to 0$ in the equation as well as the solution given in part (a) and relate it to the original unperturbed equation. Check that the limit of solutions as $h \to 0$ is a solution to the limit equation.

Solution: Applying differentiation with respect to λ from first principles we see that

$$\lim_{h\to 0} \frac{e^{(\lambda_0+h)t} - e^{\lambda_0 t}}{h} = \frac{d}{d\lambda}e^{\lambda t}\Big|_{\lambda=\lambda_0} = te^{\lambda_0 t}$$

If we let $h \to 0$ in the equation $y'' - (2\lambda_0 + h)y' + \lambda_0(\lambda_0 + h)y = 0$ we obtain the original equation $y'' - 2\lambda_0 y' + \lambda_0 y = 0$. It is not clear that the limit of solutions is a solution of the limit equation, but we might expect this anyway. Hence we need to check by differentiation and subsitution. We have, using the chain rule,

$$y(t) = te^{\lambda_0 t}, \qquad y'(t) = e^{\lambda_0 t} + \lambda_0 te^{\lambda_0 t}, \qquad y''(t) = 2\lambda_0 e^{\lambda_0 t} + \lambda_0^2 te^{\lambda_0 t}.$$

We substitute into the equation to obtain

$$\begin{aligned} &y'' - 2\lambda_0 y' + \lambda_0 y \\ &= (2\lambda_0 e^{\lambda_0 t} + \lambda_0^2 te^{\lambda_0 t}) - 2\lambda_0(e^{\lambda_0 t} + \lambda_0 te^{\lambda_0 t}) + \lambda_0^2 te^{\lambda_0 t} \\ &= (2\lambda_0 - 2\lambda_0)e^{\lambda_0 t} + (\lambda_0^2 - 2\lambda_0^2 + \lambda_0^2)te^{\lambda_0 t} \\ &= 0 \end{aligned}$$

as expected.

6. First find the general solution of each of the following non-homogeneous second-order differential equations, and then the particular solution for the given initial conditions.

(a) $y'' + 3y' + 2y = 6e^t, \quad y(0) = 1,\ y'(0) = 0.$

Solution: The auxiliary equation $\lambda^2 + 3\lambda + 2 = 0$ has roots $\lambda = -1, -2$, and so the general solution of the homogeneous equation is $y_h = Ce^{-t} + De^{-2t}$. For a particular solution, try $y_p = \alpha e^t$. Substituting this into the differential equation gives $\alpha(e^t + 3e^t + 2e^t) = 6e^t$, which implies $\alpha = 1$. So a particular integral is $y_p = e^t$, and the general solution is

$$y = Ce^{-t} + De^{-2t} + e^t.$$

The solution above gives $y(0) = C + D + 1$ and $\dot{y}(0) = -C - 2D + 1$. So $y(0) = 1$ and $\dot{y}(0) = 0$ imply that $C = -1$ and $D = 1$, and so the required particular solution is $y = -e^{-t} + e^{-2t} + e^t$.

(b) $y'' + 3y' + 2y = 6e^{-t},\ y(0) = 2,\ y'(0) = 1.$

Solution: The auxiliary equation and hence the general solution of the homogeneous equation are the same as in the last part. In this case, however, the non-homogeneous term is itself a solution of the homogeneous equation and so we will not be able to produce a particular solution of the form αe^{-t}. The standard procedure in this case is to include a factor t. So a suitable trial solution will take the form $y_p = \alpha te^{-t}$. Substitution into the differential equation gives $\alpha(t-2)e^{-t} + 3\alpha(1-t)e^{-t} + 2\alpha te^{-t} = 6e^{-t}$, which implies $\alpha = 6$. So a particular solution is $y_p = 6te^{-t}$, and the general solution is

$$y = (6t + C)e^{-t} + De^{-2t}.$$

The solution above gives $y(0) = C + D$ and $\dot{y}(0) = 6 - C - 2D$. So $y(0) = 2$ and $\dot{y}(0) = 1$ imply that $C = -1$ and $D = 3$, and so the required particular solution is $y = (6t - 1)e^{-t} + 3e^{-2t}$.

7. (a) For $\omega \neq 5$, find the general solution of the non-homogeneous differential equation,

$$\frac{d^2y}{dt^2} + 25y = 100 \sin \omega t,$$

and the particular solution subject to the initial conditions $y(0) = 0$ and $\dot{y}(0) = 0$.

Solution: The auxiliary equation $\lambda^2 + 25 = 0$ has roots $\lambda = \pm 5i$, and so the general solution of the homogeneous equation is $y_h = C\cos 5t + D \sin 5t$. Since the non-homogeneous term is sinusoidal, we try a particular solution of the form, $y_p = \alpha \sin \omega t + \beta \cos \omega t$. This will work as long as $\omega \neq \pm 5$, which we assume for the present. Now, we can save ourselves some trouble by dropping the $\cos \omega t$ term in y_p. This is permitted because there is no first-order (or any odd-order) derivative term in the differential equation and because only a $\sin \omega t$ term appears on the right-hand side. (If you have any doubt about this, keep the cosine term in y_p and find that its coefficient is zero after a calculation.) Substituting $y_p = \alpha \sin \omega t$ into the differential equation gives $-\alpha\omega^2 \sin \omega t + 25\alpha \sin \omega t = 100 \sin \omega t$, from which it follows that $\alpha = 100/(25 - \omega^2)$. Thus, a particular solution is $y_p = 100(25 - \omega^2)^{-1} \sin \omega t$, and the general solution is

$$y = C \cos 5t + D \sin 5t + \frac{100}{25 - \omega^2} \sin \omega t.$$

We want the particular solution such that $y(0) = \dot{y}(0) = 0$. Differentiation of the general solution gives

$$\dot{y} = -5C \sin 5t + 5D \cos 5t + \frac{100\omega}{25 - \omega^2} \cos \omega t.$$

The initial conditions imply that $C = 0$ and $D = -20\omega/(25 - \omega^2)$. Hence the required particular solution is

$$y = \frac{100 \sin \omega t - 20\omega \sin 5t}{25 - \omega^2}.$$

(b) For $\omega = 5$, find a particular solution of the differential equation. Then determine the particular solution with $y(0) = 0$ and $\dot{y}(0) = 0$.

Solution: In the case $\omega = 5$, a solution of the form $y_p = \alpha \sin \omega t + \beta \cos \omega t$ is a solution of the homogeneous equation. The standard trick in this case is to include a factor t, in which case $y_p = \alpha t \sin 5t + \beta t \cos 5t$. As before, we can simplify the problem by a symmetry argument. Because there is no first-order derivative in the differential equation and because the forcing term is an odd function, we can get away with restricting y_p to be an odd function. Thus $y_p = \beta t \cos 5t$. Its derivatives are $\dot{y}_p = \beta(-5t \sin 5t + \cos 5t)$ and $\ddot{y}_p = \beta(-25t \cos 5t - 10 \sin 5t)$. Substituting into the differential equation and cancelling terms shows that $\beta = -10$. Hence a particular solution is $y_p = -10t \cos 5t$, and the general solution is

$$y = (C - 10t) \cos 5t + D \sin 5t.$$

Its derivative is $\dot{y} = (50t - 5C) \sin 5t + (5D - 10) \cos 5t$. The initial conditions are satisfied by $C = 0$ and $D = 2$. Hence the required particular solution is

$$y = 2 \sin 5t - 10t \cos 5t.$$

(c) Find the corresponding particular solution of the differential equation for $\omega = 5$ by fixing t in the result of part (a) and taking the limit as ω approaches its special value.

Solution: If one puts $\omega = 5$ in the result of part (a), the solution becomes a 0/0-type indeterminate form. L'Hôpital's rule can be used to take the limit $\omega \to 5$. Here, we must hold t constant while we take derivatives with respect to ω. Thus, in the case of resonance,

$$\begin{aligned} y = \lim_{\omega \to 5} \frac{100 \sin \omega t - 20\omega \sin 5t}{25 - \omega^2} &= \lim_{\omega \to 5} \frac{(\partial/\partial\omega)(100 \sin \omega t - 20\omega \sin 5t)}{(\partial/\partial\omega)(25 - \omega^2)} \\ = \left. \frac{100t \cos \omega t - 20 \sin 5t}{-2\omega} \right|_{\omega=5} &= \frac{100t \cos 5t - 20 \sin 5t}{-10} = 2 \sin 5t - 10t \cos 5t. \end{aligned}$$

Without L'Hôpital's rule we can use differentiation from first principles. We can write

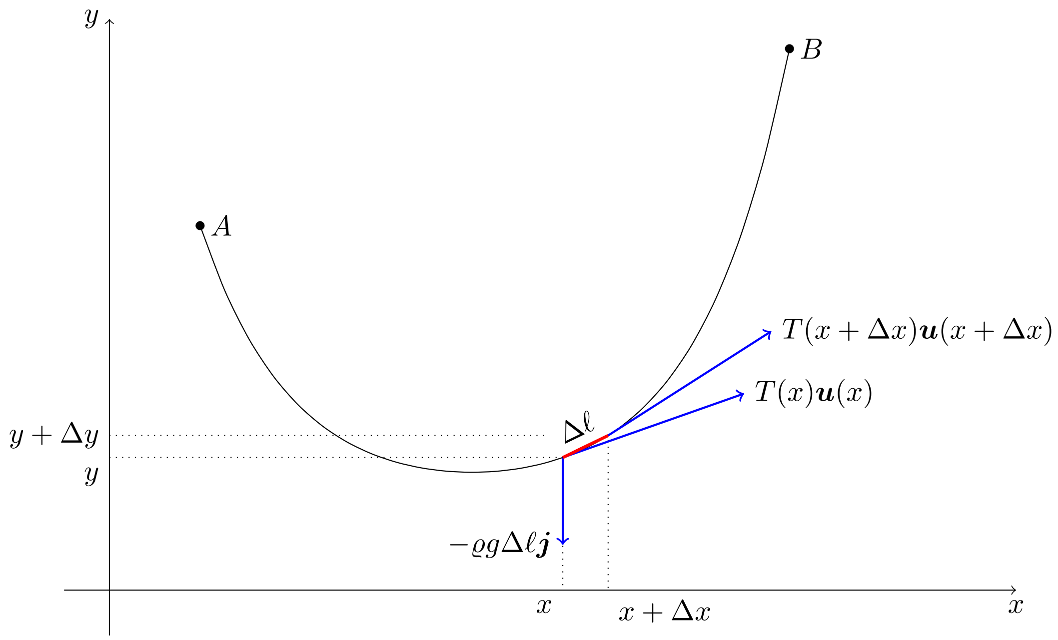
$$\begin{aligned} y &= \lim_{\omega \to 5} \frac{100 \sin \omega t - 20\omega \sin 5t}{25 - \omega^2} \\ &= \lim_{\omega \to 5} \frac{100 \sin \omega t - 100 \sin 5t - 20(\omega - 5) \sin 5t}{25 - \omega^2} \\ &= -\frac{100}{5 + \omega} \frac{\sin \omega t - \sin 5t}{\omega - 5} + \frac{20}{5 + \omega} \frac{(5 - \omega) \sin 5t}{\omega - 5} \\ &\xrightarrow{\omega \to 5} -\frac{100}{5 + 5} \frac{d}{d\omega} \sin(\omega t) \Big|_{\omega=5} + \frac{20}{5 + 5} \sin 5t \\ &= -10t \cos(5t) + 2 \sin 5t \end{aligned}$$

which is the same as before. The factor $10t$ shows that the amplitude grows without bound.

Extra questions for further practice

8. A rope of length L is suspended at two points A and B and hangs freely in-between in such a way that it does not move at all. The rope has constant mass density ϱ per unit length, that is, a section of length ℓ has mass $\varrho\ell$. We assume that the rope is perfectly flexible, that is, there is no bending force.

The only forces acting on the rope are the tension force T tangent to the rope and the gravitational force in the downwards direction. Denote the unit tangent vector along the rope by $\boldsymbol{u}$. The height of the rope above ground is given by a function $y(x)$. Denote acceleration due to gravity by g.



Consider a small section of rope of length $\Delta\ell$ between x and $x+\Delta x$. That section has mass $\varrho\Delta\ell$. We denote the unit vectors in the direction of the x-axis and the y-axis by $\boldsymbol{i}$ and $\boldsymbol{j}$, respectively.

(a) Using the fact that the sum of all forces on $\Delta\ell$ add up to zero, show that

$$\frac{d}{dx}\big(T(x)\boldsymbol{u}(x)\big) = \varrho g\sqrt{1+(y'(x))^2}\boldsymbol{j}.$$

Solution: The length of the section $\Delta\ell$ is given by $\Delta\ell = \sqrt{(\Delta x)^2+(\Delta y)^2}$, so its mass is $\varrho\sqrt{(\Delta x)^2+(\Delta y)^2}$. Hence the gravitational force on $\Delta\ell$ is given by

$$-\varrho\sqrt{(\Delta x)^2+(\Delta y)^2}\boldsymbol{j}.$$

The minus sign comes from the fact that the gravitational force points downwards, whereas $\boldsymbol{j}$ points upwards. The other forces on $\Delta\ell$ are the tension forces at the right and left ends. The tension force at the right end is

$$T(x+\Delta x)\boldsymbol{u}(x+\Delta x)$$

and that at the left end is

$$-T(x)\boldsymbol{u}(x).$$

The minus sign comes from the fact that this is a "reaction force" to the section of the rope pulling to the left. The total force on $\Delta\ell$ must be zero, so

$$T(x+\Delta x)\boldsymbol{u}(x+\Delta x) - T(x)\boldsymbol{u}(x) - \varrho\sqrt{(\Delta x)^2+(\Delta y)^2}\boldsymbol{j} = \boldsymbol{0}.$$

If we rearrange and divide by Δx we get

$$\frac{T(x+\Delta x)\boldsymbol{u}(x+\Delta x) - T(x)\boldsymbol{u}(x)}{\Delta x} = \varrho\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\boldsymbol{j}.$$

Letting $\Delta x \to 0$, using the definition of a derivative, we get the required differential equation.

(b) Explain why the unit tangent vector $\boldsymbol{u}$ is given by

$$\boldsymbol{u}(x) = \frac{1}{\sqrt{1+(y'(x))^2}}\boldsymbol{i} + \frac{y'(x)}{\sqrt{1+(y'(x))^2}}\boldsymbol{j}.$$

Solution: The slope of the tangent at every point is given by $y'(x)$. Hence the vector $\boldsymbol{i} + y'(x)\boldsymbol{j}$ points in the direction of $\boldsymbol{u}(x)$. To get the unit vector we need to divide by the length which is given by $\sqrt{1 + (y'(x))^2}$. Hence

$$\boldsymbol{u}(x) = \frac{1}{\sqrt{1 + (y'(x))^2}}\boldsymbol{i} + \frac{y'(x)}{\sqrt{1 + (y'(x))^2}}\boldsymbol{j}.$$

(c) By considering the component of the differential equation from (a) in the x-direction, that is, the direction of $\boldsymbol{i}$, show that

$$T(x) = H\sqrt{1 + (y'(x))^2}$$

for some constant H. Give a physical interpretation of H.

Solution: According to part (b) the horizontal component of $\boldsymbol{u}$ is given by

$$\frac{1}{\sqrt{1 + (y'(x))^2}}$$

Hence the horizontal component of (a) is given by

$$\frac{d}{dx}\left(\frac{T(x)}{\sqrt{1 + (y'(x))^2}}\right) = 0.$$

Hence, there exists a constant H so that

$$\frac{T(x)}{\sqrt{1 + (y'(x))^2}} = H,$$

and therefore $T(x) = H\sqrt{1 + (y'(x))^2}$ as claimed.
The horizontal component of the tension force is

$$\frac{T(x)}{\sqrt{1 + (y'(x))^2}}.$$

Using the explicit expression of T the horizontal component of the tension force has the constant value H.

(d) By considering the component of the differential equation from (a) in the y-direction, that is, the direction of $\boldsymbol{j}$, show that

$$y''(x) = \frac{\varrho g}{H}\sqrt{1 + (y'(x))^2}.$$

Solution: According to part (b) the vertical component of $\boldsymbol{u}$ is given by

$$\frac{y'(x)}{\sqrt{1 + (y'(x))^2}}$$

Hence the vertical component of (a) is given by

$$\frac{d}{dx}\left(\frac{T(x)y'(x)}{\sqrt{1 + (y'(x))^2}}\right) = \varrho g\sqrt{1 + (y'(x))^2}.$$

Substituting the solution from (c) we get

$$\frac{d}{dx}\big(Hy'(x)\big) = Hy''(x) = \varrho g\sqrt{1 + (y'(x))^2}.$$

If we divide by H we get the required differential equation.

(e) Find the general solution of the differential equation in (d). Note that the differential equation is a first order differential equation for $z(x) = y'(x)$.

Solution: Rewriting the original differential equation as a differential equation for $z(x) = y'(x)$ we get

$$\frac{dz}{dx} = \frac{\varrho g}{H}\sqrt{1+z^2}.$$

We first separate variables and write

$$\frac{dz}{\sqrt{1+z^2}} = \frac{\varrho g}{H}\,dx$$

and integrating we get

$$\int \frac{dz}{\sqrt{1+z^2}} = \int \frac{\varrho g}{H}\,dx = \frac{\varrho g x}{H} + C.$$

For the integral on the left hand side we use the substitution $z = \sinh t$. Then $dz = \cosh t\,dt$. Using that $1 + \sinh^2 t = \cosh t$ we get

$$\frac{\varrho g x}{H} + C = \int \frac{dz}{\sqrt{1+z^2}} = \int \frac{\cosh t}{\sqrt{1+\sinh^2 t}}\,dt = \int \frac{\cosh t}{\cosh t}\,dt = t = \sinh^{-1} z.$$

We do not need a constant as that constant can be merged with C. Alternatively we could use a table of standard integrals to evaluate the integral. Hence

$$z = \sinh\Big(\frac{\varrho g x}{H} + C\Big).$$

Next we recall that $z = y'$, so

$$y(x) = \int z(x)\,dx = \int \sinh\Big(\frac{\varrho g x}{H} + C\Big)\,dx = \frac{H}{\varrho g}\cosh\Big(\frac{\varrho g x}{H} + C\Big) + D.$$

The cosh curve is often called the *catenary*. The constants C, D and H could be computed in terms of the length L the mass density ϱ and the coordinates of A and B, but this is rather tedious to do for the general situation.

9. Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0,$$

expressing your answer in real form. What is the particular solution satisfying $y(0) = 1$ and $y(\pi/4) = 2$?

Solution: The auxiliary equation is $\lambda^2 - 2\lambda + 5 = 0$, which has roots $\lambda = 1 \pm 2i$, and so the general solution is

$$y = e^t(A\cos 2t + B\sin 2t).$$

Hence $y(0) = E$ and $y(\pi/4) = e^{\pi/4}F$. If $y(0) = 1$ and $y(\pi/4) = 2$ then $A = 1$ and $B = 2e^{-\pi/4}$, and hence the particular solution is

$$y = e^t\big(\cos 2t + 2e^{-\pi/4}\sin 2t\big).$$

10. Solve the following equations, giving the general solution and then the particular solution $y(x)$ satisfying the given boundary or initial conditions.

(a) $2y'' - 7y' + 5y = 0, \quad y(0) = 1,\ y'(0) = 1$

Solution: The auxiliary equation $2\lambda^2 - 7\lambda + 5 = 0$ has roots $5/2$ and 1, and so the general solution is $y(x) = Ae^{5x/2} + Be^x$, which gives $y'(x) = (5A/2)e^{5x/2} + Be^x$. Hence $y(0) = A + B$ and $y'(0) = (5A/2) + B$, so the initial conditions imply $A = 0$ and $B = 1$, and the particular solution is $y(x) = e^x$.

(b) $y'' + 4y' + 3y = 0, \quad y(-2) = 1,\ y(2) = 1$

Solution: The auxiliary equation $\lambda^2+4\lambda+3 = 0$ has roots -1 and -3, and so the general solution is $y(x) = Ae^{-x} + Be^{-3x}$. Hence $y(-2) = Ae^2 + Be^6$ and $y(2) = Ae^{-2} + Be^{-6}$, so the boundary conditions imply $Ae^2 + Be^6 = 1$ and $Ae^{-2} + Be^{-6} = 1$. Solving these simultaneous equations gives

$$A = \frac{\sinh 6}{\sinh 4} = 7.3915, \qquad B = -\frac{\sinh 2}{\sinh 4} = -0.1329,$$

and so the particular solution satisfying the boundary conditions is

$$y(x) = 7.3915e^{-x} - 0.1329e^{-3x}.$$

(c) $2y'' - 2y' + 5y = 0, \quad y(0) = 0,\ y(2) = 2$

Solution: The auxiliary equation $2\lambda^2 - 2\lambda + 5 = 0$ has roots $(1 \pm 3i)/2$, and so the general solution is $y(x) = e^{x/2}\{A\cos(3x/2) + B\sin(3x/2)\}$. Hence $y(0) = A$, and the first boundary condition implies $A = 0$. Thus $y(2) = Be\sin 3$, and so the second boundary condition implies $B = 2/(e\sin 3) = 5.2137$, and hence the particular solution satisfying the boundary conditions is $y(x) = 5.2137e^{x/2}\sin(3x/2)$.

(d) $y'' - 4y' + 4y = 0, \quad y(0) = -2,\ y(1) = 0$

Solution: The auxiliary equation $\lambda^2 - 4\lambda + 4 = 0$ has one double root $m = 2$, and so the general solution is $y(x) = (A + Bx)e^{2x}$. Hence $y(0) = A$ and the first boundary condition implies $A = -2$. Thus $y(1) = (-2+B)e^2$, and so the second boundary condition implies $B = 2$, and hence the particular solution satisfying the boundary conditions is $y(x) = 2(x-1)e^{2x}$.

11. Find the particular solution of the differential equation $y'' - 6y' + 9y = e^{3x}$ which satisfies the initial conditions $y(0) = 1$ and $y'(0) = 0$.

Solution: The auxiliary equation of the homogeneous problem is $\lambda^2-6\lambda+9 = (\lambda-3)^2 = 0$. As $\lambda = 3$ is a double root e^{3x} and xe^{3x} solve the homogeneous equation. Hecne the inhomogeneity e^{3t} solves the equation. Normally we would find a particular solution of the form Axe^{3x}, but that is a solution of the homogeneous equation as well. Hence we multiply by another x and try a particular solution of the form $y = Ax^2e^{3x}$. We note that $y'(x) = 2xAe^{3x} + 3x^2Ae^{3x}$ and $y''(x) = 2Ae^{3x} + 12Axe^{3x} + 9Ax^2e^{3x}$. Substitution into the equation yields

$$2Ae^{3x} + 12Axe^{3x} + 9x^2Ae^{3x} - 6(2Axe^{3x} + 3Ax^2e^{3x}) + 9Ax^2e^{3x} = e^{3x}.$$

If we cancel $e^{3x} \neq 0$ and collect terms according to powers of x we obtain

$$2A + (12A - 12A)x + (9A - 18A + 9A)x^2 = 2A = 1$$

Hence $A = \frac{1}{2}$ and the general solution is

$$y = \Big(C + Dx + \frac{x^2}{2}\Big)e^{3x}.$$

To make use of the initial conditions note that

$$y' = \Big(3C + 3Dx + \frac{3x^2}{2} + D + x\Big)e^{3x}.$$

Hence $y(0) = C$ and $y'(0) = 3C + D$. So the conditions $y(0) = 1$ and $y'(0) = 0$ imply that $C = 1$ and $D = -3$. Hence, the required particular solution is

$$y = \Big(1 - 3x + \frac{x^2}{2}\Big)e^{3x}.$$