Questions:

1. Let $k$ be a field, $A$ an associative $k$-algebra and $V$ an $A$-module. We say that $V$ is \textbf{cyclic} if there is some $v \in V$ such that $Av = V$.

   (a) Show that if $V$ is simple then $V$ is cyclic, but that a nonzero cyclic module need not be simple.

   \textbf{Solution:} For any $v \in V$, $Av$ is an $A$-submodule of $V$. So if $V$ is simple, meaning that it is nonzero and its only $A$-submodules are $\{0\}$ and $V$, then we must have $Av = V$ for all nonzero elements $v \in V$, not just for one element. (Conversely, if $V$ is nonzero and $Av = V$ for all nonzero elements $v \in V$, then any nonzero $A$-submodule of $V$ must be the whole of $V$, so $V$ is simple.)

   For an example of a nonzero cyclic non-simple module, take $A = k[x]$, $V = k^2$ with $x$ acting by the matrix $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. With $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we have $xv = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so $Av$ contains both elements of the standard basis of $V$ and hence equals the whole of $V$. Thus $V$ is cyclic. But $k[0]$ is a nontrivial submodule, so $V$ is not simple.

   (b) Give an example to show that an indecomposable module need not be cyclic.

   \textbf{Solution:} Take $A = k[x, y]$, $V = k^3$ with $x$ acting by the matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $y$ acting by the matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$; it is easy to see that these two matrices commute. For any $v \in V$, $Av$ is contained in the span of $v$ and $k\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, so it can never equal the whole of $V$. Thus $V$ is not cyclic. To show that $V$ is indecomposable, we assume that $V = V_1 \oplus V_2$ is a decomposition into nonzero $A$-submodules, and seek a contradiction. Since $\dim V = 3$, either $V_1$ or $V_2$ must be one-dimensional; without loss of generality, say $V_1 = kv_1$ for some nonzero $v_1 \in V$. Then the fact that $V_1$ is an $A$-submodule implies that $xv_1 \in kv_1$ and $yv_1 \in kv_1$, so $v_1$ is an eigenvector for both matrices. These matrices have 0 as their only eigenvalue, and the intersection of their 0-eigenspaces is $k\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, so $V_1 = k\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Now for any $v \in V_2$ we have $xv \in V_1$ by the form of the matrix for $x$, and $xv \in V_2$ because $V_2$ is an $A$-submodule; since $V_1 \cap V_2 = \{0\}$, this forces $xv = 0$, and similarly $yv = 0$. Thus $v$ belongs to the intersection of the 0-eigenspaces, which is $V_1$ as we have seen, and we get the contradictory conclusion that $V_2 \subseteq V_1$.

   (c) In this part, take $A = k[x]$. Let $V$ be an $n$-dimensional $k[x]$-module, and let $X \in \text{End}(V)$ be the linear transformation by which $x$ acts. Show that if $V$ is cyclic, the minimal polynomial of $X$ equals its characteristic polynomial.
Solution: Let $m(x) \in k[x]$ be the minimal polynomial of $X$; by definition, this is the unique monic polynomial which satisfies the properties that $m(X) = 0$ and $m(x) \mid p(x)$ for any $p(x) \in k[x]$ such that $p(X) = 0$. In other words, $m(x)$ generates the kernel of the representation $\rho : k[x] \to \text{End}(V)$ defined by $\rho(x) = X$: this kernel is an ideal of the principal ideal domain $k[x]$, so it has to have a single generator (and the kernel is nonzero because $\text{End}(V)$ is finite-dimensional whereas $k[x]$ is infinite-dimensional, so $m(x)$ is nonzero and can thus be chosen to be monic).

Let $\chi(x) \in k[x]$ be the characteristic polynomial of $X$; by definition, this equals $\det(x \text{id}_V - X)$, so it is monic of degree $n$. By the Cayley–Hamilton theorem, $\chi(X) = 0$, so $m(x) \mid \chi(x)$.

For any $v \in V$, we can consider the $k[x]$-module homomorphism, where $k[x]$ is regarded as a $k[x]$-module via the regular representation. So its kernel is a $k[x]$-submodule of $k[x]$, i.e. an ideal (there is no need to say left ideal, since $k[x]$ is commutative), which again must have a single generator, say $m_v(x)$. Since $m(X) = 0$, we have $m(x)v = 0$ for all $v \in V$, so $m(x)$ is a multiple of $m_v(x)$ for all $v \in V$.

If $V$ is cyclic, there is some $v \in V$ such that the image of $\varphi_v$ is all of $V$. By the fundamental theorem, this implies $\dim k[x]/k[x]m_v(x) = n$. But $\dim k[x]/k[x]m_v(x)$ equals the degree of $m_v(x)$, so $m_v(x)$ has degree $n$. Hence $m(x)$, which is a multiple of $m_v(x)$, has degree at least $n$. Then the divisibility $m(x) \mid \chi(x)$ must be an equality $m(x) = \chi(x)$, as required.

The converse of this result is also true: if $m(x) = \chi(x)$, then there exists some $v \in V$ such that $k[x]v = V$. This requires a bit more theory to prove (specifically, the theory of rational canonical forms).

Recall that if $k$ is algebraically closed, we can find a basis of $V$ relative to which the matrix of $X$ is in Jordan canonical form. The condition that $m(x) = \chi(x)$ is then equivalent to the condition that there is a single Jordan block for each eigenvalue. For example, in the special case that $X$ is diagonalizable, the condition is that $X$ has $n$ distinct eigenvalues.

2. Let $k$ be a field and let $A$ be the associative algebra generated over $k$ by elements $x, y$ with the defining relation $xy - yx = y$. In other words, $A \cong k(x, y)/I$ where $I$ is the two-sided ideal of $k(x, y)$ generated by $xy - yx - y$.

(a) Take $k = \mathbb{C}$. Let $V$ be a finite-dimensional $A$-module on which $x, y$ act via the linear transformations $X, Y$. Recall that $V$ is the direct sum of the generalized eigenspaces $V_{\lambda, \text{gen}}^X$ of $X$ as $\lambda$ runs over $\mathbb{C}$. Show that

$$V_{\lambda}^Y \subseteq V_{\lambda+1}^X$$

for all $\lambda \in \mathbb{C}$.

Solution: The given relation $xy - yx - y$ implies that $(x - (\lambda + 1))y = y(x - \lambda)$. Applying this repeatedly we deduce that $(x - (\lambda + 1))^m y = y(x - \lambda)^m$ for all $m \in \mathbb{N}$. If $v \in V_{\lambda, \text{gen}}^X$ then $(x - \lambda)^m v = 0$ for some $m$, so

$$(x - (\lambda + 1))^m y v = y(x - \lambda)^m v = 0,$$

meaning that $yv \in V_{\lambda+1}^X$ as required.
(b) Hence or otherwise show that, when \( k = \mathbb{C} \), any simple finite-dimensional \( A \)-module must be one-dimensional.

**Solution:** Let \( V \) be a nonzero finite-dimensional \( A \)-module and define \( X, Y \in \text{End}(V) \) as in the previous part. Since \( X \) has only finitely many eigenvalues, there is some eigenvalue \( \lambda \) of \( X \) such that \( \lambda + 1 \) is not an eigenvalue. Let \( v \) be a \( \lambda \)-eigenvector of \( X \); thus \( xv = \lambda v \). By the previous part we have \( yv \in V_{\lambda+1}^{X,\text{gen}} \) which is zero, so \( yv = 0 \). Thus \( \mathbb{C}v \) is preserved by both \( X \) and \( Y \), hence it is an \( A \)-submodule. If \( V \) is simple, we can conclude that \( V = \mathbb{C}v \), so \( V \) is one-dimensional.

(c) Show by example that the conclusion of the previous part need not hold if \( k \) is an arbitrary field.

**Solution:** If we suppose that \( k \) has characteristic 2, then we have a two-dimensional \( A \)-module \( k^2 \) where \( x, y \) act as multiplication by the following matrices:

\[
x : \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad y : \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

It is easy to check that the relation \( xy - yx = y \) is respected, using the fact that \( -1 = 1 \). This is a simple module because any one-dimensional submodule would have to be spanned by a common eigenvector of these two matrices, and it is clear that the eigenspaces for the first matrix are interchanged by the second.

3. Take \( k = \mathbb{C} \) and let \( n \) be a positive integer. Let \( A \) be the **Clifford algebra** over \( \mathbb{C} \) generated by \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \) with the following defining relations, for all \( i, j \):

\[
x_i x_j + x_j x_i = 0,
\]

\[
y_i y_j + y_j y_i = 0,
\]

\[
x_i y_j + y_j x_i = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise}. \end{cases}
\]

You can take for granted the fact that \( A \) is spanned by the \( 2^{2n} \) elements

\[
x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_n^{b_n}, \quad \text{where } a_i, b_i \in \{0, 1\} \text{ for each } i.
\]

This is fairly obvious, since you can use the defining relations to reorder the terms in any monomial in the generators, and the relations entail that \( x_i^2 = y_i^2 = 0 \) for all \( i \). It will follow from this exercise that these \( 2^{2n} \) elements in fact form a basis for \( A \), which is less clear in advance.

Let \( V \) be a \( 2^n \)-dimensional vector space over \( \mathbb{C} \) with basis \( \{v_K \mid K \subseteq \{1, \ldots, n\}\} \) indexed by the subsets of \( \{1, \ldots, n\}\). Define linear transformations \( X_i, Y_i \) of \( V \) for all \( i \in \{1, \ldots, n\} \) by the following rules:

\[
X_i(v_K) = \begin{cases} 0, & \text{if } i \in K, \\ (-1)^{a(i,K)} v_{K \cup \{i\}}, & \text{if } i \notin K, \end{cases}
\]

\[
Y_i(v_K) = \begin{cases} (-1)^{a(i,K)} v_{K \setminus \{i\}}, & \text{if } i \in K, \\ 0, & \text{if } i \notin K, \end{cases}
\]
where \( a(i, K) \) denotes the number of elements of \( K \) which are less than \( i \).

You can take for granted the easy fact that these linear transformations satisfy the defining relations of the Clifford algebra, i.e. \( X_i X_j + X_j X_i = 0 \) and so forth. Hence there is an algebra homomorphism \( \rho : A \to \text{End}(V) \) sending \( x_i \) to \( X_i \) and \( y_i \) to \( Y_i \), which makes \( V \) into an \( A \)-module. Let \( B \) denote the subalgebra of \( A \) generated by the elements \( x_i y_i \) for all \( i \); then we can also consider \( V \) as a \( B \)-module.

(a) Show that \( V \) is the direct sum of pairwise non-isomorphic simple \( B \)-modules.

Solution: By definition, \( V \) is the direct sum of the one-dimensional subspaces \( \mathbb{C}v_K \) as \( K \) runs over all subsets of \( \{1, \ldots, n\} \). Note that for all \( i \) and \( K \) we have

\[
X_i Y_i(v_K) = \begin{cases} v_K, & \text{if } i \in K, \\ 0, & \text{if } i \notin K. \end{cases}
\]

So each one-dimensional subspace \( \mathbb{C}v_K \) is stable under the action of \( x_i y_i \) for all \( i \), and is hence a \( B \)-submodule of \( V \), and obviously simple as a \( B \)-module.

Moreover, these \( B \)-modules \( \mathbb{C}v_K \) are pairwise non-isomorphic, because if we have a \( B \)-module isomorphism \( \mathbb{C}v_K \cong \mathbb{C}v_{K'} \) then for all \( i \) the scalar by which \( x_i y_i \) acts on \( \mathbb{C}v_K \) must equal the scalar by which it acts on \( \mathbb{C}v_{K'} \), which forces the conditions \( i \in K \) and \( i \in K' \) to be equivalent, i.e. \( K = K' \).

(b) Show that \( V \) is a simple \( A \)-module.

Solution: Suppose that \( U \) is an \( A \)-submodule of \( V \). Then \( U \) is also a \( B \)-submodule of \( V \). As seen in the previous part, \( V \) is the direct sum of the one-dimensional simple \( B \)-submodules \( \mathbb{C}v_K \), which are pairwise non-isomorphic. By the description of submodules of a semisimple module given in lectures, this implies that \( U \) is the sum of some subset of the subspaces \( \mathbb{C}v_K \), i.e. \( U \) is spanned by a subset of the basis \( \{v_K\} \) of \( V \).

Suppose that \( U \neq \{0\} \). Then \( U \) must contain at least one of the basis vectors \( v_K \). It is clear from the definition of \( X_i \) and \( Y_i \) that, using a suitable composition of these linear transformations, we can map \( v_K \) to any one of the other basis vectors \( v_{K'} \), up to sign (we use the \( X_i \)'s to add elements to \( K \) and the \( Y_i \)'s to remove elements, until we reach \( K' \)). So \( U \), being stable under all the \( X_i \) and \( Y_i \), must contain all of the basis vectors and hence equals \( V \). This shows that \( V \) is a simple \( A \)-module.

(c) Deduce that \( \rho : A \to \text{End}(V) \) is an isomorphism of algebras.

Solution: Since \( V \) is a simple \( A \)-module and \( \mathbb{C} \) is algebraically closed, the Density Theorem implies that \( \rho : A \to \text{End}(V) \) is surjective. Hence \( \dim A \geq \dim \text{End}(V) = (2^n)^2 = 2^{2n} \), but from the given spanning set we know that \( \dim A \leq 2^{2n} \), so in fact \( \dim A = 2^{2n} = \dim \text{End}(V) \). A surjective linear map between vector spaces of the same finite dimension must be bijective, so \( \rho \) is an algebra isomorphism.

The conclusion is that \( A \cong \text{End}(V) \cong \text{Mat}_{2^n} (\mathbb{C}) \).