The rigidity from infinity for Alfvén waves

Pin YU

Tsinghua University, Beijing, China

joint work with Meng-Ni LI (Tsinghua University)
Outline

- The regime of Alfvén waves in MHD
- The rigidity problems from infinity for 1D waves
- Main results
- Ideas of the proof
The MHD equations and Alfvén waves

- Magnetohydrodynamics (MHD) studies the dynamics of magnetic fields in electrically conducting fluids.
- As fluids, MHD has similar wave phenomena: sound waves and magnetoacoustic waves (restoring forces are pressure and magnetic pressure). It is hard to understand mathematically.
- A new restoring force (the magnetic tension coming from the Lorentz force) leads to new wave phenomenon (no analogue in the ordinary fluid theory): **Alfvén waves**.
- H. Alfvén was awarded the Nobel prize for his ‘*fundamental work and discoveries in magnetohydrodynamics with fruitful applications in different parts of plasma physics*’, in particular his discovery of Alfvén waves. The Alfvén waves have wide and profound applications to plasma physics, geophysics, astrophysics, cosmology and engineering.
The MHD equations

- 3-dim incompressible MHD:

\[
\begin{align*}
\partial_t v + v \cdot \nabla v &= -\nabla p + b \cdot \nabla b, \\
\partial_t b + v \cdot \nabla b &= b \cdot \nabla v, \\
\text{div } v &= 0, \\
\text{div } b &= 0,
\end{align*}
\]

\(v\) = fluid velocity vector field, \(b\) = magnetic field.

- The pressure \(p = p_{\text{fluid}} + \frac{1}{2}|b|^2\) already contains the magnetic part.
Alfvén waves

- Physically, strong magnetic background generates Alfvén waves.
- Ansatz: \( B_0 = (0, 0, 1), \ b = B_0 + \tilde{b} \) and \( |\tilde{b}| + |\nu| \sim \varepsilon. \)

\[
\begin{align*}
\partial_t \nu + \nu \cdot \nabla \nu &= -\nabla p + \overbrace{(B_0 + \tilde{b}) \cdot \nabla (B_0 + \tilde{b})}^{O(\varepsilon^2)}, \\
\partial_t \tilde{b} + \nu \cdot \nabla (B_0 + \tilde{b}) &= \overbrace{(B_0 + \tilde{b}) \cdot \nabla \nu}^{O(\varepsilon^2)}. 
\end{align*}
\]
Alfvén waves (continued)

On the linearized level, we have

\[ \partial_t v - B_0 \cdot \nabla \tilde{b} = O(\varepsilon^2), \]
\[ \partial_t \tilde{b} - B_0 \cdot \nabla v = O(\varepsilon^2). \]

This was the original description of Alfvén himself. In particular, we have

\[ -\partial^2_t v + \partial^2_3 v \approx 0, \quad -\partial^2_t \tilde{b} + \partial^2_3 \tilde{b} \approx 0. \]

- These are 1-D wave equations.
- The waves (Alfvén waves) propagate along the \( x_3 \)-axis (in two directions).
Elsässer’s formulation

Let \( B_0 = (0, 0, 1) \) and we define

\[
Z_+ = v + b, \quad Z_- = v - b, \\
z_+ = Z_+ - B_0, \quad z_- = Z_- + B_0.
\]

The incompressible MHD system becomes

\[
\begin{align*}
\partial_t z_+ + Z_- \cdot \nabla z_+ &= -\nabla p, \\
\partial_t z_- + Z_+ \cdot \nabla z_- &= -\nabla p, \\
\text{div } z_+ &= 0, \quad \text{div } z_- = 0.
\end{align*}
\]

Heuristically, \( Z_+ \sim B_0, \ Z_- \sim -B_0 \)

- \( z_+ \) propagates along \( x_3 \)-axis towards the left with speed approximately 1.
- \( z_- \) propagates along \( x_3 \)-axis towards the right with speed approximately 1.
1-D linear wave equation and its solutions

We recall the theory for 1-D free wave equation:

\[
\begin{cases}
\Box \phi = 0, \\
(\phi, \partial_t \phi) \bigg|_{t=0} = (\phi_0(x), \phi_1(x)).
\end{cases}
\]

Its solutions can be represented as

\[\phi(t, x) = \phi_+(x - t) + \phi_-(x + t),\]

i.e., superposition of left-traveling and right-traveling waves.

If we use null frame \((L, \underline{L})\) and optical functions:

\[
\begin{align*}
L &= \partial_t + \partial_x, \\
\underline{L} &= \partial_t - \partial_x,
\end{align*}
\]

\[
\begin{align*}
u &= x - t, \\
\underline{u} &= x + t,
\end{align*}
\]

It is easy to see that \(\phi_\pm\) are constant along the level curves of \(u\) or \(\underline{u}\).
Future infinities

Given a point \((0, x_0) \in \Sigma_0\), the left-traveling characteristic line \(\ell(x_0)\) is defined as

\[
\ell(x_0) = \{(u, t) \mid u = x_0, \ t \in \mathbb{R}\}.
\]

The **left future characteristic infinity** is defined as (has a manifold structure)

\[
\mathcal{F} = \{\ell(u) \mid u \in \mathbb{R}\},
\]

Similarly, we can define the **right future characteristic infinity**:

\[
\mathcal{F} = \{\ell(u) \mid u \in \mathbb{R}\},
\]
Heuristically, a right-traveling characteristic line $\ell(x_0)$ passes the point $(0, x_0) \in \Sigma_0$ and hits $\mathcal{F}$ at the point $u = x_0$. Along $\ell(x_0)$:

$$L\phi(t, u + t) = L\phi(0, u).$$

Let $t \to +\infty$. We define the scattering field $L\phi(+\infty; u)$ on $\mathcal{F}$ as

$$L\phi(+\infty; u) = \lim_{t \to +\infty} L\phi(t, u + t) = L\phi(0, u).$$

Similarly, we can define the scattering field $L\phi(+\infty; u)$ on $\mathcal{F}$. 
Rigidity theorems for free waves

The rigidity theorem says that if the scattering fields vanish at infinities, then the fields must vanish identically, i.e.,

\[
\begin{align*}
  \mathcal{L}\phi(+\infty; u) &\equiv 0, \quad \text{on } \mathcal{F}, \\
  L\phi(+\infty; u) &\equiv 0, \quad \text{on } \mathcal{F}, \\
\end{align*}
\]

\[
\Rightarrow \quad \begin{cases} 
\mathcal{L}\phi \equiv 0, \\
L\phi \equiv 0 
\end{cases} \Rightarrow \phi \equiv 0.
\]
Rigidity theorems for free waves

In the same manner, we can define the past characteristic infinities. They are depicted as the following picture.

There is another version of rigidity theorem: if the scattering field \( L\phi(+\infty; u) \) vanishes at the future infinity and the scattering field \( L\phi(-\infty; u) \) vanishes at the past infinity, i.e.,

\[
\begin{align*}
L\phi(+\infty; u) &\equiv 0, \quad \text{on } \mathcal{F}, \\
L\phi(-\infty; u) &\equiv 0, \quad \text{on } \mathcal{P},
\end{align*}
\]

then \( L\phi(0, u) = 0 \) and \( L\phi(0, u) = 0 \)
Some notations

- Assume a smooth solution \((v, b)\) or \((z_+, z_-)\) exists on the spacetime \(\mathbb{R} \times \mathbb{R}^3\) (lifespan).
- \(\Sigma_t\) is the constant time slice (\(\Sigma_0\) initial slice).
- Characteristic vector fields \(L_+\) and \(L_-\) (Analogues of null vector fields for waves.):
  \[
  L_+ = \partial_t + \partial_3, \quad L_- = \partial_t - \partial_3;
  \]
  
  \[
  u_+ = x_3 - t, \quad u_- = x_3 + t.
  \]
Given a point \((0, x_1, x_2, x_3) \in \Sigma_0\), it determines uniquely a left-traveling straight line \(\ell_-\) parameterized by

\[
\ell_- : \mathbb{R} \to \mathbb{R} \times \mathbb{R}^3, \quad t \mapsto (x_1, x_2, x_3 + t, t).
\]

We define the **left future infinity** \(\mathcal{F}_+\) as:

\[
\mathcal{F}_+ = \{ \ell_-(x_1, x_2, u_-) \mid (x_1, x_2, u_-) \in \mathbb{R}^3 \}.
\]

We use \((x_1, x_2, u_-)\) as a fixed global coordinate system on it.
The definition of scattering field at future infinities

For \( p_0 = (0, \alpha, \beta, \gamma) \in \Sigma_0 \), we look at \( p_t = (t, \alpha, \beta, \gamma + t) \) on \( \ell_+ (\alpha, \beta, \gamma) \). The MHD equations give

\[
\partial_t z_- + Z_+ \cdot \nabla z_- = -\nabla p \quad \Rightarrow
\]

\[
\frac{d}{d\tau} \left( z_- (\tau, \alpha, \beta, \gamma + \tau) \right) = -\nabla p (\tau, \alpha, \beta, \gamma + \tau) - (Z_+ \cdot \nabla z_-) (\tau, \alpha, \beta, \gamma + \tau).
\]

Thus, by integrating between \( p_0 \) and \( p_t \), we obtain

\[
z_- (t, \alpha, \beta, \gamma + t) = z_- (0, \alpha, \beta, \gamma) - \int_0^t \left( \nabla p + Z_+ \cdot \nabla z_- \right) (\tau, \alpha, \beta, \gamma + \tau) d\tau.
\]

We want to define

\[
z_- (+\infty; \alpha, \beta, \gamma) = z_- (0, \alpha, \beta, \gamma) - \int_0^{+\infty} \left( \nabla p + Z_+ \cdot \nabla z_- \right) (\tau, \alpha, \beta, \gamma + \tau) d\tau.
\]
The main theorem: rough version

The rigidity theorem says that if the scattering fields vanish at infinities, then the fields must vanish identically, i.e.,

\[
\begin{aligned}
&z_-(\infty; \alpha, \beta, \gamma) \equiv 0, \quad \text{on } \mathcal{F}_+, \\
&z_+(\infty; \alpha, \beta, \gamma) \equiv 0, \quad \text{on } \mathcal{F}_-
\end{aligned}
\]

\[\Rightarrow\]

\[
\begin{aligned}
z_+ &\equiv 0, \\
z_- &\equiv 0.
\end{aligned}
\]

The scattering fields of Alfvén waves are the waves detected from a far-away observer. Therefore, the rigidity theorems have the following physical intuition: if no waves are detected by the far-away observers, then there are no Alfvén waves at all emanating from the plasma.
The Main Estimates

Let $\delta \in (0, \frac{2}{3})$ and $N_* \in \mathbb{Z}_{\geq 5}$. There exists a universal constant $\varepsilon_0 \in (0, 1)$ such that if the initial data $(z_+(0, x), z_-(0, x))$ satisfy

$$E^{N_*}(0) = \sum_{k=0}^{N_*+1} \sum_{+, -} \| (1 + |x_3 \pm a|^2)^{1+\delta} \nabla^k z_\pm(0, x) \|_{L^2(\mathbb{R}^3)}^2 \leq \varepsilon_0^2,$$

then the ideal MHD system admits a unique global solution $(z_+(t, x), z_-(t, x))$. Moreover, there is a universal constant $C$ such that the following energy estimates hold:

$$\sum_{k=0}^{N_*+1} \sup_{t \geq 0} \| (1 + |u_\mp \pm a|^2)^{1+\delta} \nabla^k z_\mp(t, x) \|_{L^2(\mathbb{R}^3)}^2 \leq C E^{N_*}(0).$$

where $u_\mp = x_3 \pm t$. 
The Refined Main Estimates

Let $\delta \in (0, \frac{2}{3})$ and $N_* \in \mathbb{Z}_{\geq 5}$. There exists a universal constant $\varepsilon_0 \in (0, 1)$ such that if the data $(z_+(0, x), z_-(0, x))$ satisfy

$$
\mathcal{E}^{N_*}_{\pm}(0) = \sum_{k=0}^{N_*+1} \| (1 + |x_3 \pm a|^2)^{\frac{1+\delta}{2}} \nabla^k z_{\pm}(0, x) \|_{L^2(\mathbb{R}^3)}^2 \leq \varepsilon^2_{\pm, 0}
$$

$$
\mathcal{E}^{N_*}(0) = \mathcal{E}^{N_*}_{+}(0) + \mathcal{E}^{N_*}_{-}(0) \leq \varepsilon^2_0,
$$

then the global solution $(z_+(t, x), z_-(t, x))$ to the ideal MHD system satisfies the following estimates: there is a universal constant $C$ such that

$$
\sum_{k=0}^{N_*+1} \| (1 + |u_+ + a|^2)^{\frac{1+\delta}{2}} \nabla^k z_+(t, x) \|_{L^\infty_t L^2_x}^2
$$

$$
= \mathcal{E}^{N_*}_{+}(0) + O(\mathcal{E}^{N_*}_{+}(0)(\mathcal{E}^{N_*}_{-}(0))^{\frac{1}{2}}).
$$
The position parameter $a$

The position parameter $a$ tracks the centers of the Alfvén waves.

The energy estimates derived in the paper will be independent of the choice of $a$. 

\[ \left\| (1 + |u_- + a|^2)^{\frac{1+\delta}{2}} \nabla^k z_+(t, x) \right\|_{L_\infty t} L_x^2 \]
Previous works

- Alfvén waves:
  - Alfvén, 1942
  - Bardos-Sulem-Sulem, He-Xu-Y, Cai-Lei, Wei-Zhang, Xu.

- Rigidity theorems (unique continuation):
  - F. John, S. Helgason, P.D. Lax, R.S. Phillips, ···
  - Ionescu-Klainerman, Alexakis-Ionescu-Klainerman,
    Alexakis-Schlue-Shao, Alexakis-Shao
The null structure of Alfvén waves and global existence

- $z_+$ and $z_-$ travel in opposite directions (speed $\sim 1$).
- After a long time, $z_\pm$ are far apart from each other and their distance can be measured by the time $t$.
- Decay mechanism for nonlinear (not linear) terms: the quadratic nonlinearity $\nabla z_+ \wedge \nabla z_-$ must be small since $z_+$ and $z_-$ are basically supported in different regions the spatial decay will be translated into the decay in time.
The proof of the rigidity theorem

We can use energy estimates to derive \((1 \leq |\beta| \leq N_*)\):

\[
\int_{\mathbb{R}^3} |\nabla^\beta z_\pm(\pm \infty; x_1, x_2, u_{\mp}) - \nabla^\beta z_\pm( T, x_1, x_2, u_{\mp} \mp T) |^2 \langle u_{\mp} \rangle^{2\omega} \to 0.
\]

Therefore, for any \(\epsilon < \varepsilon\), we can choose a large \(T_\epsilon\), so that at time slice \(\Sigma_{T_\epsilon}\), we have the following energy estimates:

\[
\sum_{+, -} \sum_{0 \leq |\beta| \leq N_*} \int_{\mathbb{R}^3} |\nabla^\beta z_\pm( T_\epsilon, x_1, x_2, x_3) |^2 \left(1 + |x_3 \pm T_\epsilon|^2\right)^\omega dx_1 dx_2 dx_3 < \epsilon^2.
\]
The proof of the rigidity theorem

- We take $a = T\epsilon$ (the center of the waves).
- We take $(z_+(T\epsilon, x), z_-(T\epsilon, x))$ as the initial data and solve backwards in time.
- The energy estimates give

$$E^{N'}(0) \leq C\epsilon^2,$$

Hence, $z_\pm \equiv 0$. 

![Diagram](image-url)
The second rigidity theorem

There exists a $T_\epsilon > 0$ such that we have the following smallness conditions:

$$\sum_{0 \leq |\beta| \leq N_*} \int_{\mathbb{R}^3} \left| \nabla^\beta z_- (T_\epsilon, x_1, x_2, u_+ + T_\epsilon) \right|^2 \langle u_+ \rangle^{2\omega} dx_1 dx_2 du_+ < \epsilon_-^2,$$

$$\sum_{0 \leq |\beta| \leq N_*} \int_{\mathbb{R}^3} \left| \nabla^\beta z_+ (-T_\epsilon, x_1, x_2, u_- + T_\epsilon) \right|^2 \langle u_- \rangle^{2\omega} dx_1 dx_2 du_- < \epsilon_+^2,$$
The second rigidity theorem

We make use of the Refined Energy estimate: at time slice $\Sigma_0$,

$$\mathcal{E}_{+}^{N'}(0) \leq C \epsilon_+^2 + C \epsilon_+^2 \epsilon_-.$$

We can still send $\epsilon_+^2 \to 0$. 
Thank you very much!