The rigidity from infinity for Alfvén waves

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Outline

- The regime of Alfvén waves in MHD
- The rigidity problems from infinity for 1D waves
- Main results
- Ideas of the proof

The MHD equations and Alfvén waves

- Magnetohydrodynamics (MHD) studies the dynamics of magnetic fields in electrically conducting fluids.
- As fluids, MHD has similar wave phenomena: sound waves and magnetoacoustic waves (restoring forces are pressure and magnetic pressure). It is hard to understand mathematically.
- A new restoring force (the magnetic tension coming from the Lorentz force) leads to new wave phenomenon (no analogue in the ordinary fluid theory): **Alfvén waves**.
- H. Alfvén was awarded the Nobel prize for his *'fundamental work and discoveries in magnetohydrodynamics with fruitful applications in different parts of plasma physics'*, in particular his discovery of Alfvén waves. The Alfvén waves have wide and profound applications to plasma physics, geophysics, astrophysics, cosmology and engineering.

The MHD equations

- 3-dim incompressible MHD:

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p + b \cdot \nabla b, \\ \partial_t b + v \cdot \nabla b = b \cdot \nabla v, \\ \operatorname{div} v = 0, \\ \operatorname{div} b = 0, \end{cases}$$

  v =fluid velocity vector field, b =magnetic field.
- The pressure $p = p_{\text{fluid}} + \frac{1}{2}|b|^2$ already contains the magnetic part.

Alfvén waves

- Physically, strong magnetic background generates Alfvén waves.
- Ansatz: $B_0 = (0,0,1)$, $b = B_0 + \tilde{b}$ and $|\tilde{b}| + |v| \sim \varepsilon$.

$$\partial_t v + \underbrace{v \cdot \nabla v}_{O(\varepsilon^2)} = \underbrace{-\nabla p}_{O(\varepsilon^2)} + (B_0 + \tilde{b}) \cdot \nabla(\cancel{B_0} + \tilde{b}),$$
$$\underbrace{\phantom{(B_0 + \tilde{b}) \cdot \nabla(B_0 + \tilde{b})}}_{\tilde{b}\cdot\nabla\tilde{b} \sim O(\varepsilon^2)}$$

$$\partial_t \tilde{b} + \underbrace{v \cdot \nabla(\cancel{B_0} + \tilde{b})}_{O(\varepsilon^2)} = (B_0 + \underbrace{\tilde{b}) \cdot \nabla v}_{O(\varepsilon^2)}.$$

Alfvén waves (continued)

On the linearized level, we have

$$\partial_t v - B_0 \cdot \nabla \tilde{b} = O(\varepsilon^2),$$
$$\partial_t \tilde{b} - B_0 \cdot \nabla v = O(\varepsilon^2).$$

This was the original description of Alfvén himself. In particular, we have

$$-\partial_t^2 v + \partial_3^2 v \approx 0, \quad -\partial_t^2 \tilde{b} + \partial_3^2 \tilde{b} \approx 0.$$

- These are 1-D wave equations.
- The waves (Alfvén waves) propagate along the x_3-axis (in two directions).

Elsässer's formulation

Let $B_0 = (0, 0, 1)$ and we define

$$Z_+ = v + b, \qquad Z_- = v - b,$$
$$z_+ = Z_+ - B_0, \quad z_- = Z_- + B_0.$$

The incompressible MHD system becomes

$$\begin{cases} \partial_t z_+ + Z_- \cdot \nabla z_+ = -\nabla p, \\ \partial_t z_- + Z_+ \cdot \nabla z_- = -\nabla p, \\ \operatorname{div} z_+ = 0, \quad \operatorname{div} z_- = 0. \end{cases}$$

Heuristically, $Z_+ \sim B_0$, $Z_- \sim -B_0$

- z_+ propagates along x_3-axis towards the left with speed approximately 1.
- z_- propagates along x_3-axis towards the right with speed approximately 1.

1-D linear wave equation and its solutions

We recall the theory for 1-D free wave equation:

$$\begin{cases} \Box\phi = 0, \\ (\phi, \partial_t\phi)\big|_{t=0} = (\phi_0(x), \phi_1(x)). \end{cases}$$

Its solutions can be represented as

$$\phi(t,x) = \phi_+(x-t) + \phi_-(x+t),$$

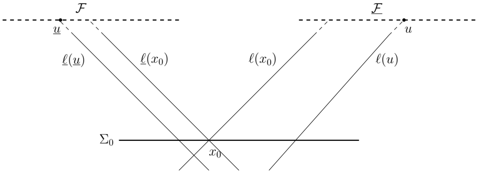
i.e., superposition of left-traveling and right-traveling waves.

If we use null frame $(L, \underline{L})$ and optical functions:

$$\begin{cases} L = \partial_t + \partial_x, \\ \underline{L} = \partial_t - \partial_x, \end{cases} \qquad \begin{cases} u = x - t, \\ \underline{u} = x + t, \end{cases}$$

It is easy to see that $\phi_\pm$ are constant along the level curves of u or $\underline{u}$.

Future infinities



Given a point $(0, x_0) \in \Sigma_0$, the left-traveling characteristic line $\underline{\ell}(x_0)$ is defined as

$$\underline{\ell}(x_0) = \left\{(\underline{u}, t)\middle|\underline{u} = x_0, t \in \mathbb{R}\right\}.$$

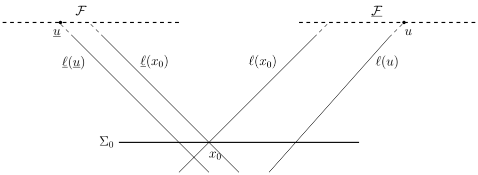
The left future characteristic infinity is defined as (has a manifold structure)

$$\mathcal{F} = \left\{\underline{\ell}(\underline{u})\middle|\underline{u} \in \mathbb{R}\right\},$$

Similarly, we can define the right future characteristic infinity:

$$\underline{\mathcal{F}} = \left\{\ell(u)\middle|u \in \mathbb{R}\right\},$$

Scattering field



Heuristically, a right-traveling characteristic line $\ell(x_0)$ passes the point $(0, x_0) \in \Sigma_0$ and hits $\underline{\mathcal{F}}$ at the point $u = x_0$. Along $\ell(x_0)$:

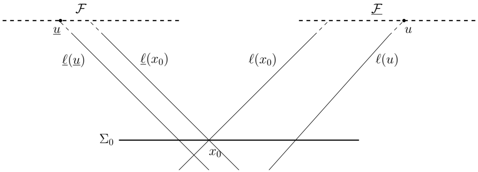
$$\underline{L}\phi(t, u+t) = \underline{L}\phi(0, u).$$

Let $t \to +\infty$. We define the scattering field $\underline{L}\phi(+\infty; u)$ on $\underline{\mathcal{F}}$ as

$$\underline{L}\phi(+\infty; u) = \lim_{t \to +\infty} \underline{L}\phi(t, u+t) = \underline{L}\phi(0, u).$$

Similarly, we can define the scattering field $L\phi(+\infty; \underline{u})$ on $\mathcal{F}$.

Rigidity theorems for free waves

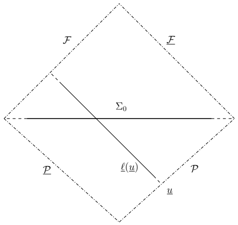


The rigidity theorem says that if the scattering fields vanish at infinities, then the fields must vanish identically, i.e.,

$$\begin{cases} \underline{L}\phi(+\infty; u) \equiv 0, & \text{on } \underline{\mathcal{F}}, \\ L\phi(+\infty; \underline{u}) \equiv 0, & \text{on } \mathcal{F}, \end{cases} \Rightarrow \begin{cases} \underline{L}\phi \equiv 0, \\ L\phi \equiv 0 \end{cases} \Rightarrow \phi \equiv 0.$$

Rigidity theorems for free waves

In the same manner, we can define the past characteristic infinities. They are depicted as the following picture.



There is another version of rigidity theorem: if the scattering field $\underline{L}\phi(+\infty; u)$ vanishes at the future infinity and the scattering field $L\phi(-\infty; \underline{u})$ vanishes at the past infinity, i.e.,

$$\begin{cases} \underline{L}\phi(+\infty; u) \equiv 0, & \text{on } \underline{\mathcal{F}}, \\ L\phi(-\infty; \underline{u}) \equiv 0, & \text{on } \mathcal{P}, \end{cases}$$

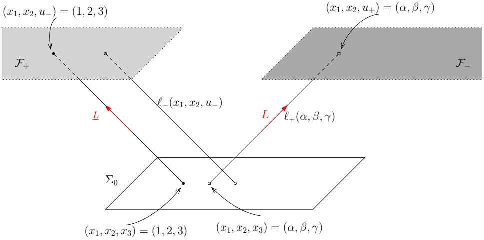
then $\underline{L}\phi(0, u) = 0$ and $L\phi(0, \underline{u}) = 0$

Some notations

- Assume a smooth solution (v, b) or (z_+, z_-) exists on the spacetime $\mathbb{R} \times \mathbb{R}^3$(lifespan).
- Σ_t is the constant time slice (Σ_0 initial slice).
- Characteristic vector fields L_+ and L_- (Analogues of null vector fields for waves.):

$$L_+ = \partial_t + \partial_3, \quad L_- = \partial_t - \partial_3;$$
$$u_+ = x_3 - t, \quad u_- = x_3 + t.$$

The future infinities



Given a point $(0, x_1, x_2, x_3) \in \Sigma_0$, it determines uniquely a left-traveling straight line ℓ_- parameterized by

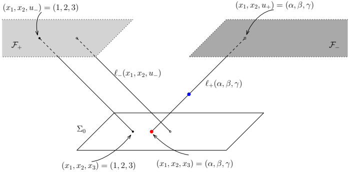
$$\ell_- : \mathbb{R} \to \mathbb{R} \times \mathbb{R}^3, \quad t \mapsto (x_1, x_2, x_3 + t, t).$$

We define the the left future infinity $\mathcal{F}_+$ as:

$$\mathcal{F}_+ = \left\{ \ell_-(x_1, x_2, u_-) \middle| (x_1, x_2, u_-) \in \mathbb{R}^3 \right\}.$$

We use (x_1, x_2, u_-) as a fixed global coordinate system on it.

The definition of scattering field at future infinities



For $p_0 = (0, \alpha, \beta, \gamma) \in \Sigma_0$, we look at $p_t = (t, \alpha, \beta, \gamma + t)$ on $\ell_+(\alpha, \beta, \gamma)$. The MHD equations give

$$\partial_t z_- + Z_+ \cdot \nabla z_- = -\nabla p \quad \Rightarrow$$

$$\frac{d}{d\tau}\Big(z_-(\tau, \alpha, \beta, \gamma+\tau)\Big) = -\nabla p(\tau, \alpha, \beta, \gamma+\tau) - \big(z_+ \cdot \nabla z_-\big)(\tau, \alpha, \beta, \gamma+\tau).$$

Thus, by integrating between p_0 and p_t, we obtain

$$z_-(t, \alpha, \beta, \gamma+t) = z_-(0, \alpha, \beta, \gamma) - \int_0^t \big(\nabla p + z_+ \cdot \nabla z_-\big)(\tau, \alpha, \beta, \gamma+\tau) d\tau.$$

We want to define

$$z_-(+\infty; \alpha, \beta, \gamma) = z_-(0, \alpha, \beta, \gamma) - \int_0^{+\infty} \big(\nabla p + z_+ \cdot \nabla z_-\big)(\tau, \alpha, \beta, \gamma+\tau) d\tau.$$

The main theorem: rough version

The rigidity theorem says that if the scattering fields vanish at infinities, then the fields must vanish identically, i.e.,

$$\begin{cases} z_-(\infty;\alpha,\beta,\gamma)\equiv 0, & \text{on } \mathcal{F}_+, \\ z_+(\infty;\alpha,\beta,\gamma)\equiv 0, & \text{on } \mathcal{F}_-, \end{cases} \Rightarrow \begin{cases} z_+\equiv 0, \\ z_-\equiv 0. \end{cases}$$

The scattering fields of Alfvén waves are the waves detected from a far-away observer. Therefore, the rigidity theorems have the following physical intuition: if no waves are detected by the far-away observers, then there are no Alfvén waves at all emanating from the plasma.

The Main Estimates

Let $\delta \in (0, \frac{2}{3})$ and $N_* \in \mathbb{Z}_{\geqslant 5}$. There exists a universal constant $\varepsilon_0 \in (0,1)$ such that if the initial data $\big(z_+(0,x), z_-(0,x)\big)$ satisfy

$$\mathcal{E}^{N_*}(0) = \sum_{+,-} \sum_{k=0}^{N_*+1} \Big\|(1+|x_3 \pm a|^2)^{\frac{1+\delta}{2}} \nabla^k z_\pm(0,x)\Big\|^2_{L^2(\mathbb{R}^3)} \leqslant \varepsilon_0^2,$$

then the ideal MHD system admits a unique global solution $\big(z_+(t,x), z_-(t,x)\big)$. Moreover, there is a universal constant C such that the following energy estimates hold:

$$\sum_{k=0}^{N_*+1} \sup_{t \geqslant 0} \Big\|(1+|u_\mp \pm a|^2)^{\frac{1+\delta}{2}} \nabla^k z_\pm(t,x)\Big\|^2_{L^2(\mathbb{R}^3)} \leqslant C\mathcal{E}^{N_*}(0).$$

where $u_\mp = x_3 \pm t$.

The Refined Main Estimates

Let $\delta \in (0, \frac{2}{3})$ and $N_* \in \mathbb{Z}_{\geqslant 5}$. There exists a universal constant $\varepsilon_0 \in (0,1)$ such that if the data $\big(z_+(0,x), z_-(0,x)\big)$ satisfy

$$\mathcal{E}_{\pm}^{N_*}(0) = \sum_{k=0}^{N_*+1} \big\|(1+|x_3 \pm a|^2)^{\frac{1+\delta}{2}} \nabla^k z_{\pm}(0,x)\big\|_{L^2(\mathbb{R}^3)}^2 \leqslant \varepsilon_{\pm,0}^2$$

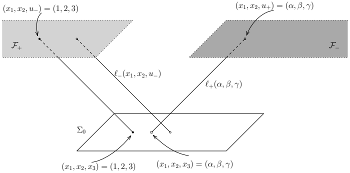
$$\mathcal{E}^{N_*}(0) = \mathcal{E}_{+}^{N_*}(0) + \mathcal{E}_{-}^{N_*}(0) \leqslant \varepsilon_0^2,$$

then the global solution $\big(z_+(t,x), z_-(t,x)\big)$ to the ideal MHD system satisfies the following estimates: there is a universal constant C such that

$$\sum_{k=0}^{N_*+1} \big\|(1+|u_- + a|^2)^{\frac{1+\delta}{2}} \nabla^k z_+(t,x)\big\|_{L_t^\infty L_x^2}^2$$

$$= \mathcal{E}_{+}^{N_*}(0) + O(\mathcal{E}_{+}^{N_*}(0)(\mathcal{E}_{-}^{N_*}(0))^{\frac{1}{2}}).$$

The position parameter a



$$\left\| (1 + |u_- + a|^2)^{\frac{1+\delta}{2}} \nabla^k z_+(t, x) \right\|^2_{L^\infty_t L^2_x}$$

- The position parameter a tracks the centers of the Alfvén waves.
- The energy estimates derived in the paper will be independent of the choice of a.

Previous works

- Alfvén waves:
  - Alfvén, 1942
  - Bardos-Sulem-Sulem, He-Xu-Y, Cai-Lei, Wei-Zhang, Xu.
- Rigidity theorems (unique continuation):
  - F. John, S. Helgason, P.D. Lax, R.S. Phillips, $\cdots$
  - Ionescu-Klainerman, Alexakis-Ionescu-Klainerman, Alexakis-Schlue-Shao, Alexakis-Shao

The null structure of Alfvén waves and global existence

- z_+ and z_- travel in opposite directions (speed ~ 1).
- After a long time, $z_\pm$ are far apart from each other and their distance can be measured by the time t.
- Decay mechanism for nonlinear (not linear) terms: the quadratic nonlinearity $\nabla z_+ \wedge \nabla z_-$ must be small since z_+ and z_- are basically supported in different regions **the spatial decay will be translated into the decay in time.**

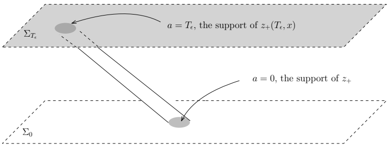
The proof of the rigidity theorem

We can use energy estimates to derive ($1 \leqslant |\beta| \leqslant N_*$):

$$\int_{\mathbb{R}^3} \left|\nabla^\beta z_\pm(+\infty; x_1, x_2, u_\mp) - \nabla^\beta z_\pm(T, x_1, x_2, u_\mp \mp T)\right|^2 \langle u_\mp \rangle^{2\omega} \to 0.$$

Therefore, for any $\epsilon < \varepsilon$, we can choose a large T_ϵ, so that at time slice Σ_{T_ϵ}, we have the following energy estimates:

$$\sum_{+,-} \sum_{0 \leqslant |\beta| \leqslant N_*} \int_{\mathbb{R}^3} \left|\nabla^\beta z_\pm(T_\epsilon, x_1, x_2, x_3)\right|^2 \left(1 + |x_3 \pm T_\epsilon|^2\right)^\omega dx_1 dx_2 dx_3 < \epsilon^2.$$

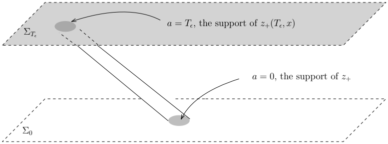


The proof of the rigidity theorem

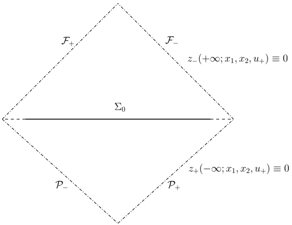
- We take $a = T_\epsilon$ (the center of the waves) .
- We take $\big(z_+(T_\epsilon, x), z_-(T_\epsilon, x)\big)$ as the initial data and solve backwards in time.
- The energy estimates give

$$\mathcal{E}^{N'_*}(0) \leqslant C\epsilon^2,$$

Hence, $z_\pm \equiv 0$.



The second rigidity theorem

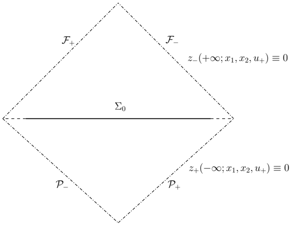


There exists a $T_\epsilon > 0$ such that we have the following smallness conditions:

$$\sum_{0\leqslant|\beta|\leqslant N_*} \int_{\mathbb{R}^3} \left|\nabla^\beta z_-(T_\epsilon, x_1, x_2, u_+ + T_\epsilon)\right|^2 \langle u_+\rangle^{2\omega} dx_1 dx_2 du_+ < \epsilon_-^2,$$

$$\sum_{0\leqslant|\beta|\leqslant N_*} \int_{\mathbb{R}^3} \left|\nabla^\beta z_+(-T_\epsilon, x_1, x_2, u_- + T_\epsilon)\right|^2 \langle u_-\rangle^{2\omega} dx_1 dx_2 du_- < \epsilon_+^2,$$

The second rigidity theorem



We make use of the Refined Energy estimaste: at time slice Σ_0,

$$\mathcal{E}_+^{N'_*}(0) \leqslant C\epsilon_+^2 + C\epsilon_+^2 \varepsilon_{-,0}.$$

We can still send $\epsilon_+^2 \to 0$.

Thank you very much!