THE VANISHING DISCOUNT PROBLEM FOR SYSTEMS OF HAMILTON-JACOBI EQUATIONS

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Asia-Pacific Analysis and PDE seminar. May 18, 2020
Vanishing discount problem

Convex, coercive HJ equations

Ergodic problem

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Systems of HJ equations

Appendix
Vanishing discount problem

Scalar Case: We consider the Hamilton-Jacobi equation

\[(P_\lambda) \quad \lambda v(x) + H(x, Dv(x)) = 0 \quad \text{in } \mathbb{T}^n.\]

Here

\[
\begin{cases}
    v = v^\lambda \quad \text{the unknown function on } \mathbb{T}^n, \\
    Dv = (v_{x_1}, \ldots, v_{x_n}), \\
    \lambda > 0 \quad \text{a given constant, discount factor,} \\
    H \quad \text{a given function of } (x, p) = (x, Dv(x)).
\end{cases}
\]

**Problem:** asymptotic behavior of $v^\lambda$ as $\lambda \to 0$.  

p.1
Convex, coercive HJ equations

Hypotheses:

(H0) Continuity: \( H \in C(\mathbb{T}^n \times \mathbb{R}^n) \).

(H1) \( H \) is convex,
\[
p \mapsto H(x, p) \text{ is convex.}
\]

(H2) \( H \) is coercive,
\[
\lim_{|p| \to \infty, x \in \mathbb{T}^n} \min H(x, p) = \infty.
\]

Property of \( H \):
\[
H(x, p) \geq \delta |p| - C \quad (\exists \delta > 0, \exists C > 0).
\]

Example: \( H(x, p) = |p|^m - f(x), \ m \geq 1, \ f \in C(\mathbb{T}^n) \).
Theorem 1  For each $\lambda > 0$ problem $(P_\lambda)$ has a unique solution $v^\lambda$. Furthermore,

$$(\lambda v^\lambda)_{\lambda > 0} \text{ is uniformly bounded},$$

$$(v^\lambda)_{\lambda > 0} \text{ is equi-Lipschitz continuous}.$$

- If $C_0 \geq |H(x, 0)|$, then

$$\lambda(C_0/\lambda) + H(x, 0) \geq 0, \quad \lambda(-C_0/\lambda) + H(x, 0) \leq 0,$$

and, by comparison, $-C_0/\lambda \leq v^\lambda(x) \leq C_0/\lambda$.

- Since $H(x, p) \geq \delta|p| - C$, we have

$$\delta|Dv^\lambda(x)| \leq C + \lambda\|v^\lambda\|_\infty.$$
Notation. Lagrangian of $H$:

$$L(x, \xi) := \sup_{p \in \mathbb{R}^n} [\xi \cdot p - H(x, p)].$$

Properties: $L$ is convex and lower semicontinuous on $\mathbb{T}^n \times \mathbb{R}^n$.

- $L(x, \xi) \geq -H(x, 0),$
- $L(x, \xi) \geq A|\xi| - H(x, A\xi/|\xi|)$
  $$\geq A|\xi| - \max_{|p| \leq A} H(x, p) \quad \forall A > 0,$$
- $L(x, \xi) \leq \sup_p (|\xi||p| - \delta |p| + C) = C \quad \forall \xi \in B_\delta.$$

Recall here that $H(x, p) \geq \delta |p| - C$.  

p.4
**Ergodic problem**

Formal expansion of the solution of \((P_\lambda)\):

\[
v^\lambda(x) \approx a_0(x)\lambda^{-1} + a_1(x) + a_2(x)\lambda + \cdots.
\]

Plug this into \((P_\lambda)\):

\[
a_0(x) + a_1(x)\lambda + a_2(x)\lambda^2 + \cdots
+ H(x, Da_0(x)\lambda^{-1} + Da_1(x) + Da_2(x)\lambda + \cdots) \approx 0.
\]

We deduce that

\[
Da_0(x) = 0 \quad \text{i.e.} \quad a_0(x) \equiv a_0 \quad \text{(constant)},
\]

\[
a_0 + H(x, Da_1(x)) = 0.
\]

The ergodic problem or additive eigenvalue problem:
The problem of finding a constant $c \in \mathbb{R}$ and a function $u \in C(\mathbb{T}^n)$ satisfying
\[(E) \quad H(x, Du(x)) = c \quad \text{in } \mathbb{T}^n.
\]
A classical result:

**Theorem 2 (Lions-Papanicolaou-Varadhan, 1987)**

Under (H0), (H2), there exists a solution $(c, u) \in \mathbb{R} \times C(\mathbb{T}^n)$ of (E). Moreover, the constant $c$ is unique.

- The constant $c$ is called the **critical value**, additive eigenvalue, or **ergodic constant**.

Their proof is to show that for some $(c, u) \in \mathbb{R} \times C(\mathbb{T}^n)$,

\[
\begin{align*}
-\lambda v^\lambda(x) & \to c \quad \text{uniformly on } \mathbb{T}^n, \\
v^\lambda(x) + \lambda^{-1}c & \to u(x) \quad \text{uniformly on } \mathbb{T}^n
\end{align*}
\]

along a subsequence.
Main question: does the whole family \( \{v^\lambda + \lambda^{-1}c\}_{\lambda > 0} \) converge to a function as \( \lambda \to 0^+ \)?

- The ergodic problem (E) has multiple solutions. If \( u \) is a solution of (E), then \( u + \text{const} \) is a solution. Consider the case

\[
Du \cdot (Du - D\psi) = 0 \quad \text{in} \quad \mathbb{T}^n, \quad \text{with} \quad \psi \in C^1(\mathbb{T}^n).
\]

We have many solutions:

\[
u = C_1, \quad u = \psi + C_2, \quad u = \min\{C_1, \psi + C_2\}.
\]
● Ergodic problem (E) arises in the ergodic optimal control, the homogenization of HJ equations, and the large-time behavior of solutions of evolutionary HJ equations.

A decisive result on the main question:

**Theorem 3 (Davini-Fathi-Iturriaga-Zavidovique, 2016)**

Assume (H0)–(H2). Let $c$ be the critical value. Then, for some function $v^0 \in C(\mathbb{T}^n)$, as $\lambda \to 0+$,

$$v^\lambda(x) + \lambda^{-1}c \to v^0(x) \text{ in } C(\mathbb{T}^n).$$

● If $H$ is not convex, the convergence of the whole family does not hold in general. A counterexample by B. Ziliotto (2019).
Related work:
1) A. Davini, A. Fathi, R. Iturriaga, M. Zavidovique, Coercive, convex HJ equation on $\mathbb{T}^n$ (closed manifold).
2) E. S. Al-Aidarous, E. O. Alzahrani, HI, A. M. M. Younas, Coercive, convex HJ equation on a bounded domain with the Neumann type BC.
3) H. Mitake, H. V. Tran
Viscous HJ equation on $\mathbb{T}^n$, with coercive and convex Hamiltonian. (2nd-order degenerate elliptic PDEs.)
4) D. Gomes, H. Mitake, H. V. Tran
Coercive, quasi-convex HJ equation on $\mathbb{T}^n$.
5) HI, H. Mitake, H. V. Tran,
2nd-order fully nonlinear, convex, degenerate elliptic PDEs on $\mathbb{T}^n$ or on a bounded domain with BC.
6) B. Ziliotto,
A counterexample, with non-convex Hamiltonian.
● Use of Mather measures.
An approach to Theorem 3

We review the proof of Theorem 3 (Davini et al.).

\[ \mathcal{P} = \mathcal{P}(\mathbb{T}^n \times \mathbb{R}^n) \text{ all Borel probability measures on } \mathbb{T}^n \times \mathbb{R}^n. \]

\[ \mathcal{P}_1 = \mathcal{P}_1(\mathbb{T}^n \times \mathbb{R}^n) \text{ all } \mu \in \mathcal{P} \text{ such that } \]

\[ \langle \mu, |\xi| \rangle := \int_{\mathbb{T}^n \times \mathbb{R}^n} |\xi| \mu(dx d\xi) < \infty. \]

(The function \((x, \xi) \mapsto |\xi|\) is denoted by \(|\xi|\))

Fix \((z, \lambda) \in \mathbb{T}^n \times [0, \infty)\).

\[ \mathcal{E}(z, \lambda) \text{ (closed measures)} \]

\[ := \{ \mu \in \mathcal{P}_1 \mid \lambda \psi(z) = \langle \mu, \xi \cdot D\psi + \lambda \psi \rangle \ \forall \psi \in C^1(\mathbb{T}^n) \}. \]

Note that

\[ \lambda u(x) + H(x, Du(x)) = \sup_{\xi} (\lambda u(x) + \xi \cdot Du(x) - L(x, \xi)). \]
When $\lambda = 0$, the defining condition reads

$$0 = \langle \mu, \xi \cdot D\psi \rangle \quad \forall \psi \in C^1(\mathbb{T}^n).$$

So, we write $\mathcal{C}(0)$ for $\mathcal{C}(z, 0)$.

**Theorem 4** Assume (H0)–(H2). If $\lambda > 0$, then

$$\lambda v^\lambda(z) = \min_{\mu \in \mathcal{C}(z, \lambda)} \langle \mu, L \rangle.$$

- Any minimizer $\mu$ of the optimization problem above is called a discounted Mather measure. $\mathcal{M}(z, \lambda) = \mathcal{M}(z, \lambda, L)$.

**Theorem 5** Assume (H0)–(H2). Let $c$ be the critical value. Then

$$-c = \min_{\mu \in \mathcal{C}(0)} \langle \mu, L \rangle.$$
Any minimizer $\mu$ of the optimization problem
\[
\min_{\mu \in \mathcal{E}(0)} \langle \mu, L \rangle.
\]
is called a Mather measure. $\mathcal{M} = \mathcal{M}(L)$.

We assume henceforth that $c = 0$. (Replace $H$ by $H - c$ if needed.)
The family $(v^\lambda)_{\lambda > 0}$ is equi-Lipschitz and uniformly bounded on $\mathbb{T}^n$ ($\Rightarrow$ relatively compact in $C(\mathbb{T}^n)$ by A^2 theorem).

(Uniform boundedness) Let $v_0 \in C(\mathbb{T}^n)$ be a solution of (E).
Let $C > 0$ be a constant such that $\|v_0\|_\infty \leq C$, and note that $v_0 + C$ (resp. $v_0 - C$) is a supersolution (resp. a subsolution) of $(P_\lambda)$.

By the comparison theorem, which is valid for $(P_\lambda)$ with $\lambda > 0$,
\[
v_0 - C \leq v^\lambda \leq v_0 + C \quad \forall \lambda > 0.
\]
all accumulation points of \((v^\lambda)_{\lambda>0}\) in \(C(\mathbb{T}^n)\) as \(\lambda \to 0+\).

By the observation above, \(\mathcal{V} \neq \emptyset\).

To show Theorem 3 (Davini et al.), it is enough to prove that \(#(\mathcal{V}) \leq 1\).

The main part of the proof (Theorem 3):

(Claim 1) \(\langle \mu, v \rangle \leq 0\) \quad \forall v \in \mathcal{V}, \forall \mu \in \mathcal{M}\).

(Claim 2) For \(\forall v, w \in \mathcal{V}, \forall z \in \mathbb{T}^n, \exists \mu \in \mathcal{M}\) s.t.

\[
w(z) \leq v(z) + \langle \mu, w \rangle.
\]

Claims 1 and 2 show that \(v, w \in \mathcal{V} \Rightarrow v = w\). I.e., \(#\mathcal{V} \leq 1\).

Proof (sketch) of Claims 1 and 2

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Davini et al. have obtained two representations of the limit function of \((v^\lambda)\). Here is one of them.

**Theorem 6** Assume \((H0)-(H2)\) and that \(c = 0\). Let \(v^0 \in C(\mathbb{T}^n)\) be the limit function of \((v^\lambda)\), that is,

\[
v^0 = \lim_{\lambda \to 0^+} v^\lambda \quad \text{in} \quad C(\mathbb{T}^n).
\]

Then

\[
v^0(x) = \max\{w(x) \mid w \in \mathcal{S}, \langle \mu, w \rangle \leq 0 \forall \mu \in \mathcal{M}\},
\]

where \(\mathcal{S}\) denotes the set of all solutions of \((E)\).
Remarks. Davini et al. have proved Theorem 4 by using techniques from optimal control or dynamical systems (value functions, the Hopf-Lax-Oleinik formula). Mitake-Tran use the adjoint method introduced by L. C. Evans. Mitake-Tran-HI use the convex duality argument similar to those used by Gomes (Duality principles for fully nonlinear elliptic equations, 2005) and Mikami-Thieullen (Duality theorem for the stochastic optimal control problem, 2006). A feature of this approach by Mitake-Tran-HI is that it belongs to functional analysis and is easily adopted to different situations, for instance, 2nd-order elliptic equations, nonlocal equations, systems of PDEs without going into detailed studies of the underlying dynamics. Siconolfi-HI use the convex duality in the form of the Hahn-Banach theorem.
The measures $\mu \in \bigcup_{z,\lambda} \mathcal{M}(z, \lambda, L)$ are supported in a common compact subset of $\mathbb{T}^n \times \mathbb{R}^n$. This is a consequence of the fact that $\sup_{\lambda > 0} \|Dv^\lambda\|_\infty < \infty$ (equi-Lipschitz). The set $\bigcup_{z,\lambda} \mathcal{M}(z, \lambda, L)$ is relatively compact in the topology of the weak convergence in the sense of measures.
Systems of HJ equations

Some recent results with Liang Jin.

The problem is now the \( m \)-system

\[
\begin{align*}
\lambda v_1^\lambda + H_1(x, Dv_1^\lambda, v^\lambda) &= 0 \quad \text{in } \mathbb{T}^n, \\
\vdots & \\
\lambda v_m^\lambda + H_m(x, Dv_m^\lambda, v^\lambda) &= 0 \quad \text{in } \mathbb{T}^n.
\end{align*}
\]

We write for the system above simply

\[(P_\lambda) \quad \lambda v^\lambda + H(x, Dv^\lambda, v^\lambda) = 0 \quad \text{in } \mathbb{T}^n,
\]

where \( v^\lambda = (v_i^\lambda) \) and \( H = (H_i) \).

Assume

(1) \( H_i \in C(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^m) \).

(2) \( H_i \) is coercive, that is,

\[
\lim_{|p| \to \infty} H_i(x, p, u) = \infty \quad \text{uniformly for } (x, u) \in \mathbb{T}^n \times B_R^m, \ \forall R > 0.
\]
(3) \((p, u) \mapsto H_i(x, p, u)\) is convex for any \(x \in \mathbb{T}^n\).

(4) \(H = (H_i)\) is monotone, that is, for \(u, v \in \mathbb{R}^m\),

\[(u-v)_k = \max_i (u-v)_i \geq 0 \implies H_k(x, p, u) \geq H_k(x, p, v).\]

(5) \(H(x, Du, u) = 0\) has a solution \(u \in C(\mathbb{T}^n)^m\).

**Theorem 7** Assume (1)-(5) above. Then, as \(\lambda \to 0^+\), we have

\[v^\lambda \to v^0 \text{ in } C(\mathbb{T}^n)^m\]

for some \(v^0 \in C(\mathbb{T}^n)^m\).

Davini-Zavidovique (2019) have studied the case where the coupling is linear and the coupling coefficients are constants.
Examples (coupling)

(E1) \[
\begin{align*}
\lambda u_1 + |D u_1| + u_1 - u_2 &= f_1(x), \\
\lambda u_2 + |D u_2|^2 + u_2 - u_1 &= f_2(x).
\end{align*}
\]

(E2) \[
\begin{align*}
\lambda u_1 + |D u_1| + (u_1 - u_2)^+ &= f_1(x), \\
\lambda u_2 + |D u_2| + (u_2 - u_1)^+ &= f_2(x).
\end{align*}
\]

(E3) \[
\begin{align*}
\lambda u_1 + |D u_1| + u_1 &= f_1(x), \\
\lambda u_2 + |D u_2|^2 + u_2 &= f_2(x).
\end{align*}
\]
Some ideas for the proof.

- Set $\mathbb{I} = \{1, \ldots, m\}$ and

$$L_i(x, \xi, \eta) = \sup_{(p,u)} [\xi \cdot p + \eta \cdot u - H_i(x, p, u)],$$

$$Y_i = \{\eta \in \mathbb{R}^m \mid \sum_{j \in \mathbb{I}} \eta_j \geq 0, \, \eta_j \leq 0 \text{ for } j \neq i\}.$$ 

**Theorem 8** Assume (1)–(3). Then,

$H$ monotone $\iff L_i(x, \xi, \eta) = \infty$ for $\eta \in \mathbb{R}^m \setminus Y_i$
• When $\lambda > 0$, we set $T^\lambda(\eta) = 1 + \lambda^{-1} \sum_j \eta_j$ for $\eta \in \mathbb{R}^m$.

Note that

$$T^\lambda(\eta) \geq 1 \quad \forall \eta \in Y_i, \; i \in \mathbb{I},$$

$$H^\lambda_{\phi+\lambda}T^\lambda_1(x, D(u + 1), u + 1) = H^\lambda_\phi(x, Du, u),$$

where $1 = (1, \ldots, 1) \in \mathbb{R}^m$ and

$$H^\lambda_\phi(x, pu) = \left(\lambda u_i + \sup_{(\xi, \eta)} (\xi \cdot + \eta \cdot u - \phi_i(x, \xi, \eta))\right)_{i \in \mathbb{I}}.$$

$\mathcal{P}(\lambda)$ the set of collections $\mu = (\mu_i)_{i \in \mathbb{I}}$ of nonnegative Borel measures $\mu_i$ on $\mathbb{T}^n \times \mathbb{R}^n \times Y_i$ such that

$$\langle \mu_i, |\xi| + |\eta| \rangle < \infty \quad \forall i \in \mathbb{I} \quad \text{and} \quad \sum_{i \in \mathbb{I}} \langle \mu_i, T^\lambda \rangle = 1.$$ 

$\mathcal{P}(0)$ the set of collections $\mu = (\mu_i)$ of nonnegative Borel measures $\mu_i$ on $\mathbb{T}^n \times \mathbb{R}^n \times Y_i$ such that

$$\langle \mu_i, |\xi| + |\eta| \rangle < \infty \quad \text{and} \quad \sum_{i \in \mathbb{I}} \langle \mu_i, 1 \rangle \leq 1.$$
Fix \((z, k, \lambda) \in \mathbb{T}^n \times I \times [0, \infty)\).

\(\mathcal{C}(z, k, \lambda)\), closed measures all \(\mu = (\mu_i) \in \mathcal{P}(\lambda)\) such that

\[
\lambda \psi_k(z) = \sum_{i \in I} \langle \mu_i, \xi \cdot D\psi_i + \eta \cdot \psi + \lambda \psi_i \rangle \quad \forall \psi \in C^1(\mathbb{T}^n)^m.
\]

Theorem 9 Assume (1)–(4). Then, if \(\lambda > 0\),

\[
\lambda v^\lambda_k(z) = \min_{\mu \in \mathcal{C}(z, k, \lambda)} \sum_{i \in I} \langle \mu_i, L_i \rangle.
\]

Discounted Mather measures \(\mathcal{M}(z, k, \lambda)\).

Proof (sketch). We have \(\|(v^\lambda, Dv^\lambda)\|_\infty < \infty\), We may assume that for some \(R > 0\),

\[
\begin{cases} 
L_i(x, \xi, \eta) = +\infty & \text{if } (\xi, \eta) \notin K_i, \\
L_i \in C(\mathbb{T}^n \times K_i),
\end{cases}
\]

where

\[
K_i = \overline{B}_R^n \times (\overline{B}_R^m \cap Y_i), \quad i \in \mathbb{I}.
\]
\( \mathcal{F}(\lambda) \) all pairs \( u = (u_i)_{i \in I} \in C(\mathbb{T}^n)^m \) and
\( \phi = (\phi_i)_{i \in I} \in \prod_{i \in I} C(\mathbb{T}^n \times K_i) \) such that

\[
\lambda u(x) + H_\phi(x, Du(x), u(x)) \leq 0 \quad \text{in } \mathbb{T}^n,
\]

where \( H_\phi = (H_{\phi,i})_{i \in I} \) and

\[
H_{\phi,i}(x, p, v) = \max_{(\xi, \eta) \in K_i} [p \cdot \xi + v \cdot \eta - \phi_i(x, \xi, \eta)].
\]

Our claim now is: Theorem 9 holds when we replace \( \mathcal{C}(z, k, \lambda) \) by \( \mathcal{C}_K(z, k, \lambda) := \{ \mu = (\mu_i) \in \mathcal{C}(z, k, \lambda) \mid \text{supp } \mu_i \subset \mathbb{T}^n \times K_i \} \).

Similarly, \( \mathcal{P}_K(\lambda) \) for \( \lambda \geq 0 \).
Set

\[ \mathcal{G}(z, k, \lambda) = \{ \phi - \lambda u_k(z) T^\lambda \mathbf{1} \mid (u, \phi) \in \mathcal{F}(\lambda) \}, \]

where \( \mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^m \).

This is a closed convex cone in \( \prod_{i \in \mathbb{I}} C(\mathbb{T}^n \times K_i) \) with vertex at the origin.

**Theorem 10** Let \( (z, k, \lambda) \in \mathbb{T}^n \times \mathbb{I} \times (0, \infty) \) and \( \mu \in \mathcal{P}_K(\lambda) \). Then, \( \mu \in \mathcal{C}_K(z, k, \lambda) \) if and only if

\[ \sum_{i \in \mathbb{I}} \langle \mu_i, g_i \rangle \geq 0 \quad \forall g = (g_i) \in \mathcal{G}(z, k, \lambda). \]
Proof (pictorial) \((\exists \nu \in \mathcal{M}(z, k, \lambda))\)

\[ G(z, k, \lambda) \]

\[ t(L - \lambda \nu_k(z)T^\lambda 1), \ t \geq 0 \]

\[ \prod_{i \in I} C(\mathbb{T}^n \times K_i) \]

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Thank you for your attention!
Appendix
Theorem 11 Let $\chi, u \in C(\mathbb{T}^n)$. Let $(z, \lambda) \in \mathbb{T}^n \times [0, \infty)$. Assume (H0)–(H2) and that $u$ is a subsolution of $\lambda u + H(x, Du) = \chi$ in $\mathbb{T}^n$. Then

$$\lambda u(z) \leq \langle \mu, L + \chi \rangle \quad \forall \mu \in \mathcal{C}(z, \lambda).$$

Proof (sketch). Assume that $u \in C^1$. Then

$$\lambda u(x) + \xi \cdot Du(x) \leq L(x, \xi) + \chi(x),$$

which implies

$$\lambda u(z) = \langle \mu, \lambda u + \xi \cdot Du \rangle \quad (\because \mu \in \mathcal{C}(z, \lambda))$$

$$\leq \langle \mu, L + \chi \rangle \quad \forall \mu \in \mathcal{C}(z, \lambda). \quad \square$$
Claim 1: Let $v \in \mathcal{V}$ and $\mu \in \mathcal{M}$. If we set $\chi := -\lambda v^\lambda$, then

$$H(x, Dv^\lambda) = \chi \quad \text{in } \mathbb{T}^n,$$

and, by Theorem 11,

$$0 \leq \langle \mu, L + \chi \rangle = \langle \mu, L - \lambda v^\lambda \rangle$$

$$= \langle \mu, L \rangle - \langle \mu, \lambda v^\lambda \rangle = -\lambda \langle \mu, v^\lambda \rangle,$$

$$= 0$$

and

$$\langle \mu, v^\lambda \rangle \leq 0.$$

In the limit as $\lambda \to 0+$, we get Claim 1.
Claim 2: Fix any \( v, w \in \mathcal{V} \) and \( z \in \mathbb{T}^n \). Choose a sequence \( \lambda_j \to 0^+ \) such that

\[
v^{\lambda_j} \to v \quad \text{in} \quad C(\mathbb{T}^n).
\]

By Theorem 4, we may choose a discounted Mather measure \( \mu_j \in \mathcal{M}(z, \lambda_j) \). Observe that

\[
\lambda_j w + H(x, Dw) = \lambda_j w,
\]

and, by Theorem 11,

\[
\lambda_j w(z) \leq \langle \mu_j, L + \lambda_j w \rangle = \langle \mu_j, L \rangle + \lambda_j \langle \mu_j, w \rangle = \lambda_j v^{\lambda_j}(z)
\]

\[
= \lambda_j v^{\lambda_j}(z) + \lambda_j \langle \mu_j, w \rangle.
\]
Dividing the above by $\lambda_j$ and taking the limit along a subsequence of $(\lambda_j)$, we get

$$w(z) \leq v(z) + \langle \mu, w \rangle$$

for some $\mu \in \mathcal{M}$ and, hence, $w(z) \leq v(z)$. 
Since \((v^\lambda, L) \in \mathcal{F}(\lambda)\), we have 
\(L - \lambda v_k^\lambda(z)T^\lambda 1 \in \mathcal{G}(z, k, \lambda)\) and, for all \(\mu \in \mathcal{C}(z, k, \lambda)\),

\[
0 \leq \sum_{i \in \mathcal{I}} \langle \mu_i, L_i - \lambda v_k^\lambda(z)T^\lambda \rangle = -\lambda v_k^\lambda(z) + \sum_{i \in \mathcal{I}} \langle \mu_i, L_i \rangle.
\]

\(\exists \nu \in \mathcal{C}(z, k, \lambda)\) minimizer: Note that if \(\|\phi\|_\infty < 1\), then 
\((v^\lambda, L + 1 + \phi) \in \mathcal{F}(\lambda)\). This implies that \(\text{int} \mathcal{G}(z, k, \lambda) \neq \emptyset\).

We may show that 
\(L - \lambda v_k^\lambda(z)T^\lambda 1 \in \partial \mathcal{G}(z, k, \lambda)\) By the Hahn-Banach theorem, 
\(\exists \nu \in (\prod_{i \in \mathcal{I}} C(K_i))^*\) such that \(\nu \neq 0\) and

\[
\langle \nu, L - \lambda v_k^\lambda(z)T^\lambda 1 \rangle \leq \langle \nu, g \rangle \ \forall g \in \mathcal{G}(z, k, \lambda).
\]

Since \(t(L - \lambda v_k^\lambda(z)T^\lambda 1) \in \mathcal{G}(z, k, \lambda)\), we see that

\[
\langle \nu, L - \lambda v_k^\lambda(z)T^\lambda 1 \rangle = 0.
\]

For \(\phi = (\phi_i)\), if \(\phi_i \geq 0 \ \forall i \in \mathcal{I}\), then 
\((v^\lambda, L + \phi) \in \mathcal{F}(\lambda)\). This, with the Riesz theorem, implies that \(\nu_i \geq 0\) and are Radon measures.