Degeneration of the spectral gap with negative Robin parameter

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What is a spectral gap?

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Heat distribution $u(x,t)$ flowed by heat equation:

$$u(x,t) = c_1 e^{-\lambda_1 t} u_1(x) + O(e^{-\lambda_2 t}), \quad \text{as } t \to \infty.$$

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Answer: Lower bound on gap $\rightsquigarrow$ Lower bound convergence rate!

The Robin Laplacian

Robin eigenvalue problem ($\alpha \in \mathbb{R}$):

$$\begin{cases} -\Delta u_j = \lambda_j u_j, & \text{on } \mathcal{D} \subset \mathbb{R}^n \\ \partial_\nu u_j + \alpha u_j = 0, & \text{on } \partial\mathcal{D}. \end{cases}$$

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Discrete spectrum

$$\lambda_1(\mathcal{D}, \alpha) < \lambda_2(\mathcal{D}, \alpha) \leq \lambda_3(\mathcal{D}, \alpha) \leq \cdots \to \infty$$

when $\mathcal{D}$ is a bounded domain (Lipschitz boundary).

Spectral gap: $\lambda_2(\mathcal{D}, \alpha) - \lambda_1(\mathcal{D}, \alpha) > 0$

Lower bounds: $\alpha = 0$ and ∞ (Neumann and Dirichlet)

I = interval of length $\mathrm{Diam}(\mathcal{D})$

Theorem (Payne, Weinberger 1960/ Andrews, Clutterbuck 2011)

Let $n \geq 2$. If $\mathcal{D} \subset \mathbb{R}^n$ is a convex domain then

$$(\lambda_2 - \lambda_1)(\mathcal{D}, 0) \geq (\lambda_2 - \lambda_1)(I, 0) = \frac{\pi^2}{\mathrm{Diam}(\mathcal{D})^2}$$

and

$$(\lambda_2 - \lambda_1)(\mathcal{D}, \infty) \geq (\lambda_2 - \lambda_1)(I, \infty) = \frac{3\pi^2}{\mathrm{Diam}(\mathcal{D})^2},$$

and equality is attained in the limit of rectangular boxes collapsing to a line segment.

Conjectured lower bound for $0 \leq \alpha \leq +\infty$

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Conjecture (Robin Gap Conjecture; Andrews, Clutterbuck, Hauer)

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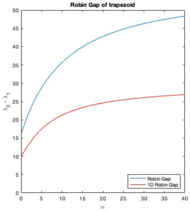
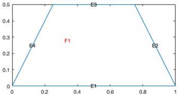
Holds for rectangular boxes and equality is attained in the limit of rectangular boxes collapsing to a line segment (Laugesen 2019).





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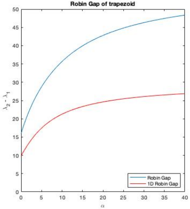

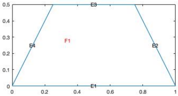

Question: Could the conjecture hold for $\alpha < 0$?

Gap conjecture fails to extend to $\alpha < 0$

Figure: *Double cone* domain $\mathcal{D}_\theta$ in $\mathbb{R}^2$ and $\mathbb{R}^3$

$$\mathcal{D}_\theta = \{(x, y) \in (-1, 1) \times \mathbb{R}^{n-1} : |y| < \tan(\theta/2)(1 - |x|)\}, \quad \text{for } \theta \in (0, \pi),$$

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Fix $\alpha < 0$.

Theorem (Kielty 2021)

If $\alpha < 0$ then $(\lambda_2 - \lambda_1)(\mathcal{D}_\theta) \to 0$ as $\theta \to 0$. Moreover, there exists a constant $C > 0$ such that

$$(\lambda_2 - \lambda_1)(\mathcal{D}_\theta) \leq C \exp\{-4(1 - \epsilon)|\alpha|/\theta\}, \quad \textit{for all } \theta \textit{ sufficiently small.}$$

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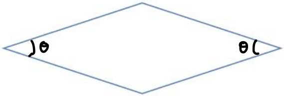

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$\implies (\lambda_2 - \lambda_1)(\mathcal{D}_\theta) < (\lambda_2 - \lambda_1)(I)$ for small θ

$\implies$ Robin gap conjecture does not extend to $\alpha < 0$

Double cone numerics in 2-dimensions

Theorem (Kielty 2021)

If $\alpha < 0$ *then* $(\lambda_2 - \lambda_1)(\mathcal{D}_\theta) \to 0$ *as* $\theta \to 0$. *Moreover, there exists a constant* $C > 0$ *such that*

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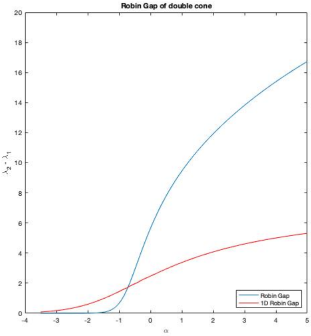
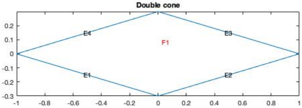
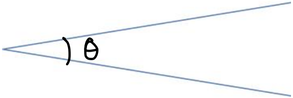




Heuristic for small gaps — infinite cone $\mathcal{C}_\theta$



Infinite cone w/opening angle θ

$$\mathcal{C}_\theta = \{(x,y) \in (0,\infty) \times \mathbb{R}^{n-1} : |y| < \tan(\theta/2)x\}$$

has ground state:

$$\phi_\theta(x,y) = A_\theta e^{\alpha x/\sin(\theta/2)}, \quad \text{when } \alpha < 0 \text{ and } \theta < \pi,$$

concentrates at vertex, as $\theta \to 0$.

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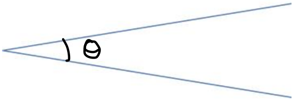

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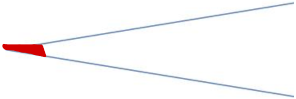
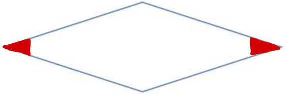
In fact, all e.f. $(n = 2)$ concentrate at vertex, as $\theta \to 0$ (Khalile & Pankrashkin 2016).

Intuition: Robin b.c. $\leftrightarrow$ potential $\alpha\delta_{\partial\mathcal{D}}$ (attractive for $\alpha < 0$)

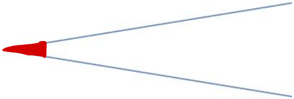
Heuristic for small gaps (cont.)

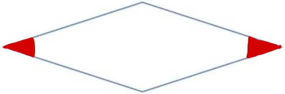




Heuristic: e.f. of $\mathcal{D}_\theta$ also concentrate at vertices so expect that $\lambda_j(\mathcal{D}_\theta) \approx \lambda_j(\mathcal{C}_\theta \sqcup \mathcal{C}_\theta)$, as $\theta \to 0$.

Heuristic for small gaps (cont.)





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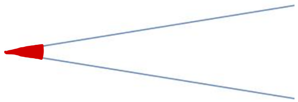
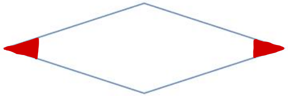
$$\lambda_1(\mathcal{C}_\theta \sqcup \mathcal{C}_\theta) = \lambda_2(\mathcal{C}_\theta \sqcup \mathcal{C}_\theta) \implies (\lambda_2 - \lambda_1)(\mathcal{C}_\theta \sqcup \mathcal{C}_\theta) = 0$$

$$\overset{\text{heuristic}}{\implies} (\lambda_2 - \lambda_1)(\mathcal{D}_\theta) \to 0, \quad \text{as } \theta \to 0.$$

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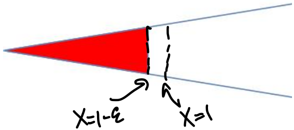
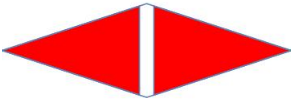
$$\overset{\text{heuristic}}{\implies} (\lambda_2 - \lambda_1)(\mathcal{D}_\theta) \to 0, \quad \text{as } \theta \to 0.$$

Proof: trial function argument $\rightsquigarrow \lambda_2(\mathcal{D}_\theta) \leq \lambda_1(\mathcal{C}_\theta) + E(\theta)$ and $\lambda_1(\mathcal{D}_\theta) \geq \lambda_1(\mathcal{C}_\theta) - E(\theta)$, with $E(\theta)$ exponential small.

Upper bound on $\lambda_2(\mathcal{D}_\theta)$

Transplant a cutoff ground state $\chi\phi_\theta$ onto each vertex of $\mathcal{D}_\theta$ to make trial function $\psi_\theta = (\chi\phi_\theta) \circ F - (\chi\phi_\theta) \circ G$.





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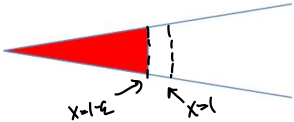

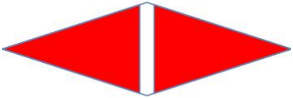

$$\mathcal{D}_\theta \text{ symmetric } \implies u_1 \text{ symmetric } \implies \psi_\theta \perp u_1$$

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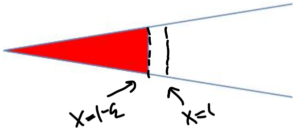

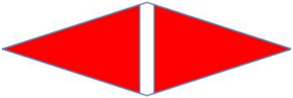

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$$R_{\mathcal{D}_\theta}[f] = \frac{\int_{\mathcal{D}_\theta} |\nabla f|^2 \, dx + \alpha \int_{\partial \mathcal{D}_\theta} f^2 \, dS}{\int_{\mathcal{D}_\theta} f^2 \, dx}, \quad \text{for } f \in H^1(\mathcal{D}_\theta).$$

$$\lambda_2(\mathcal{D}_\theta) \le R_{\mathcal{D}_\theta}[\psi_\theta] = R_{\mathcal{C}_\theta}[\chi\phi_\theta] \le \lambda_1(\mathcal{C}_\theta) + C \exp\{-4(1-\epsilon)|\alpha|/\theta\},$$

as $\theta \to 0$.

ϕ & u ground states of $\mathcal{C}_\theta$ & $\mathcal{D}_\theta$, $\Delta_\tau(\cdot) = \tau^{-1}\mathrm{div}(\tau\nabla(\cdot))$ with $\tau = \phi^2$, and $v = u/\phi$:

$$\begin{cases} -\Delta_\tau v = \mu_1 v, & \text{on } \mathcal{T}_\theta \\ \partial_\nu v + (\alpha/\sin(\theta/2))v = 0, & \text{on } \Gamma_\theta \\ \partial_\nu v = 0, & \text{on } \Sigma_\theta, \end{cases}$$

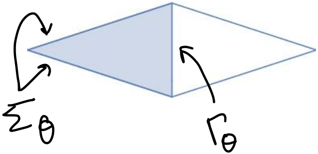

Figure: Truncated cone $\mathcal{T}_\theta$

$$\mu_1 = \mu_1(\mathcal{T}_\theta) = \lambda_1(\mathcal{D}_\theta) - \lambda_1(\mathcal{C}_\theta) < 0$$

Goal: Get lower bound to show $\mu_1(\mathcal{T}_\theta) \to 0$ as $\theta \to 0$.

Lower bound on $\lambda_1(\mathcal{D}_\theta)$: Ratio of e.f.

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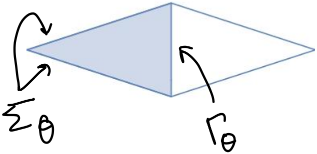

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Limiting problem unclear: $\mathcal{T}_\theta$ collapsing, boundary parameter tending to $-\infty$, and $\tau = e^{2\alpha x/\sin(\theta/2)}$ concentrating at vertex

Solution: "Push out" problem on $\mathcal{T}_\theta$ to a radial problem on spherical sector, extend to $B(1)$, rescale to $B(\theta^{-1})$.

Define "push out" $P(\mathcal{T}_\theta) = \mathcal{S}_\theta$ stretch $\mathcal{T}_\theta$ linearly in radial direction and $\sigma = \tau \circ P = e^{\alpha r/\tan(\theta/2)}$

$$\begin{cases} -\Delta_\sigma w = \mu_1 w, & \text{on } \mathcal{S}_\theta \\ \partial_\nu w + (\beta/\sin(\theta/2))w = 0, & \text{on } \tilde{\Gamma}_\theta \\ \partial_\nu w = 0, & \text{on } \Sigma_\theta, \end{cases}$$

$\beta = (1 + o(1))\alpha$, as $\theta \to 0$

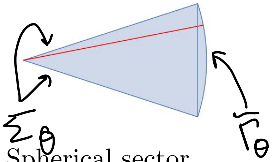

Figure: Spherical sector $\mathcal{S}_\theta = \mathcal{C}_\theta \cap B(1)$

Lower bound on $\lambda_1(\mathcal{D}_\theta)$: Push out to $\mathcal{S}_\theta$

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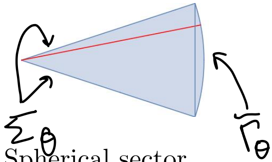

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Use $v \circ P \in H^1(\mathcal{S}_\theta)$ as trial function & $P \approx Id$ in $C^1(\overline{\mathcal{S}_\theta}; \mathbb{R}^n)$

$$\implies \mu_1(\mathcal{S}_\theta) \leq R_{\mathcal{S}_\theta}[v \circ P] \leq (1 + o(1))R_{\mathcal{T}_\theta}[v] = (1 + o(1))\mu_1(\mathcal{T}_\theta)$$

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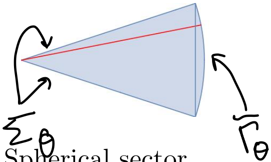

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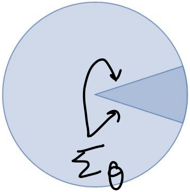
Use $v \circ P \in H^1(\mathcal{S}_\theta)$ as trial function & $P \approx Id$ in $C^1(\overline{\mathcal{S}_\theta}; \mathbb{R}^n)$

$$\implies \mu_1(\mathcal{S}_\theta) \le R_{\mathcal{S}_\theta}[v \circ P] \le (1 + o(1))R_{\mathcal{T}_\theta}[v] = (1 + o(1))\mu_1(\mathcal{T}_\theta)$$

Map P used to analyze Dirichlet e.v. of thin triangles (Freitas 2007)

Lower bound on $\lambda_1(\mathcal{D}_\theta)$: Schrödinger on a ball

Neumann conditions on Σ_θ and σ radial $\implies$ ground state on $\mathcal{S}_\theta$ extends to ball $B(1)$



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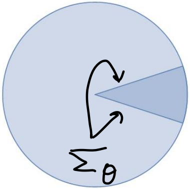

Rescale radial by $r \to \theta^{-1} r$ then let $\varphi(r) = \tilde{w}(r)/e^{-\alpha r}$:

$$\begin{cases} (-\Delta - \frac{(n-1)|\alpha|}{r} + \alpha^2)\varphi = \nu_1 \varphi, & \text{on } B(\theta^{-1}) \\ (\partial_r + \gamma)\varphi = 0, & \text{on } \partial B(\theta^{-1}). \end{cases}$$

where $\nu_1(\theta) = \theta^2 \cdot \mu_1(\mathcal{S}_\theta)$ and $\gamma = \gamma(\theta) = o(1)$.

First e.v. of $-\Delta - (n-1)|\alpha|/r + \alpha^2$ on $\mathbb{R}^n$ is 0

$\implies$ expect that $\nu_1(\theta) \to 0$, as $\theta \to 0$.

Lower bound on $\lambda_1(\mathcal{D}_\theta)$: Special functions

Ground state of $-\Delta - (n-1)|\alpha|/r$ with e.v. E has form

$$\varphi(r) = M(a_0, b_0; 2\sqrt{|E|}r)e^{-\sqrt{|E|}r}, \quad \text{on } B(\theta^{-1}),$$

where $M(a_0, b_0; \cdot)$ is a *confluent hypergeometric function.*

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Estimates on power series for $\rho \mapsto M(a_0, b_0; \rho)$ and IVT

$$\implies -C\exp\{-4(1-\epsilon)|\alpha|/\theta\} \le \nu_1(\theta) < 0$$
$$\implies -C\exp\{-4(1-\epsilon)|\alpha|/\theta\} \le \lambda_1(\mathcal{D}_\theta) - \lambda_1(\mathcal{C}_\theta) < 0.$$

Combine w/ upper bound on $\lambda_2(\mathcal{D}_\theta)$
$\implies (\lambda_2 - \lambda_1)(\mathcal{D}_\theta) \le C\exp\{-4(1-\epsilon)|\alpha|/\theta\}$ □

Questions and open problems

$$\text{RGC: } (\lambda_2 - \lambda_1)(\mathcal{D}, \alpha) \geq (\lambda_2 - \lambda_1)(I, \alpha), \quad \text{for } \alpha \in [0, +\infty].$$

Theorem (Laugesen 2019)

Let $\alpha \in (-\infty, +\infty]$. If $\mathcal{R} \subset \mathbb{R}^n$ is a rectangular box then

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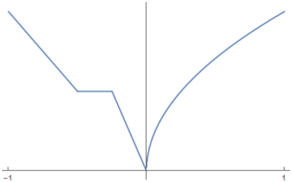
Theorem (Ashbaugh & Kielty 2020)

Let $\lambda_j(V, \alpha)$ be the eigenvalues of $-\frac{d^2}{dx^2} + V$ with α-Robin b.c. If V is a symmetric single-well potential then

$$(\lambda_2 - \lambda_1)(V, \alpha) \geq (\lambda_2 - \lambda_1)(0, \alpha), \quad \text{for each } \alpha \in (-\infty, +\infty].$$

Open problems



Single-well potential with centered transition point.

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Let $\lambda_j(V, \alpha)$ *be the eigenvalues of* $-\frac{d^2}{dx^2} + V$ *with* α*-Robin b.c. If* V *is convex or single-well w/ centered transition point then*

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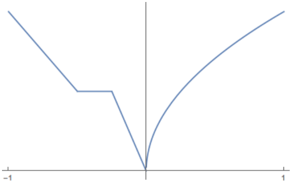
Known for:

- V convex when $\alpha \geq -1/L$ (Andrews, Clutterbuck, Hauer 2020)
- V single-well w/ centered transition point when $\alpha \geq 0$ (Ashbaugh, Kielty 2020)

Open problems



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Thank you for your attention!