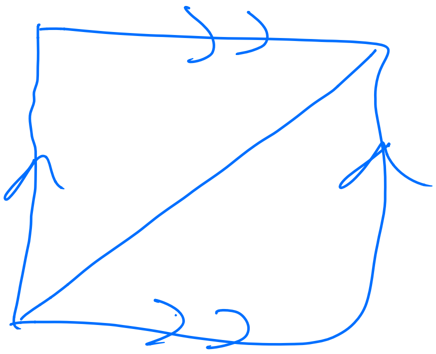
Triangulation Complexity and Mapping Tori

<u>"Defn"</u> A triangulation of an n-manifold M



$\cong T^2$

is a cell complex T and homeo $\varphi: T \to M$ where T is constructed by taking a collection of n-simplices and identifying faces.

<u>Defn.</u> The triangulation complexity of a closed 3-manifold M is the minimal number of tetrahedra in a triangulation. $\Delta(M)$

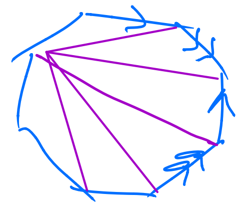
Calculate:

- homology
- unknottedness
- generate interesting surfaces

Regina

Fact. For any fixed n, there are only finitely many M with $\Delta(M) \le n$

Ex.



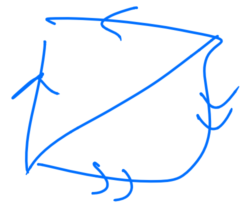
4g sides

genus g

$$\Delta(\Sigma_g) \le 4g-2$$

By Euler characteristic argument, this is the best you can do

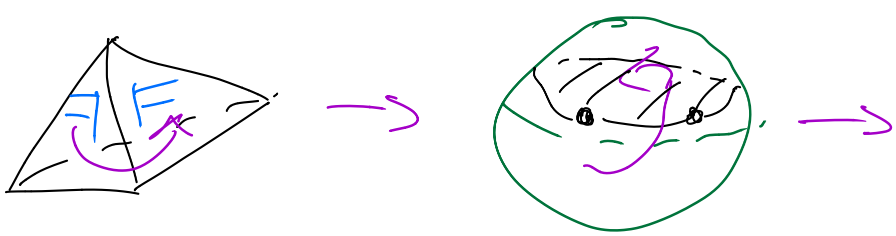
$$\Delta(S^2) = 2.$$



$\Delta(\Sigma_g)$ must be even

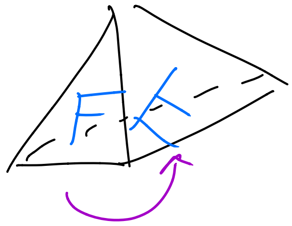
3-manifolds:

$$\Delta(S^3) = 1$$



S^3

$$\Delta(D^2 \times S^1) = 1$$



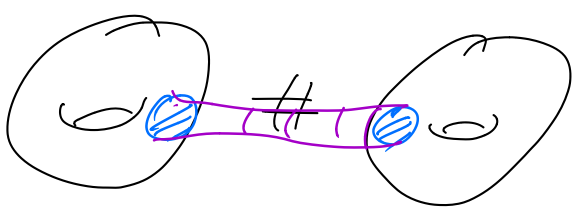
$\cong D^2 \times S^1$

Relating $\Delta(M)$ to geometry/dynamics

How does $\Delta(M)$ behave under connect sum?

Def. Let M, N be n-manifolds. The connect sum $M \# N$ is constructed by removing an n-ball from each and identify the resulting boundary components.

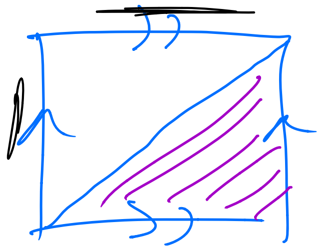
$$\Sigma_g \# \Sigma_h \cong \Sigma_{g+h}$$



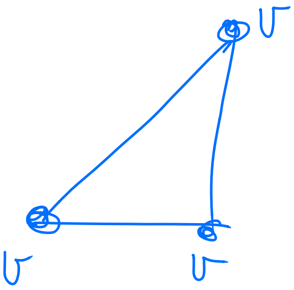
Q: $\Delta(M \# N) \leq \Delta(M) + \Delta(N) - 2$??

A: no as the closure of a tetrahedron in M is not necessarily a 3-ball

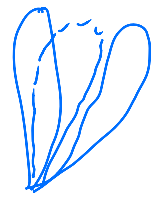
$$T^2 \# T^2 \cong \Sigma_2$$



$\cong$



$\cong$



$\#$ itself $\cong T^2$

$$\Delta(\Sigma_{g+h}) = \Delta(\Sigma_g) + \Delta(\Sigma_h) + 2$$

Q: what happens with 3-manifolds

Open

Hyperbolic volume

Suppose closed orientable 3-manifold M is hyperbolic

i.e. atlas of charts into $\mathbb{H}^3$ with transition maps that are hyperbolic isometries.

As M closed, equivalent definition

$M \cong \mathbb{H}^3/\Gamma$ ← discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$

By Mostow rigidity, $\mathrm{vol}(M)$ is an invariant of M.

hyperbolic volume

Th. (Gromov, Thurston)

$$\Delta(M) \geq \frac{\mathrm{vol}(M)}{\text{volume of a regular ideal tetrahedron in } \mathbb{H}^3}$$

Q: $\Delta(M) \sim \mathrm{vol}(M)$

A: no, by hyperbolic Dehn surgery.

Let N is a finite volume hyp. manifold with $\partial N = T^2$.

Dehn surgery on N is gluing in a solid torus along ∂N.

Parametrised by a slope $\in \mathbb{Q}$.

Th. (Thurston)

For all but finitely many $p/q \in \mathbb{Q}$, resulting $N_{p/q}$ is hyperbolic.

Furthermore, $\mathrm{vol}(N_{p/q}) < \mathrm{vol}(N)$ and we get infinitely many topological manifolds.

Mapping Tori

Let Σ be a closed surface and $\phi : \Sigma \to \Sigma$ is a homeomorphism.

Def The mapping torus of ϕ, M_ϕ, is

$$\Sigma \times I \Big/ \phi$$

Q: how does the dynamics of ϕ relate to the geometry of M_ϕ?

Upper bound:

Def. A 2-2 Pachner move on a triangulation of Σ is

Fact. If T_1 and T_2 are one-vertex triangulations of Σ, you can move between them using 2-2 moves.

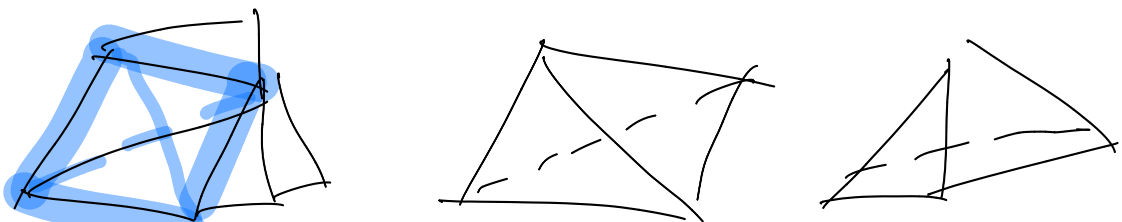
Def $Tr(\Sigma)$ has vertices triangulations of Σ and edges 2-2 moves.

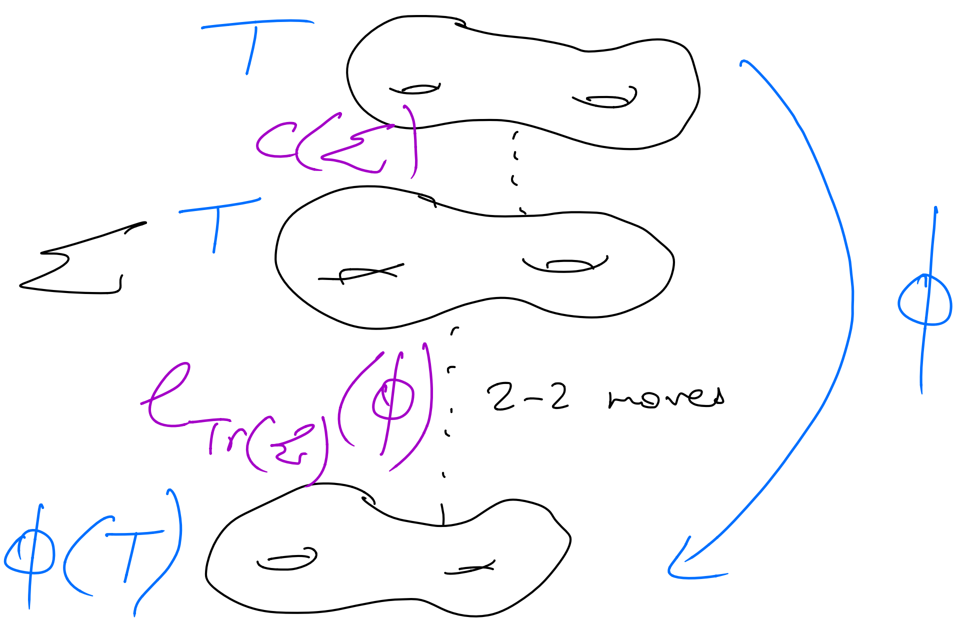
$Tr(\Sigma)$ is connected.

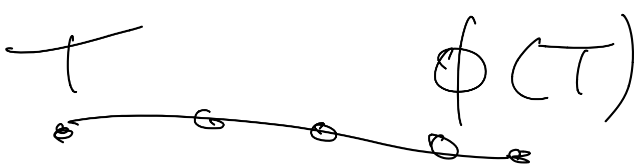
Def. The translation length of ϕ in $Tr(\Sigma)$, $\ell_{Tr(\Sigma)}(\phi)$, is

$$\inf_{x \in Tr(\Sigma)} \{ d(x, \phi(x)) \}.$$

Lemma. $\Delta(M_\phi) \leq c(\Sigma) + \ell_{Tr(\Sigma)}(\phi)$.







Th. (Lackenby–Purcell, '19)
When M_ϕ is hyperbolic, there are constants c and d depending only on Σ s.t.

$$c \le \frac{\Delta(M_\phi)}{\ell_{Tr(\Sigma)}(\phi)} \le d.$$

Th. If $\Sigma = T^2$ and ϕ is Anosov, then

Dehn twist

$$c \le \frac{\Delta\left(M_{t_c^n \phi}\right)}{m+n} \le d$$

Stephan Tillmann

ϕ pA ✓
periodic ✓
reducible   free a multi-curve