Are there purely topological explanations of the number of equilibria of pendulum systems?

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$$\boxed{\begin{array}{ccccc} A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 \\ A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 \\ A_3 & \longrightarrow & B_3 & \longrightarrow & C_3 \end{array}} \Longrightarrow D$$

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- If a feature of a complex dynamical system remains invariant, it naturally raises a question: how are such non-causal topological explanations even applicable to a rich and complex physical system?
- These explanations are also called as Distinctively Mathematical Explanations (DMEs).

Plan for the talk

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3. Conclusion

Causal explanation concerning the double pendulum

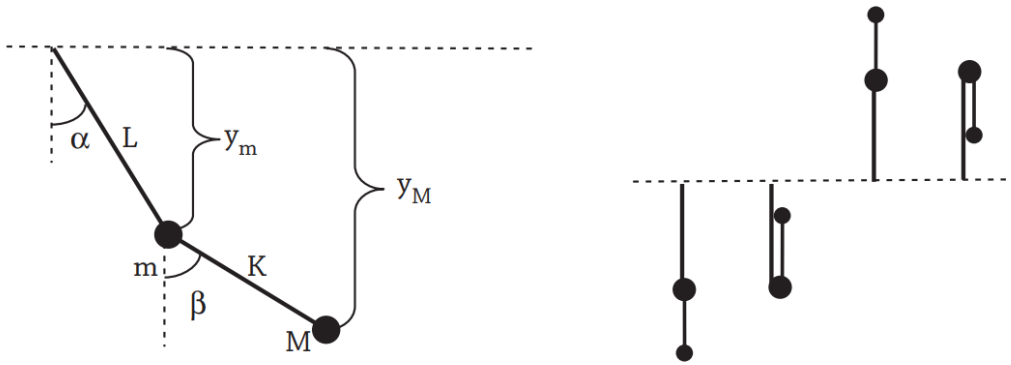

Figure: A double pendulum with stiff rods: Lange (2016)

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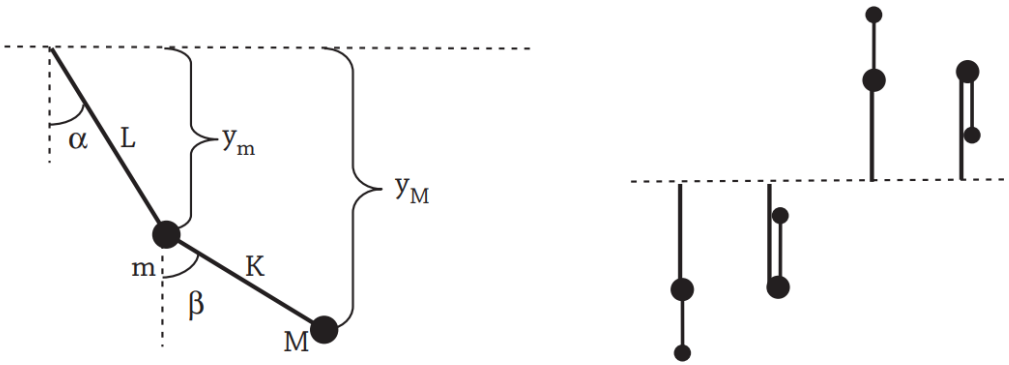

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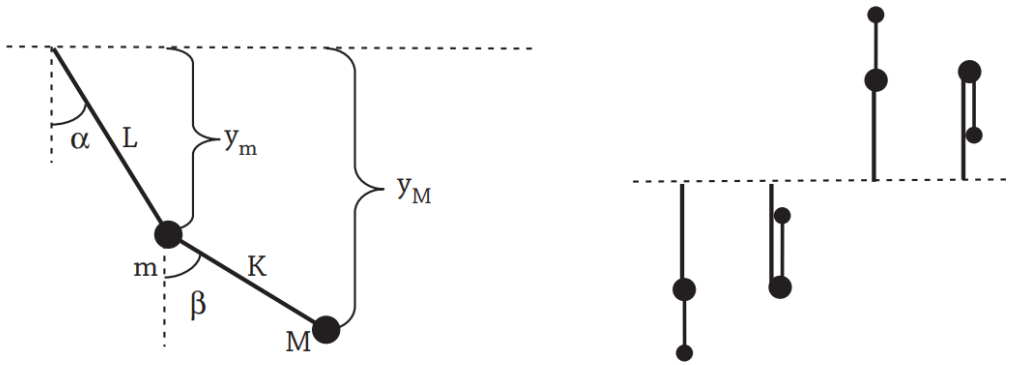

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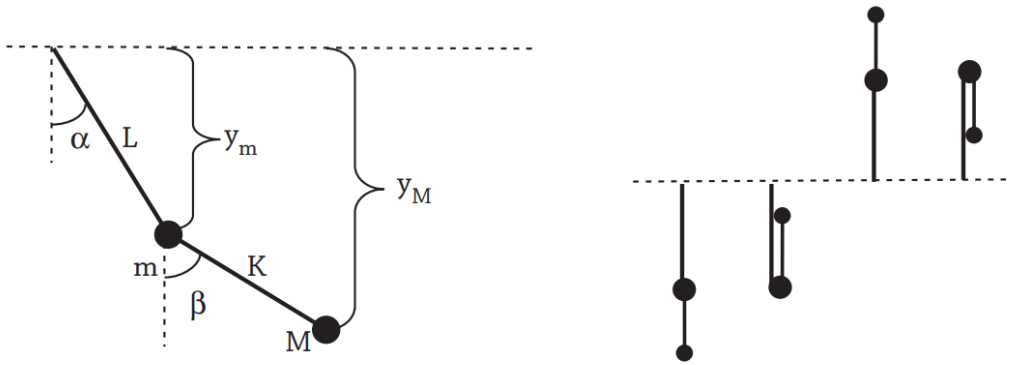

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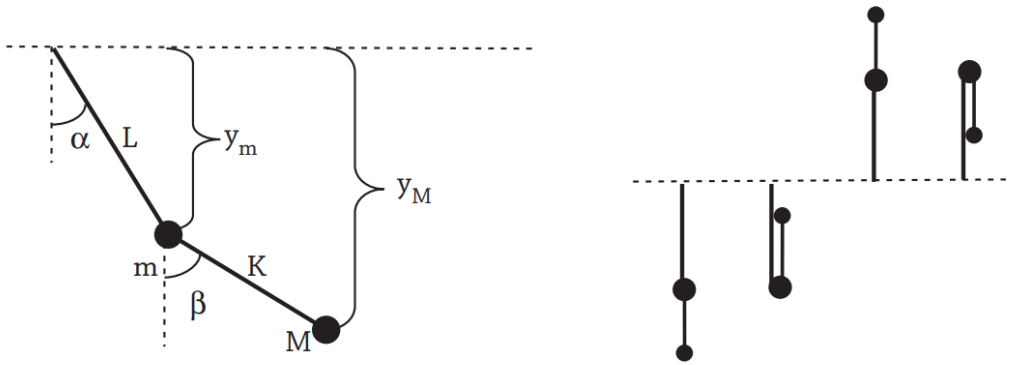

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Equilibrium positions where $\sin\alpha$ and $\sin\beta$ are simultaneously zero: $(0,0), (0,\pi), (\pi,0)$ and (π,π)

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(D2): There are four or more equilibrium positions of any double pendulum because for any n-tuple pendulum, the lower bound on its number of equilibria with k stable directions is equal to the kth Betti number of its distorted configuration space (n-dimensional torus), where a stable direction implies a pendulum rod pointing downward.

Review: Algebraic Topology and Dynamical Systems

A configuration space is a manifold which represents all possible points that a physical system may realise given its geometrical constraints.

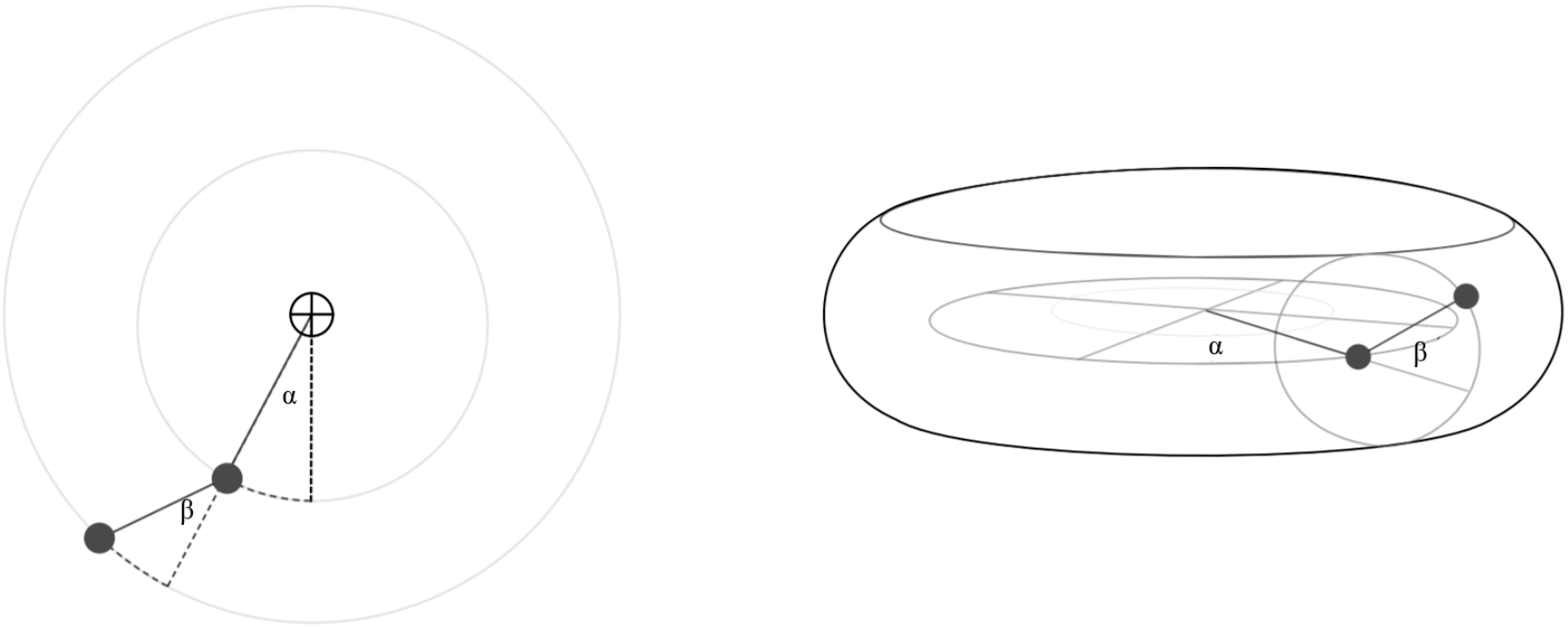

Figure: Torus configuration space: double pendulum (adapted from Lessig 2012)

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Euler characteristic

For a smooth, compact, and orientable surface: $\chi = N_{max} - N_{saddle} + N_{min} = 2 - 2g$

Example

The number of holes in a sphere is 0, and so $g = 0$ and $\chi = 2$. For a torus, the number of holes is 1, so $g = 1$ and $\chi = 0$.

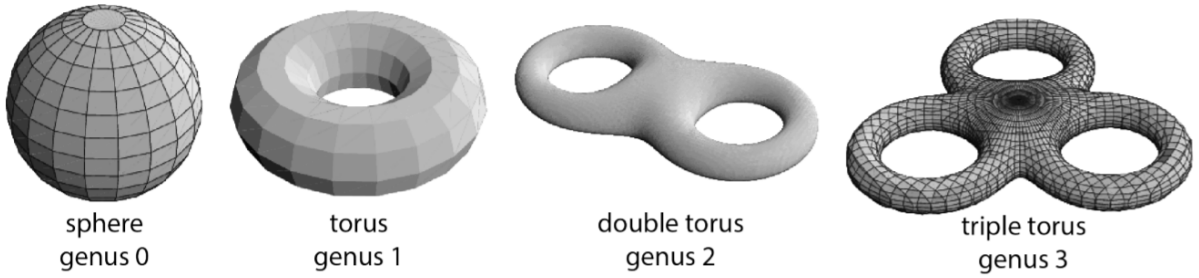
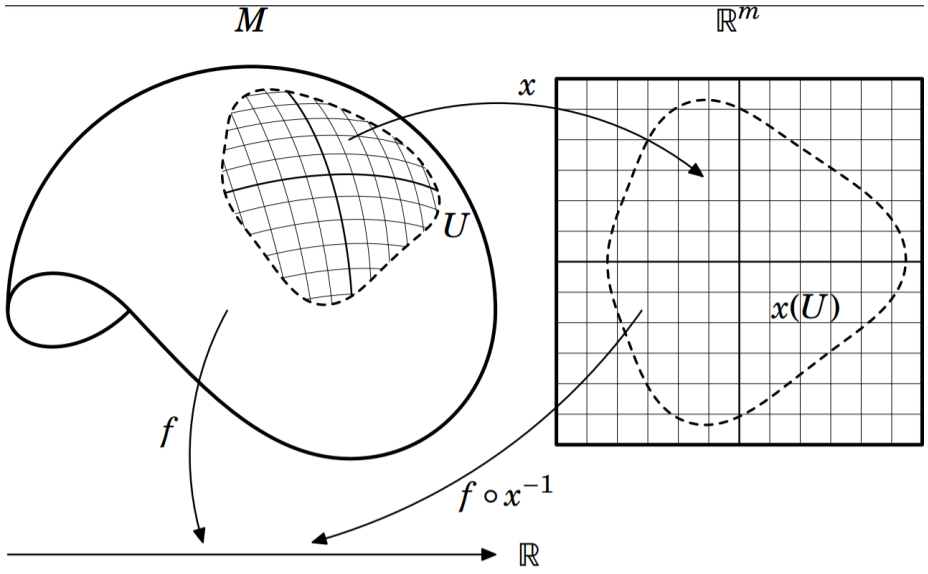


Figure: Retrieved from http://mathworld.wolfram.com

DME (D1): Lange's strategy using Euler characteristic



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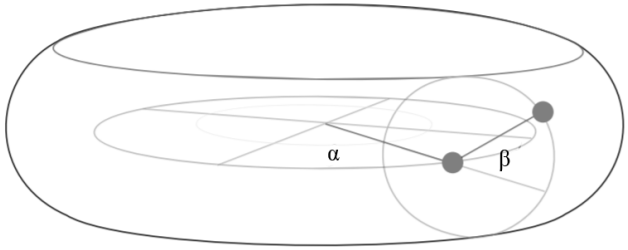

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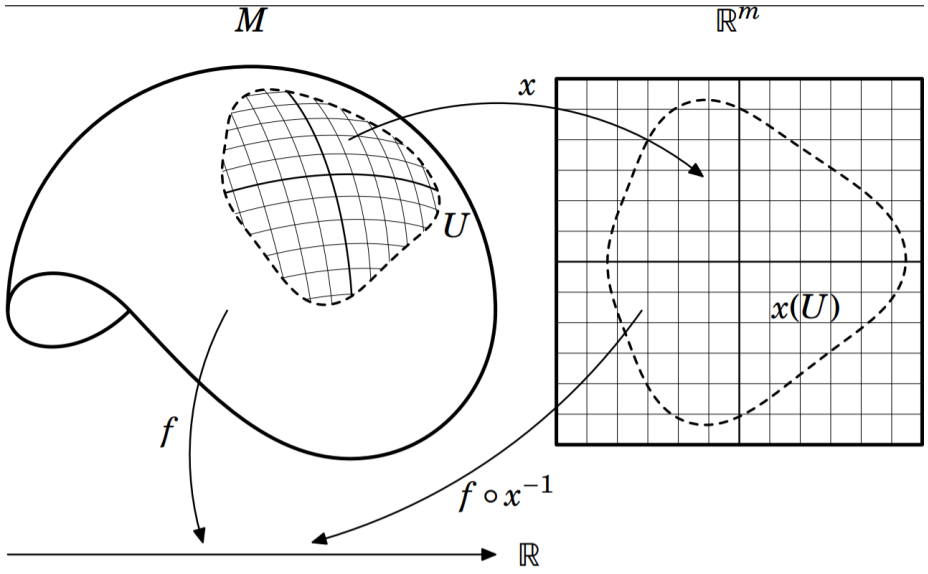

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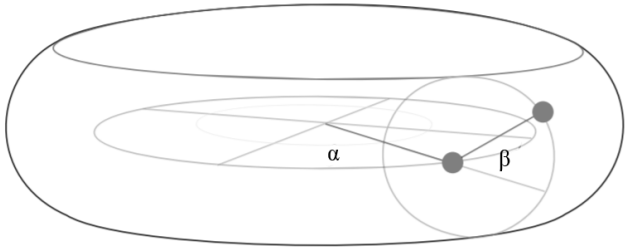
- If the potential energy function $U(\alpha, \beta)$ remains finite and continuous, then it can be "represented by distorting the torus so that each point (α, β)'s height equals $U(\alpha, \beta)$. Any such distortion remains a surface of genus $g = 1$"



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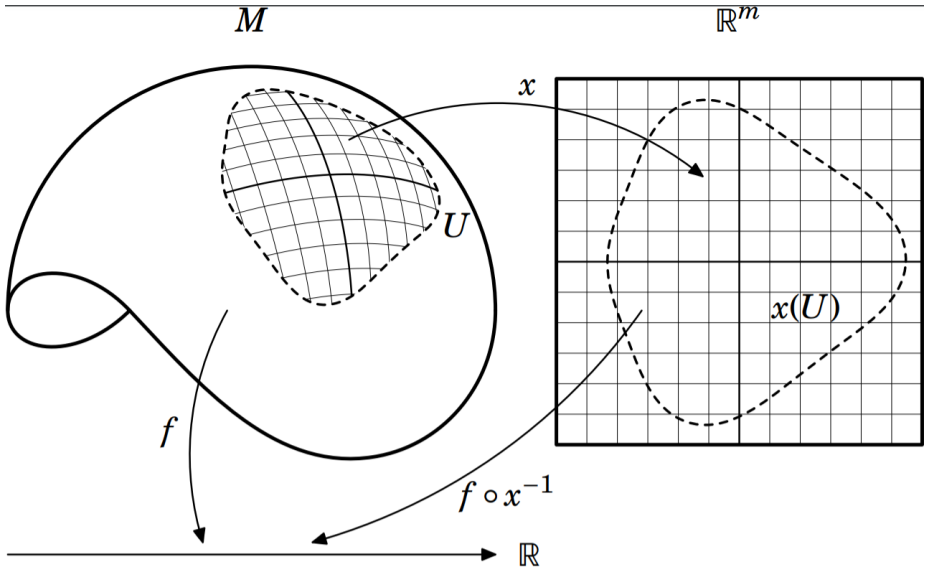

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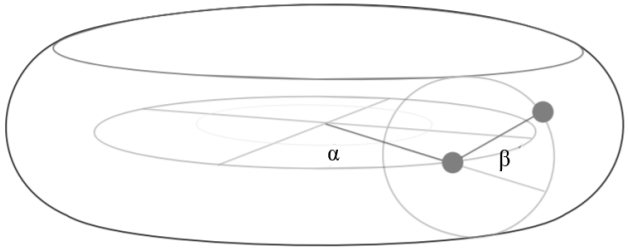

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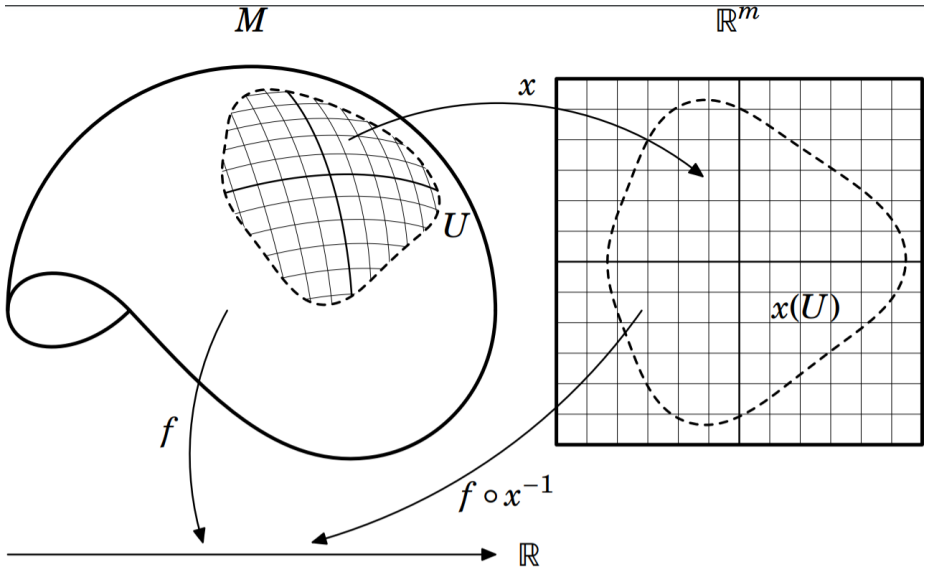

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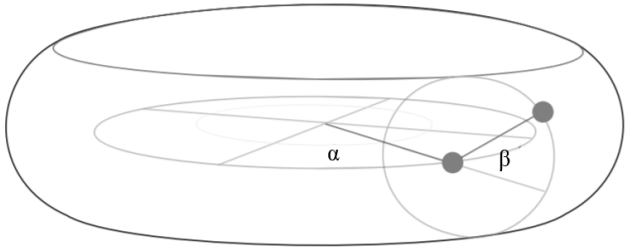

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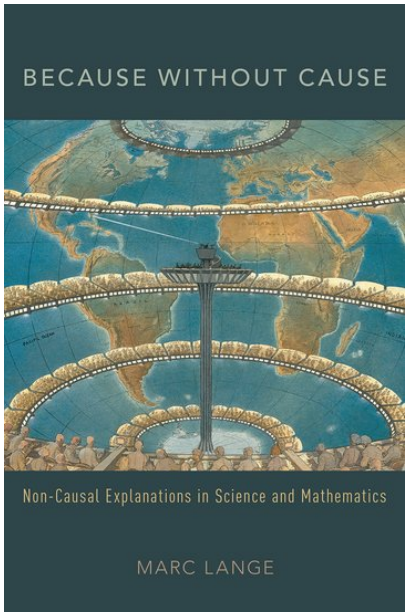
- With at least one minima and one maxima, there must be at least two saddle points for the invariance to hold, implying a total of four or more critical points for $U(\alpha, \beta)$.

Why Lange claims it is a DME?

- (D1) [arguably] applies to any kind of double pendulum, with stiff rods, springs, non-stiff roads, compound pendulums etc.
- No causal explanation [arguably] reveals why all such double pendulums have four or more number of equilibrium positions.



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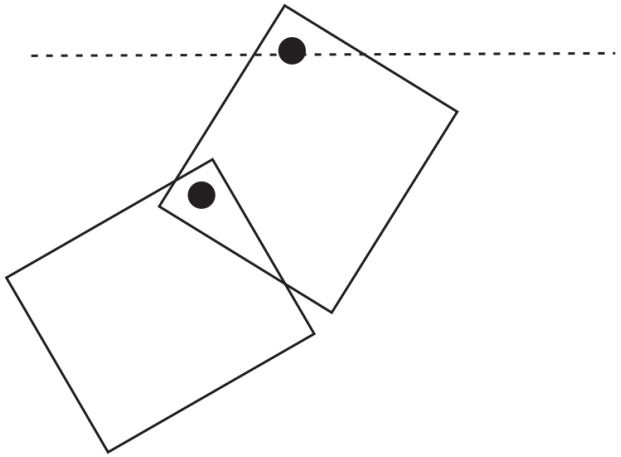

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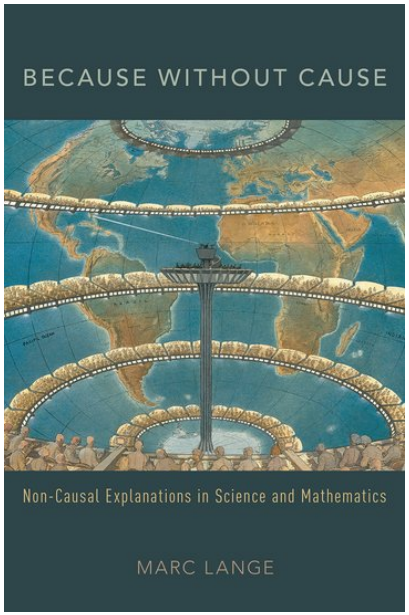

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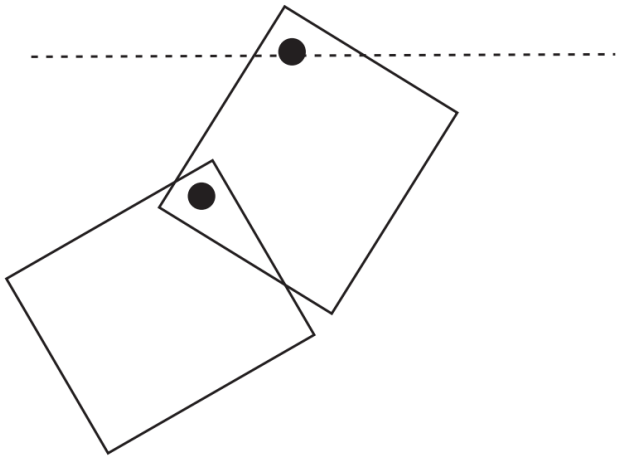

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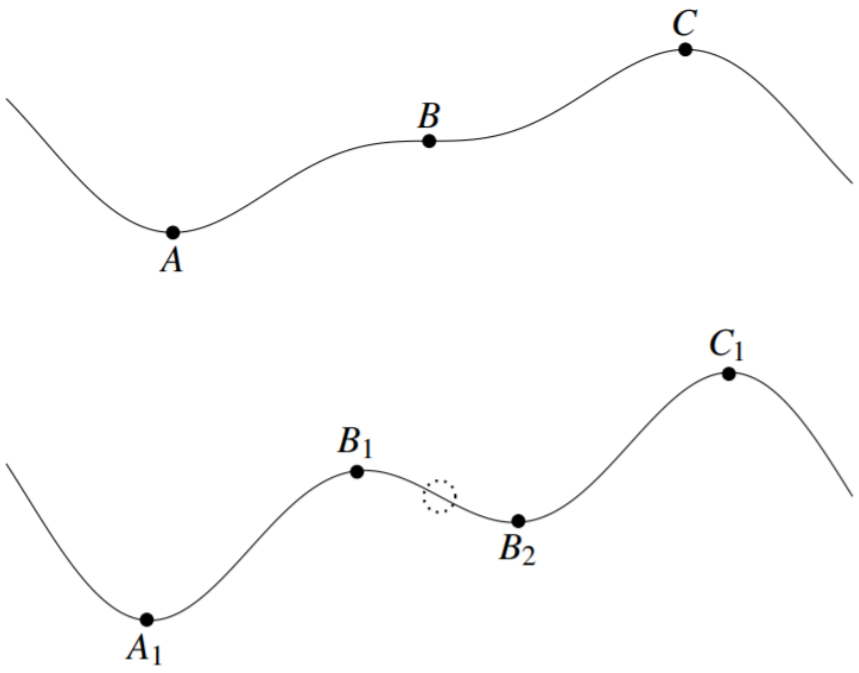
- (D1) only appeals to the potential energy function and the Newtonian force-acceleration framework, which Lange (2016) maintains to be a framework law holding with a greater necessity than other contingent forces.

Background: Extended (D1) - Morse Theory

Morse theory: Its primary interest is to understand how the shape of the manifold constrains the distribution of the critical points of differentiable functions defined over the manifold (Matsumoto, 2001).



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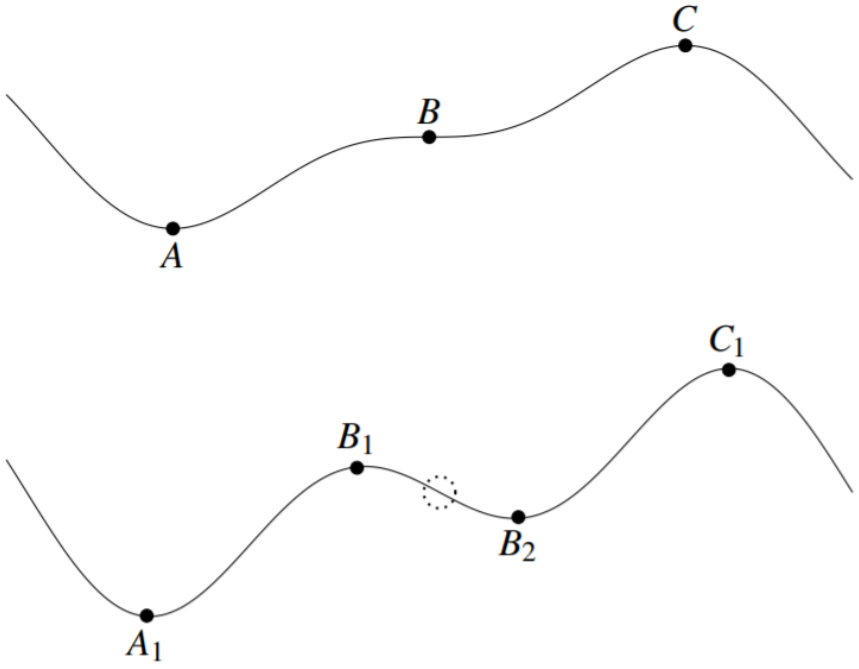

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Critical points are classified by their *index* which, in a nutshell, is the total number of downward directions one can move starting from the critical point.

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- Index calculation invokes causal processes (in Lange's sense of the term).

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Morse functions

If S is a smooth surface, then a Morse function is any function $f : S \to \mathbb{R}$ if every critical point p of f satisfies these two criteria: (a) p is isolated or that p is the only critical point in its immediate neighbourhood, and (b) p is a non-degenerate point.

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Example

$f_1 = x^2$ is a Morse function since at the critical point $x = 0$, $f''(0) = 2 \neq 0$.
$f_2 = x^3$ is a non-Morse function since at the critical point $x = 0$, $f''(0) = 6x = 0$.

Extended (D1): Exposition

- The configuration space of a simple pendulum is a circle with $m = 1$ and $g = 1$. Equation (6) yields: $(-1)^0 N_0 + (-1)^1 N_1 = 2 - 2 \times 1$ which is equal to $N_0 - N_1 = 0$ or to say that $N_{min} - N_{max} = 0$. Since a circle (or even a distorted circle which may represent the P.E. function) has at least one minima and one maxima, it satisfies this equality.

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For an n-torus $g = 1$, therefore, extended (D1) for n-tuple pendulums looks like:

$$\sum_{k=0}^{k=m} (-1)^k N_k = 0. \tag{3}$$

A major worry with both (D1) and its extension

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- So, even as the number of critical points of n-tuple pendulums are arguably constrained by the Morse equality, one cannot use this result to find out a non-trivial lower bound on the number of equilibrium positions of such pendulums.
- Is there a better topological strategy that avoids these problems?

(D2) background: An improved DME

	•			
β_0	1	1	1	1
β_1	0	1	0	2
β_2	0	0	1	1
β_3	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure: From Gholizadeh et. al. (2018): k-th betti number for topological surfaces

- Betti numbers are topological invariants for compact manifolds - Morse functions can be defined and studied over such manifolds.
- The k-th betti number shows the number of k-dimensional holes on a topological surface.

(D2): Using Betti numbers

(D2) There are four or more equilibrium positions of any *n*-**tuple pendulum** because the lower bound on its number of equilibria with k stable directions is equal to the kth Betti number of its configuration space (n-dimensional torus).

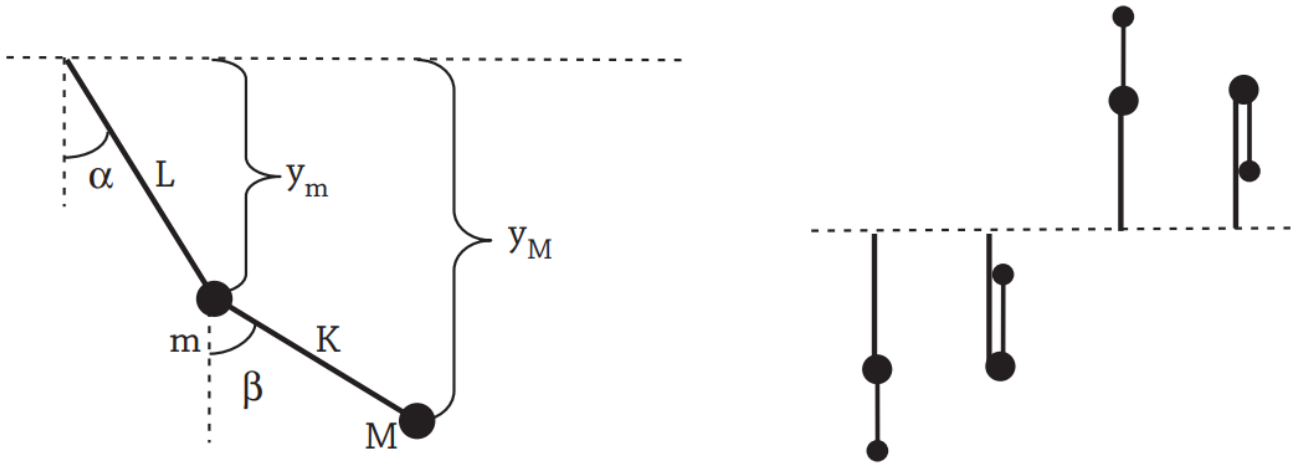

Figure: A rod pointing downwards connotes a stable direction for that rod.

(D2): Exposition

- For a torus, $\beta_0 = 1$ shows the number of connected surfaces, and $\beta_1 = 2$, and $\beta_2 = 1$ correspond to the number of k-th dimensional holes (i.e. 1D and 2D) in the torus.

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- We have $\binom{2}{1}$ or 2 stable directions when only 1 rod is pointing down which is equal to $\beta_1 = 2$, or $\binom{2}{2}$ or 1 stable directions when both rods are pointing down which is equal to $\beta_2 = 1$, and also $\binom{2}{0}$ or 1 stable direction which is equal to $\beta_0 = 1$ when 0 rods are pointing down.

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- Not only does this strategy gives the lower bound on the number of stable positions for each k-th betti number, but it also tells us the total number of equilibrium positions (lower bound) of any n-tuple pendulum system.

Using induction and Pascal's identity for binomials:

$$\sum_{k=0}^{k=n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots\dots\dots + \binom{n}{n} = 2^n$$

Why (D1), its extension and (D2) all fail as DMEs?

A major presumption in using the configuration space, to modally predict the number of equilibrium positions of the system, is that the P.E. function, of such a complex dynamical system, is always a Morse function.

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a. one or more critical points of the P.E. function become degenerate, which makes it difficult (and in some cases impossible) to calculate the index of these degenerate critical points, and/or
b. the P.E. function may no longer be mappable to a smooth or compact manifold, and thus Morse inequalities fail to be applicable.

Case (a): Smooth manifolds - (D1) and extension fail

We perturb the length of the rod L is such a way that the distorted configuration space of the simple pendulum remains smooth and compact (a restriction we relax in the next case), and its P.E. function, thus, remains definable over the configuration space.

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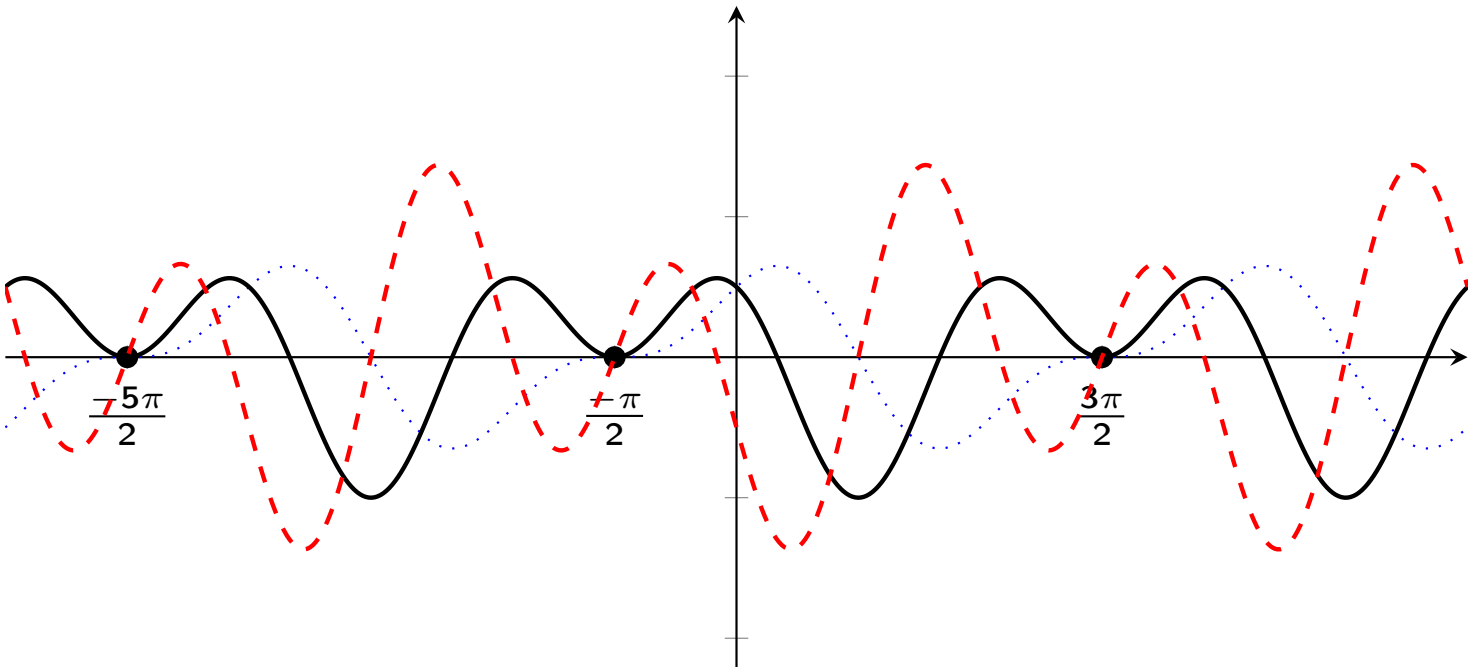

$U_{pert_1}(\alpha) = (1 + \sin\alpha)\cos\alpha$

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Figure: Degeneracy in perturbed P.E. function: $U_{pert_1}(\alpha)$ degenerate at the inflection points $(2n\pi - \frac{\pi}{2})$ for $n \in \mathbb{Z}$, where the first order partial derivative meets the second order partial derivative.

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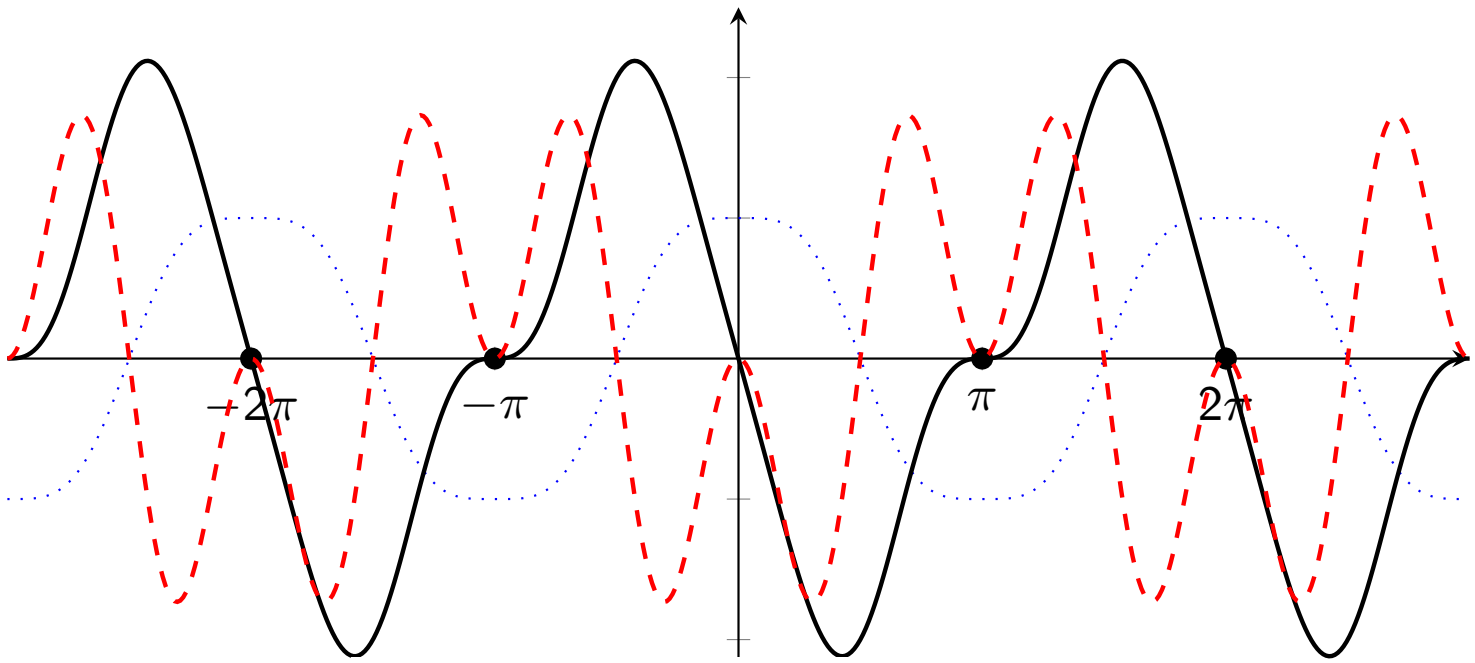

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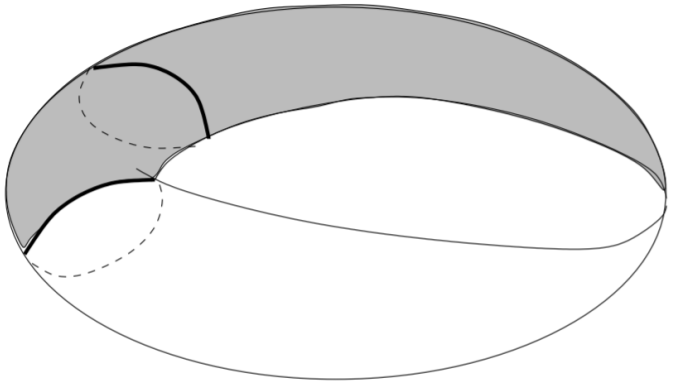
A note on Degenerate tori

A torus may be distorted in way that it has only three critical points.



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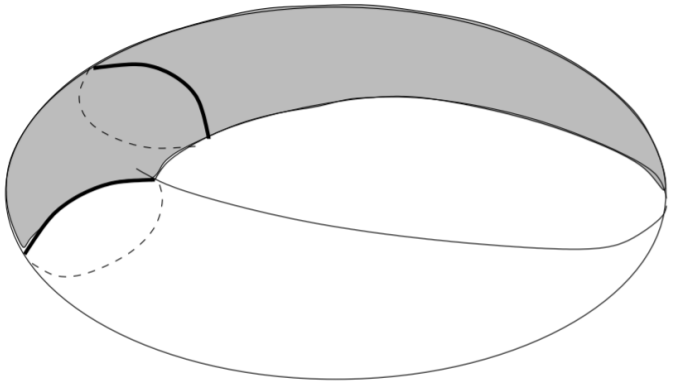

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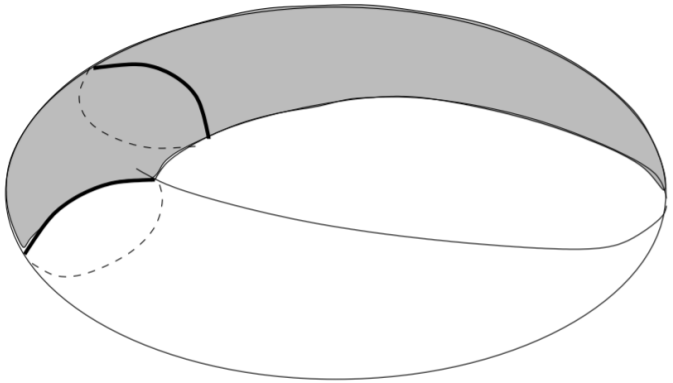

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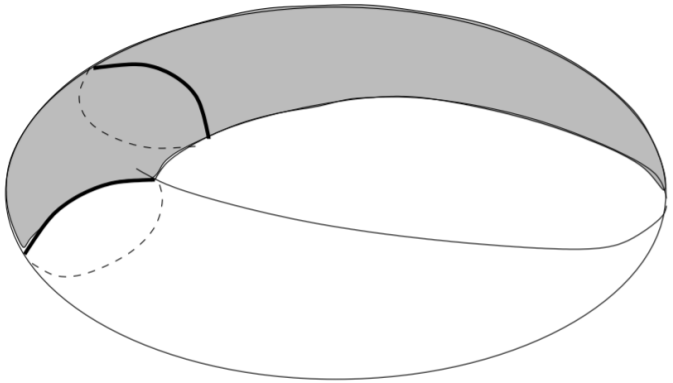

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- Whether such immersions are legitimate P.E. functions, pertaining to pendulum systems, is a question that is beyond the scope of this talk.
- But if such immersions are legitimate P.E. functions, then it is not necessary that all double pendulums have four or more equilibrium positions, even without an external force.

Case (b): non-smooth manifolds - (D1), its extension and (D2) fail

If the conditions concerning case (a) are relaxed, the differentiable P.E. function fails to be mappable to a smooth and compact manifold, and the strategy of studying differentiable functions on such manifolds fail.

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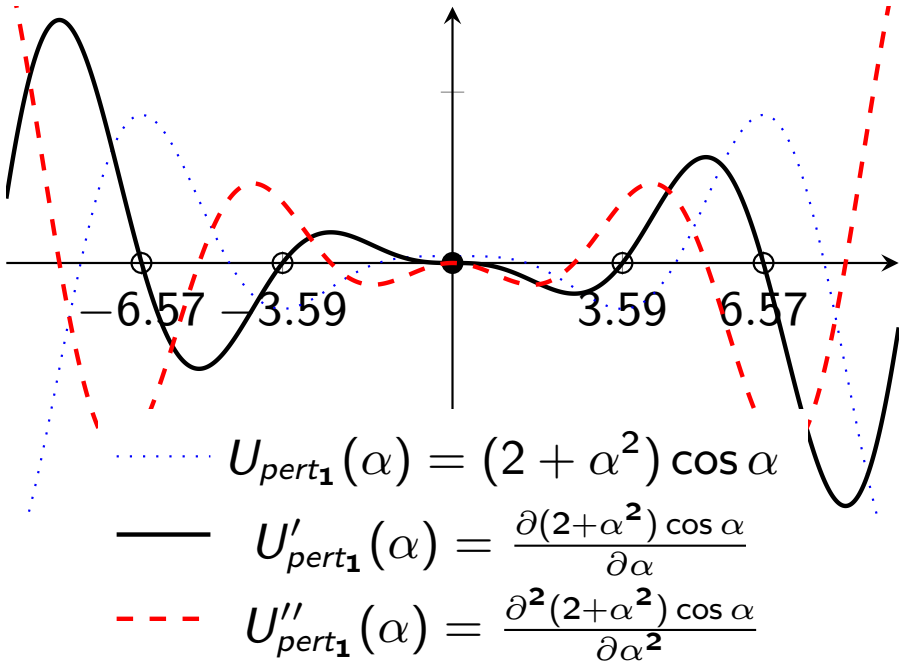

Figure: Non-compact manifolds have only one critical point for a simple pendulum and there is additional degeneracy at $\alpha = 0$

Case (b): Non-smooth manifolds - (D1), its extension and (D2) fail

With non-stiff rods, α^3 perturbation in the length of the rod result in:

Input interpretation:

$$\frac{\partial((L+a^3)\cos(a))}{\partial a} = \frac{\partial^2((L+a^3)\cos(a))}{\partial a^2} = 0$$

Result:

$$3a^2\cos(a) - (a^3+L)\sin(a) = -(a^3-6a+L)\cos(a) - 6a^2\sin(a) = 0$$

Plot of solution set:

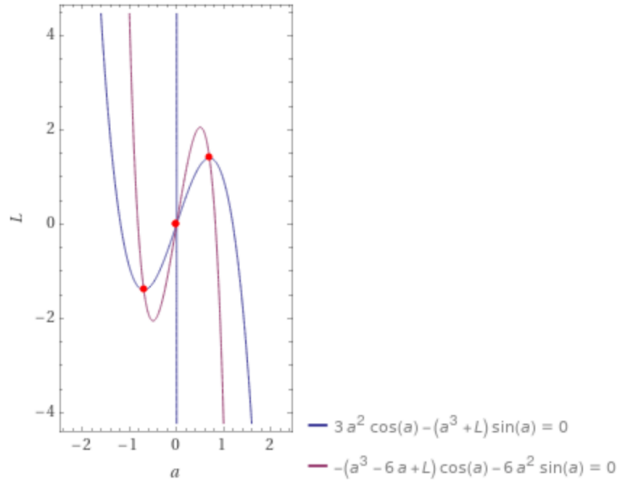

Figure: Sourced from http://wolframalpha.com

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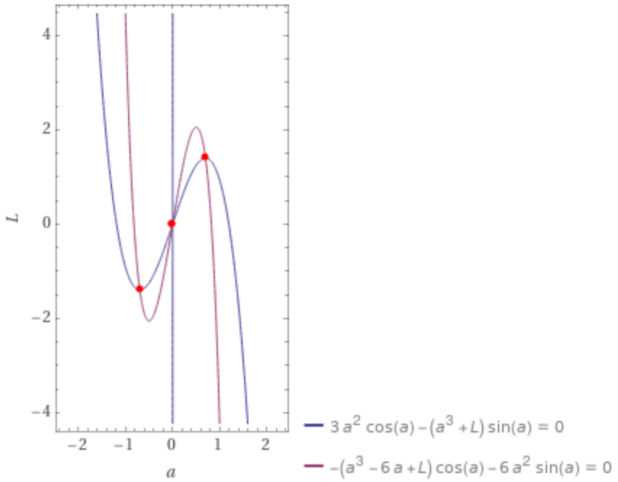

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Result:

$$a(3a+2)\cos(a) - (a^3+a^2+L)\sin(a) = -(a^3+a^2-6a+L-2)\cos(a) - 2a(3a+2)\sin(a) = 0$$

Integer solution:

$$a = 0, \quad L = 2$$

Figure: Sourced from http://wolframalpha.com

Case (b): Non-smooth manifolds - (D1) and extension fail

The P.E. function (ignoring the stored potential energy in the rod) for general quadratic perturbations is: $U(\alpha) = -mg(p\alpha^2 + q\alpha + r)\cos\alpha$.

Input interpretation:

$$\frac{\partial((p\,a^2 + q\,a + r)\cos(a))}{\partial a} = \frac{\partial^2((p\,a^2 + q\,a + r)\cos(a))}{\partial a^2} = 0$$

Result:

$$\cos(a)\,(2\,a\,p + q) - \sin(a)\,(a^2\,p + a\,q + r) = \cos(a)\,(-((a^2 - 2)\,p + a\,q + r)) - 2\sin(a)\,(2\,a\,p + q) = 0$$

Solution:

$$0.159155\,(a - 3.14159) \notin \mathbb{Z}, \quad \tan^4(0.5\,a) + 6\tan^2(0.5\,a) + 1 \neq 0,$$

$$q \approx -\frac{2\,p\,(a + a\tan^4(0.5\,a) + 2\tan^3(0.5\,a) + 6\,a\tan^2(0.5\,a) - 2\tan(0.5\,a))}{\tan^4(0.5\,a) + 6\tan^2(0.5\,a) + 1},$$

$$r \approx (a^2\,p + a^2\,p\tan^4(0.5\,a) + 6\,a^2\,p\tan^2(0.5\,a) + 2\,p\tan^4(0.5\,a) + 4\,a\,p\tan^3(0.5\,a) - 4\,p\tan^2(0.5\,a) - 4\,a\,p\tan(0.5\,a) + 2\,p)/(\tan^4(0.5\,a) + 6\tan^2(0.5\,a) + 1)$$

Real solution:

$$a \approx 3.14159\,n, \quad q \approx -6.28319\,n\,p, \quad r \approx 9.8696\,p\,n^2 + 2\,p, \quad n \in \mathbb{Z}$$

Figure: Sourced from http://wolframalpha.com

Generalization of degeneracy conditions: Extension (D1)

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If $D^2(H) = 0$, then either of $\frac{\partial^2 U}{\partial \alpha^2}$ or $\frac{\partial^2 U}{\partial \beta^2}$ is 0. These are also the points where $\frac{\partial U}{\partial \alpha}$ and $\frac{\partial U}{\partial \beta}$ should be simultaneously zero for this to be a non-Morse function.

Generalization of degeneracy conditions: Extension (D1)

For a double pendulum with α^2 and β^2 perturbations:

$$U(\alpha, \beta) = -mg(L + \alpha^2)\cos\alpha - Mg((L + \alpha^2)\cos\alpha + (K + \beta^2)\cos\beta)$$

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The resulting partial differential equations are:

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But, ∂U_α and ∂U_β are the same equations with only different symbols, so what solves the former will also solve the latter. We could just keep U'_α. Since this is also the same equation as the U' for the simple pendulum, its double partial derivative with respect to α or β will behave exactly as before.

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- These are essentially the same conditions (just replace the constants) that make the P.E. function of the simple pendulum a non-Morse function.
- Thus, for any n-tuple pendulum, with independent perturbations in each rod, the degeneracy conditions will be just this: $\frac{\partial^2 U}{\partial \alpha^2} = \frac{\partial U}{\partial \alpha} = 0$. Hence the same family of solutions that make a simple pendulum P.E. function degenerate will be applicable to any n-tuple pendulum.
- (D1), (D2) and its extensions, thus, generally seem to fail as DMEs for n-tuple pendulums.

Case (b): A non-smooth configuration space

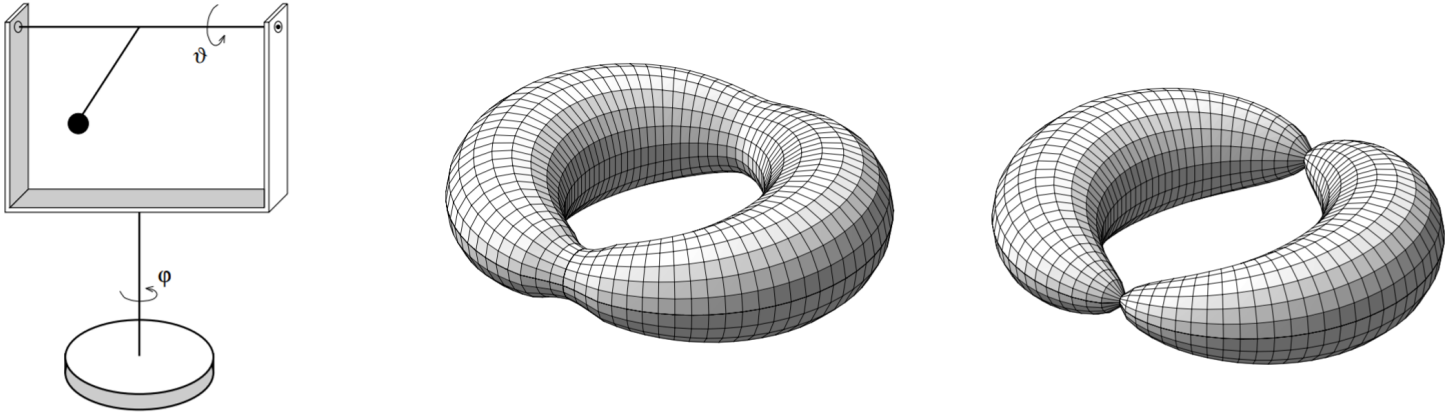

Figure: (left) Spherical pendulum suspended in a frame which rotates about a vertical axis and the resulting configuration spaces (centre and right) for two different values of the moment of inertia of the frame.

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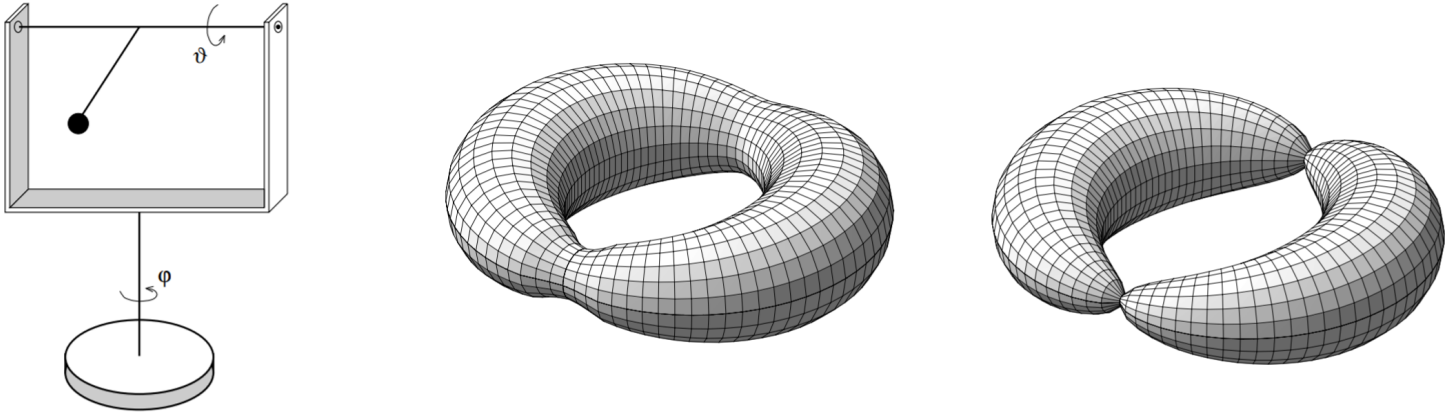

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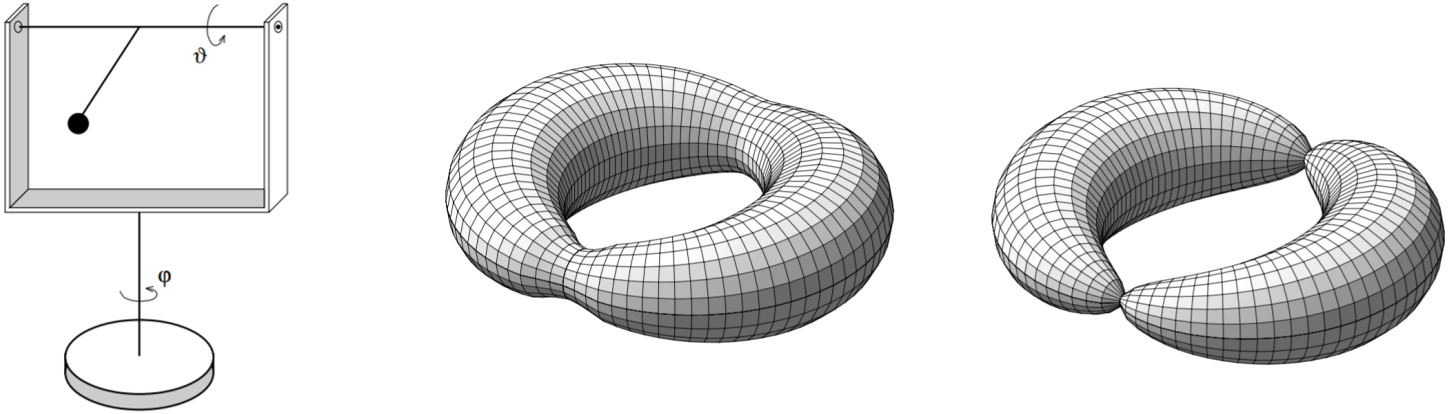

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- They show how the configuration space of S^2 is actually bifurcated into a non-smooth manifold of two tori T^2, due to external parameters (such as moment of inertia), which recover the S^2 as a limiting case on certain values of the external parameter.

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- Bismut (1986), improvising on that of Witten (1982), for Morse-Bott functions where the critical point is required to be non-degenerate only in a direction normal to the tangent space of the manifold at the critical point, and not necessarily in every direction
- Castrigiano et al. (2019) using Catastrophe theory

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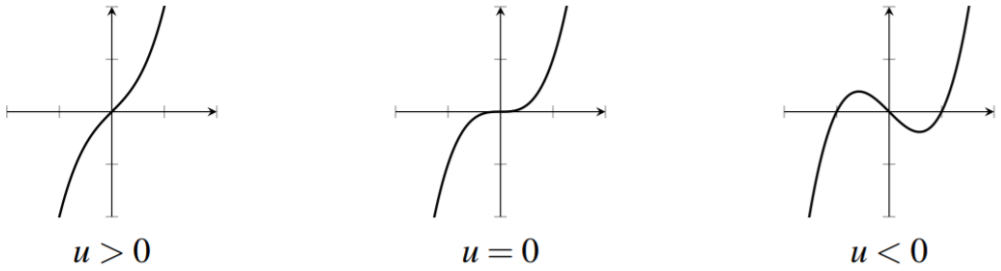

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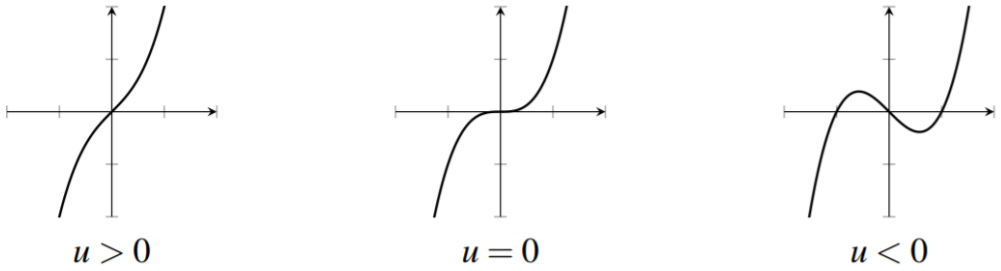

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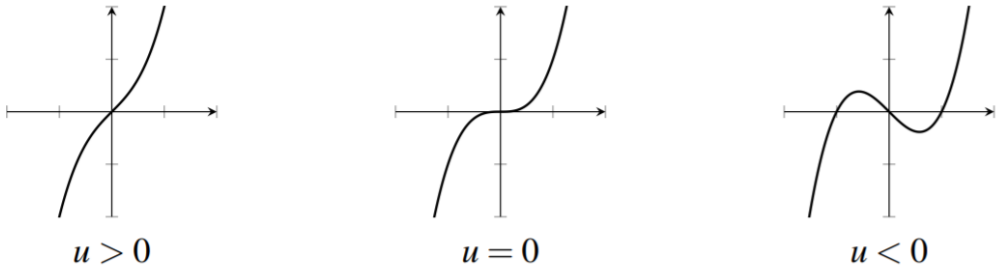

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- Second theorem of Thom: The classification above remains stable under small perturbations.

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- Even if such a P.E. function can be classified accordingly, the index of the critical points of such functions obtained after perturbing the degenerate points needs to be assessed manually using the Hessian of the function, which goes against the core idea of non-causal reasoning.

Summary of proposed DMEs and conclusion

Explanation	**Applicability**	**Modality**	**Is it a DME?**
D1 and D2	n-tuple pendulum ($n \leq 2$ for D1)	Morse function: strong, Non-Morse function: may fail	No, weak modality in perturbations
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- Case generalisable to any integrable Hamiltonian system using Kolmogorov–Arnold–Moser theorem (under research).

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