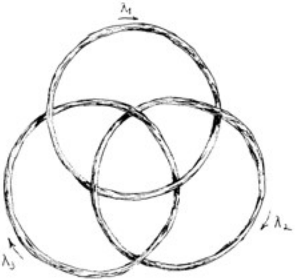
Computing the Thurston Norm

an approach by Normal Surface Theory

David G. Hilder



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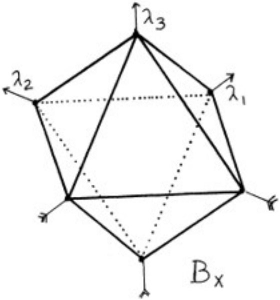
The University of Queensland
Australia



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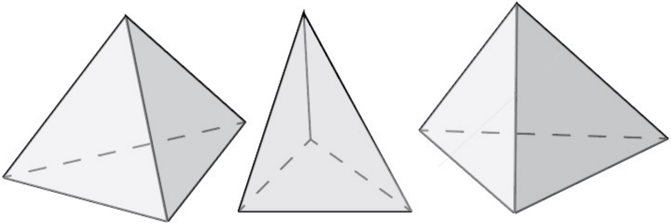
Triangulations

Triangulations on 3-manifolds are built from n tetrahedra:



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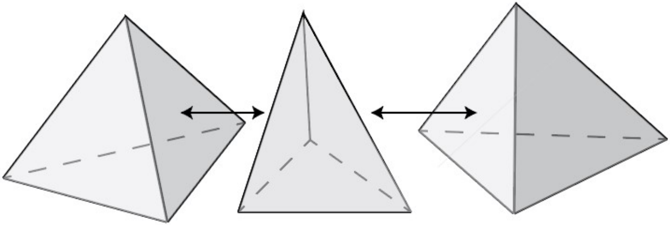

We *"glue"* each tetrahedron together by identifying their faces to form a space representing some 3-manifold.

Motivation and Background

Algorithms in computational topology input a **triangulation** $\mathcal{T}$ on a **3-manifold** M and often test for various topological properties by looking for embedded surfaces within $\mathcal{T}$ (eg. unknot recognition).

The field sees great success in solving a number of topological problems, but the algorithmic efficiency remains key.

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The Main Problem

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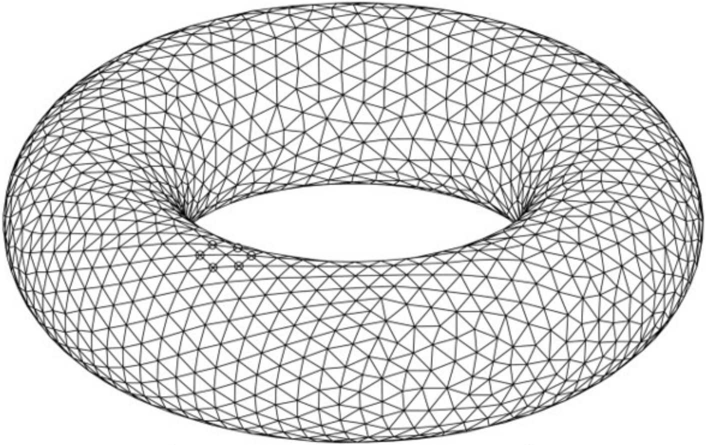
The Main Problem

Algorithmic complexity increases *exponentially* with the number of tetrahedra n in a 3-manifold triangulation. (Consider: $\mathbb{Z}^{14n}$)

- We want to keep n at a minimum as well as the **exponential base**:
  - Reduces running time;
  - No topological information lost.

A Triangulation of $\mathbb{T}^2$



n = ... would rather not know!

Thurston Norm

Definition

Let $\mathcal{T}$ be a 3-manifold triangulation. Topological complexity is the minimum complexity on a surface.

- Thurston norm developed by William Thurston in 1986 as a fundamental measure of **topological complexity**.
- It is a norm defined for a certain class of 3-manifolds with non-trivial second homology $H_2(M)$.

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Remark

The Thurston norm measures the complexity of *surfaces representing a class of the second homology* for certain 3-manifolds.

Thurston Norm

Definition

Let $\mathcal{T}$ be a triangulation of a 3-manifold M and S_i be the i-th embedded surface. Then:

$$||\phi||_T = \min_S \sum_{i=1}^{n} \chi_-(S_i)$$

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- Defined on: $||\cdot||_T : H_2(M) \to \mathbb{R}$
- χ_- measures genus excluding torus and sphere contributions.
- The second homology group $H_2(M)$ is an abelian group. Classes $c \in H_2(M)$ are represented by maps of oriented 2-simplices.
- The Thurston norm $||\phi||_T$ is a rough extension of the knot genus.

Thurston Norm Unit Ball

- The Thurston norm unit ball $\mathcal{B}$ is a compact polyhedron formed in a space $\mathbb{Z}^d$, where $d = rank(H_2(M))$.

- Each point on $\mathcal{B}$ is the projection of relevant surfaces to $H_2(M)$.

- The convex hull of points obtained forms the polytope $\mathcal{B}$.

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Definition

For a 3-manifold triangulation $\mathcal{T}$ and the space of relevant embedded surfaces $NS(\mathcal{T})$, we define the homology map h:

$$h(\mathcal{T}) : NS(\mathcal{T}) \rightarrow H_2(M)$$

Thurston Norm Unit ball

- Each point of the norm unit ball represents a class in $H_2(M)$.
  - Every embedded surface in $\mathcal{T}$ with $\chi < 0$ representing a class $c \in H_2(M)$ will be a point on the norm unit ball
  - Can be represented as a vector in $\mathbb{Z}^d$, forming a vertex of $\mathcal{B}_T$

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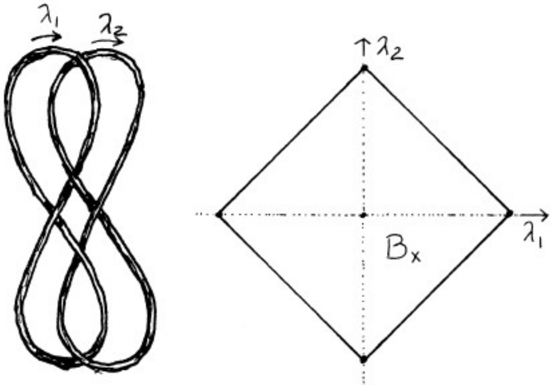
- The boundary of $\mathcal{B}$ gives where the norm is equal to 1.



([1], Thu. 1986)

Thurston norm computationally

Question

Given a 3-manifold triangulation $\mathcal{T}$, how do we construct the Thurston norm unit ball and hence the Thurston norm of any homology class?

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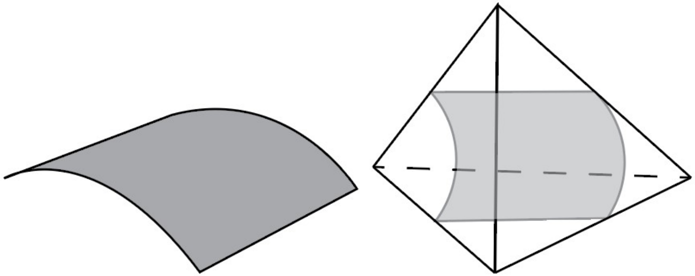
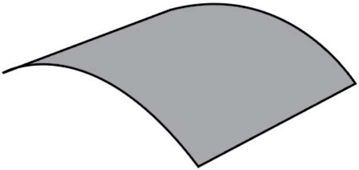
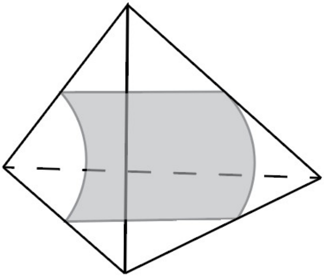
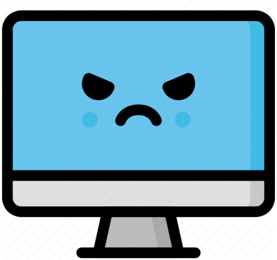
The Problem

How can we enumerate embedded surfaces in $\mathcal{T}$ when their geometry cannot be input into a computer?









Normal Surfaces

- **Normal surfaces** were first introduced by Hellmuth Kneser in 1929 to describe **embedded surfaces**

Remark

We say an embedded surface can be *isotoped* to *normal position*.

(i.e. We apply a *"normal"* structure to a surface.)

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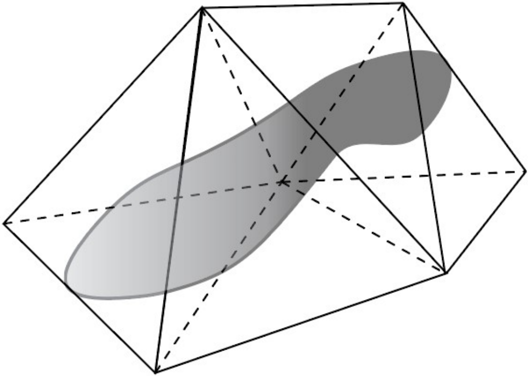
- Normal surfaces 'discretises' the notion of surfaces
- Represents a surface as a union of discrete **discs**.

Proto-lemma

Every *"interesting"* embedded surface in a 3-manifold triangulation can be represented by an equivalent embedded normal surface.

Example of a Normal Surface



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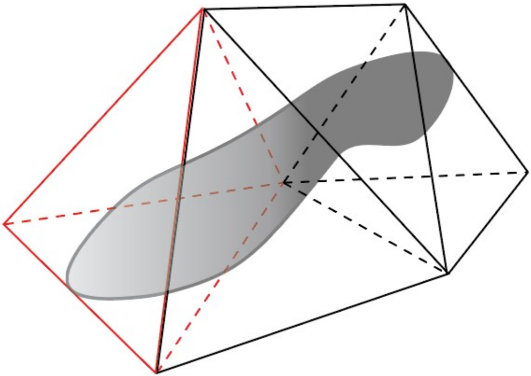

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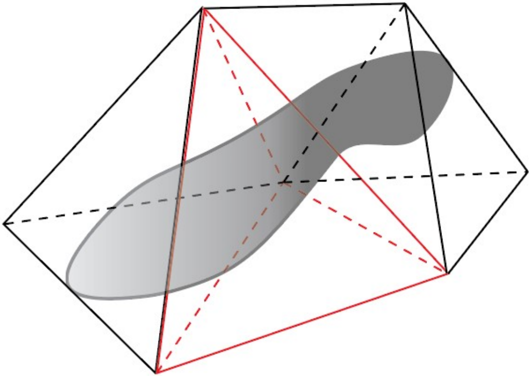

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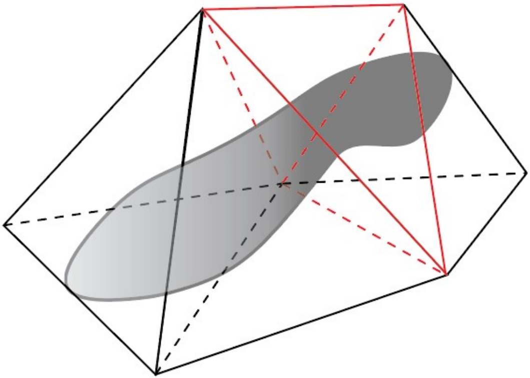

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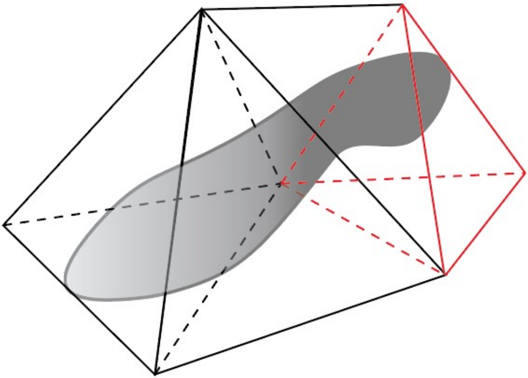

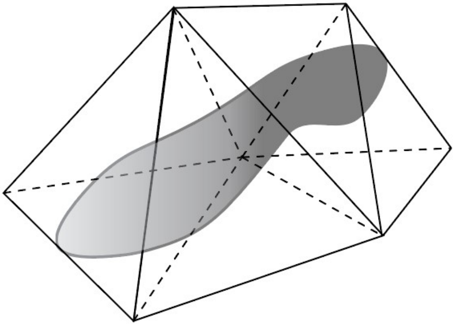

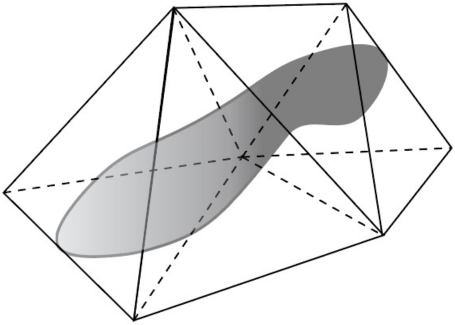

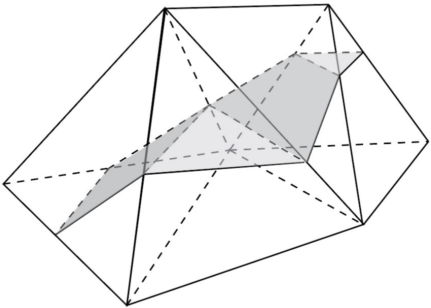

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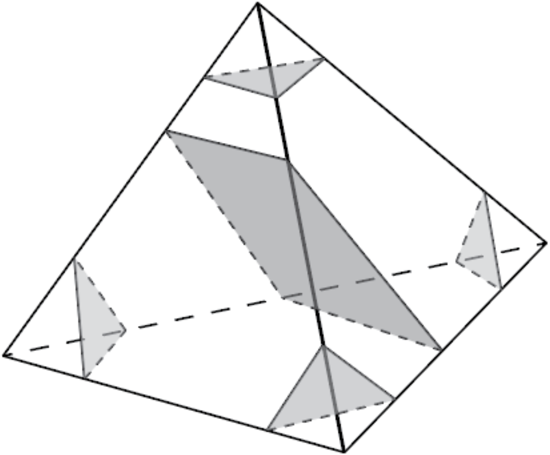

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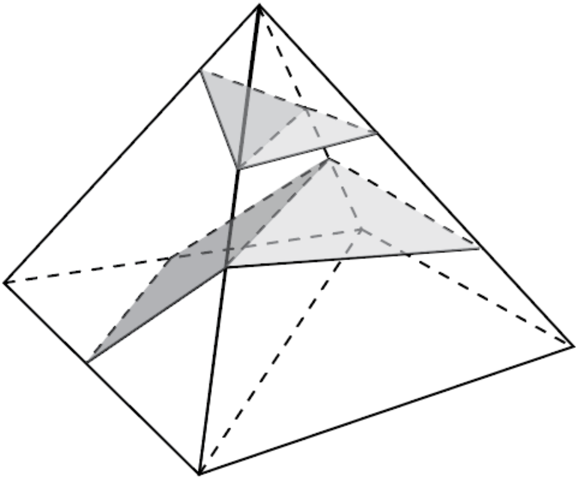

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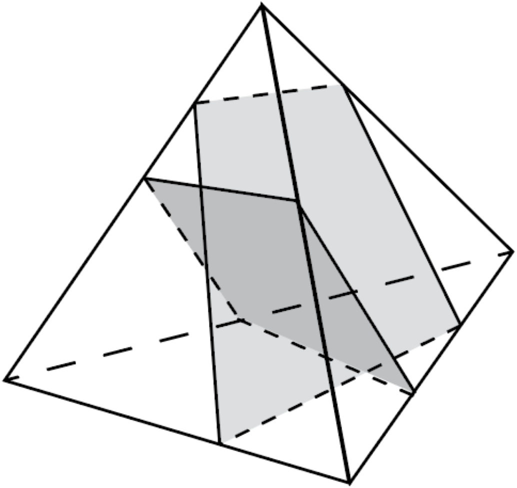

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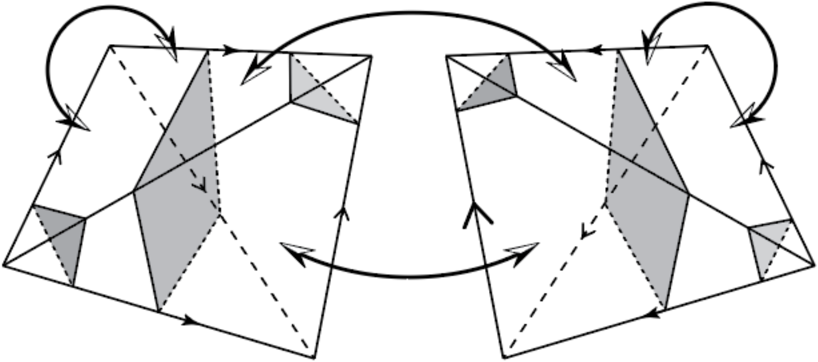
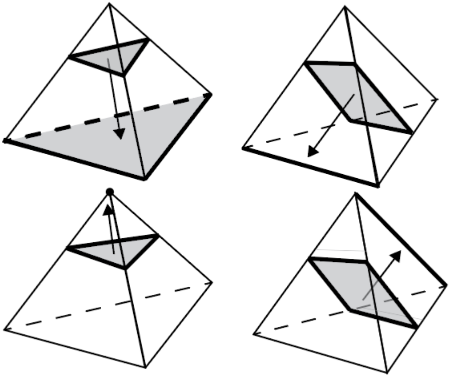


Transversely Oriented Normal Surfaces



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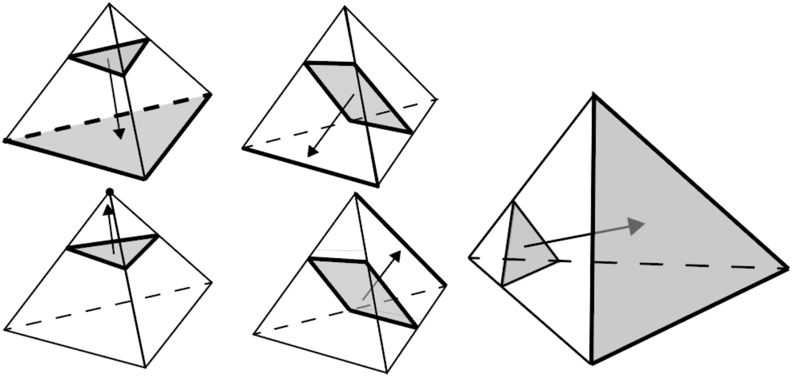

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The Key Result

We can solve topological 3-manifold problems by methods in integer linear programming.

- Opened a wealth of potential to solve topological problems via linear programming
- Utilised in a variety of algorithms analysing topological properties
  - 3-sphere recognition, homeomorphism problem, etc.

Our Approach

- In 2009 Tillman and Cooper outlined first approach to the Thurston norm via normal surfaces; this is where we pick up from!
  - Take a triangulation $\mathcal{T}$
  - Find basis of all normal surfaces where $\chi < 0$
  - Find contributions to homology $H_2(M)$ and represent as a point on the norm unit ball

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Lemma

An integer vector representation of the i-th surface $v_i \in \mathbb{Z}^{14n}$ will have an integer vector representation of its second homology class.

Homology and Euler Char. as linear functions

- Respectively Lemma 3 and Proposition 4 (Tillman, Cooper 2009)

Lemma

There is a linear map:

$$\chi^* : NS^\nu(\mathcal{T}) \to \mathbb{R}$$

which agrees with the Euler Characteristic $\chi(S)$ for a transversely oriented normal immersion.

Proposition

There is a surjective homomorphism:

$$h : NS^\nu(\mathcal{T}) \to H_2(M; \mathbb{R})$$

which agrees with the homology class $[S]$ of a surface S for a transversely oriented normal immersion.

Algorithm

- This algorithm inputs a triangulation $\mathcal{T}$ of a 3-manifold M
- By enumerating a *certain class* of normal surfaces, we can determine their contribution to the Thurston norm
- The algorithm outputs the points of the Thurston norm unit ball $\mathcal{B}_T$
- Taking the convex hull of these points forms $\mathcal{B}_T$

Example Computation

Let $\mathcal{T} = M159(4,1)$.

- $\mathcal{T}$ contains properties found in many standard triangulations.
- $\mathcal{T}$ complex enough to allow for non-trivial algorithmic testing.

Basic info: $n = 13$; $H_2(M) = \mathbb{Z}$; $d = rank(H_2(M)) = 1$.

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Of these surfaces, we check which contribute to $H_2(M)$: we obtain *86* normal surfaces.

Example Computation

Recall

Recall every normal surface contributing to $H_2(M)$ will be represented by some vector in $\mathbb{Z}^d$.

Hence we will obtain 86 vectors in the space $\mathbb{Z}^1$.

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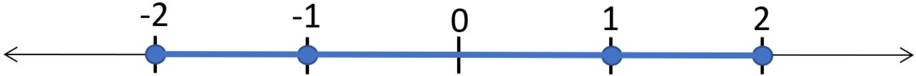
Hence we will obtain 86 vectors in the space $\mathbb{Z}^1$.

Our output yields 86 vectors such that: $v_i \in \{-2, -1, 1, 2\}$.

Visually, we get the Thurston norm unit ball:



Example Computation

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For our manifold M, $d = rank(H_2(M)) = 1$, so we get points on a line.
If $d = rank(H_2(M)) = 2$, each result would be a vector $(x, y) \in \mathbb{Z}^2$.
If $d = rank(H_2(M)) = 3$, each result would be a vector $(x, y, z) \in \mathbb{Z}^3$.

Summary

Revisiting the Main Problem

Algorithmic complexity increases *exponentially* for the number of tetrahedra n in a 3-manifold triangulation, as well as the base of the exponential.

- Thurston norm $||\cdot||_{\mathcal{T}}$ as a measure of topological complexity,
- Normal surfaces discretise embedded surfaces and allows us to use methods in linear programming,
- Apply algorithm: input 3-manifold triangulation $\mathcal{T}$; obtain the Thurston norm unit ball $\mathcal{B}_{\mathcal{T}}$,
- Hence can determine the Thurston norm of any homology class $\rightarrow$ minimum value of complexity of a surface representing that class.

Further Considerations

- Reducing computational complexity:
  - Further constraints on normal surfaces;
  - Different classes of normal surfaces;
  - Programming shortcuts,
- Exploring bounds on the Thurston norm,
- Permitting a larger class of 3-manifolds to input,
- Further algorithms:
  - Determining fibred faces of $\mathcal{B}_T$;
  - Computing relation to Alexander invariants.

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[2] **B. A. Burton, R. Budney, and W. Pettersson**, *Regina: Software for Low-Dimensional Topology*, 1999–2021.

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