Kummer varieties of polarised abelian varieties

Robert Carls

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In this talk we discuss the material presented in [1, §4.3-4.4]. We construct the *quotient variety* of a projective variety under the action of a finite group of automorphisms. We prove that the automorphism group G of a polarised abelian variety $(A, \mathcal{C})$ is finite. The quotient variety A/G is called the *Kummer variety* of $(A, \mathcal{C})$.

1 Quotients under a finite group of automorphisms

Let X be a projective k-variety, where k is a field, and let G be a finite group acting on X, i.e. we are given a homomorphism $\rho : G \to \mathrm{Aut}_k(X)$. We set $\rho_g = \rho(g)$.

Theorem 1.1 *There exists a k-variety Y and a k-morphism $\pi : X \to Y$ which is finite surjective and G-invariant, i.e. $\forall g \in G$ one has $\pi \circ \rho_g = \pi$. The morphism π has the following universal property: For every G-invariant k-morphism $\alpha : X \to Z$, where Z is a k-variety, there exists a unique k-morphism $\tau : Y \to Z$ such that $\alpha = \tau \circ \pi$.*

Proof. In the following we sketch a proof of Theorem 1.1. We claim that we can cover X by finitely many affine open subvarieties U such that $g(U) = U$ for all $g \in G$. We can assume that X is embedded in $\mathbb{P}^n_k$ for some $n \geq 1$. Let x be a point of X. We choose a hyperplane H_x contained in $\mathbb{P}^n_k$ such that $\rho_g(x) \notin H_x \cap X$ for all $g \in G$. We denote the complement of $H_x \cap X$ in X by V_x. We set

$$U_x = \bigcap_{g \in G} \rho_g(V_x).$$

Then U_x is an open affine subvariety of X such that $g(U_x) = U_x$ and $\rho_g(x) \in U_x$ for all $g \in G$. We have $X = \cup_{x \in X} U_x$. The claim now follows from the quasi-compactness of X with respect to the Zariski topology.

By the above we can construct the quotient locally and obtain a global quotient by gluing affine pieces. Now assume that X is an affine variety. Let $A = \Gamma(X, \mathcal{O}_X)$. The morphism ρ induces an action of G on A. Let $B = A^G$

denote the subring of G-invariant elements of A. We claim that A is integral over B. For $a \in A$ we define

$$\chi_a(T) = \prod_{g \in G} \big(T - g(a)\big).$$

Obviously we have $\chi_a(T) \in B[T]$ and $\chi_a(a) = 0$. This proves the above claim.

We claim that B is of finite type over k. We have $A = k[a_1, \ldots, a_n]$ with $a_1, \ldots, a_n \in A$. Let B' be the k-algebra generated by the coefficients of the polynomials $\{\chi_{a_i}\}_{i=1,\ldots,n}$. The k-algebra A is integral and of finite type over B'. It follows that A is a finitely generated B'-module. The ring B' is noetherian. We conclude that the submodule B is a finitely generated B'-module. This proves our claim.

By general theory the ring B equals the ring of global sections of the structure sheaf of an affine variety Y. The natural inclusion $B \subseteq A$ of k-algebras induces a k-morphism $\pi : X \to Y$. The universal property of π follows from the following observation. The set of G-invariant k-morphisms $X \to Z$, where Z is affine, is in bijection with the set of k-algebra morphisms $C \to A$ such that the image of C lies in B.

The ring A is a finite B-module since A is integral and of finite type over B. Hence the morphism π is finite.

In the following we prove that π is surjective. We claim that for every prime $\mathfrak{p} \lhd B$ there exists a prime $\mathfrak{q} \lhd A$ such that $\mathfrak{p} = \mathfrak{q} \cap B$. Suppose $\mathfrak{p} \lhd B$ is prime. The ring extension $B_\mathfrak{p} \hookrightarrow A_\mathfrak{p}$ induced by the natural inclusion $B \subseteq A$ is integral. Let $\mathfrak{m}$ be a maximal ideal of $A_\mathfrak{p}$. We set $\mathfrak{n} = \mathfrak{m} \cap B_\mathfrak{p}$. Then $\mathfrak{n}$ is a prime ideal. The extension $B_\mathfrak{p}/\mathfrak{n} \hookrightarrow A_\mathfrak{p}/\mathfrak{m}$ is integral. Let $0 \neq x \in B_\mathfrak{p}/\mathfrak{n}$. Since $x^{-1} \in A_\mathfrak{p}/\mathfrak{m}$ it follows that there exist $b_1, \ldots, b_k \in B_\mathfrak{p}/\mathfrak{n}$, where $k \geq 1$, such that

$$x^{-k} + b_1 x^{-(k-1)} + \ldots + b_k = 0.$$

As a consequence we have

$$x^{-1} = -(b_1 + \ldots + b_k x^{k-1}) \in B_\mathfrak{p}/\mathfrak{n}.$$

We conclude that $B_\mathfrak{p}/\mathfrak{n}$ is a field and that the ideal $\mathfrak{n}$ is maximal. Hence $\mathfrak{n} = B_\mathfrak{p}\mathfrak{p}$. We set $\mathfrak{q} = \mathfrak{m} \cap A$. It follows that $\mathfrak{q} \cap B = \mathfrak{p}$. This proves our claim and completes the proof of the theorem. □

Note that the universal property of the quotient implies that it is unique up to isomorphism.

Remark 1.2 *We use the notation of Theorem 1.1. Assume that the action of G on X is free, i.e. all orbits have length $\#G$. Then the quotient morphism $\pi : X \to Y$ is étale.*

2 The Kummer variety

Let $(A, \mathcal{C})$ be a polarised abelian variety over a field k. We write $\mathrm{Aut}_k(A, \mathcal{C})$ for the k-automorphisms of A which preserve the polarisation $\mathcal{C}$.

Proposition 2.1 *The group* $\mathrm{Aut}_k(A, \mathcal{C})$ *is finite.*

Proof. Let D be an ample divisor which is contained in $\mathcal{C}$. Let $\varphi_D : A \to \check{A}$ be the the unique isogeny having as kernel the finite group

$$K(D) = \{x \in A | t_x^* D \overset{\text{lin}}{\sim} D\}$$

Let

$$(\gamma \mapsto \gamma^*) \in \mathrm{End}_{\mathbb{Q}}\big(\mathrm{End}_k^0(A)\big)$$

denote the Rosati involution (compare Alex' talk). Let $m = \deg(\varphi_D)$ and $\gamma \in \mathrm{Aut}_k(A, \mathcal{C})$. Then we have

$$[m] \circ \check{\gamma} \circ \varphi_D \circ \gamma = [m] \circ \varphi_D.$$

We conclude that $\gamma\gamma^* = 1$ in $\mathrm{End}_k^0(A)$. As a consequence we have

$$\mathrm{tr}(\gamma\gamma^*) = 2g$$

where $g = \dim A$. Here we take the trace with respect to the l-adic representation for some prime $l \neq \mathrm{char}(k)$. We claim that there exist only finitely many $\beta \in \mathrm{End}_k(A)$ satisfying $\mathrm{tr}(\beta\beta^*) = 2g$. Our claim follows from the following two facts which we state without proof.

- The quadratic form $\gamma \mapsto \mathrm{tr}(\gamma\gamma^*)$ is positive definite.
- The ring $\mathrm{End}(A)$ is a free $\mathbb{Z}$-module of finite rank.

This completes the proof of the proposition. □

Now assume that $k = \mathbb{C}$. Since A is projective (compare David's talk) the quotient W_A of A under the action of the finite group $\mathrm{Aut}(A, \mathcal{C})$ exists by Theorem 1.1. The variety W_A is called the *Kummer variety* of $(A, \mathcal{C})$. Let $\pi : A \to W_A$ denote the quotient morphism.

Proposition 2.2 *The Kummer variety* W_A *has the following properties:*

1. *The variety* W_A *can be defined over the field of moduli* k_0 *of* $(A, \mathcal{C})$*.*
2. *The morphism* π *is defined over every field of definition for* $(A, \mathcal{C})$ *containing* k_0*.*
3. *Assume we are given a field of definition* k *for* $(A, \mathcal{C})$ *containing* k_0*, a* $\sigma \in \mathrm{End}(\mathbb{C})$ *and an isomorphism of polarised abelian varieties*

   $$\eta : (A, \mathcal{C}) \xrightarrow{\sim} (A^\sigma, \mathcal{C}^\sigma).$$

   Then the equality $F = F^\sigma \circ \eta$ *holds.*

3 Example

Consider the complex elliptic curve E given by

$$y^2 = x(x-1)(x-\lambda)$$

where $\lambda \in \mathbb{C} \setminus \{0,1\}$. Let j_E denote the j-invariant of E and assume that $j_E \neq 0, 1728$. The zero section 0_E gives an ample divisor on E. Let $\mathcal{C}$ denote the induced polarisation. We claim that the Kummer variety W_E equals $\mathbb{P}^1_{\mathbb{C}}$ and the quotient map $\pi : E \to W_E = \mathbb{P}^1_{\mathbb{C}}$ is given by the rational map $(x,y) \mapsto (x:1)$. The morphism π is of degree 2 and is ramified at all points contained in $E[2]$. Since $j \notin 0, 1728$ it follows that

$$\mathrm{Aut}_{\mathbb{C}}(E, \mathcal{C}) = \mathrm{Aut}_{\mathbb{C}}(E) = \{\pm \mathrm{id}\}.$$

Note that the morphism π is invariant under the action of $\{\pm\mathrm{id}\}$. It remains to prove that π has the universal property of a quotient. Let $\alpha : E \to Z$ be a non-constant morphism of varieties which is invariant under the automorphism $-\mathrm{id}$. We can assume that Z is a non-singular curve. It follows by Hurwitz' theorem that $g(Z) \leq 1$. Here $g(Z)$ denotes the genus of Z. We claim that $g(Z) = 0$. Suppose $g(Z) = 1$. Then α decomposes as an isogeny followed up by a translation and hence cannot be invariant under the automorpism $-\mathrm{id}$. This contradicts our assumptions and implies the claim. By the above we can assume that $Z = \mathbb{P}^1_{\mathbb{C}}$. Let α be given by

$$(x,y) \mapsto \left(\frac{s(x,y)}{t(x,y)} : 1 \right)$$

where $s, t \in \mathbb{C}[X,Y]$. Since $s(x,y) = s(x,-y)$ and $t(x,y) = t(x,-y)$ for all points $(x,y) \in E(\mathbb{C})$ we conclude that the polynomials s and t contain only even powers of Y. Substituting $x(x-1)(x-\lambda)$ for y^2 we can assume that $s, t \in \mathbb{C}[X]$. We define a morphism $\tau : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ by setting

$$(w:1) \mapsto \left(\frac{s(w)}{t(w)} : 1 \right).$$

It follows that $\tau \circ \pi = \alpha$.

In David's talk it was proven that the field of moduli k_0 of $(E, \mathcal{C})$ equals $\mathbb{Q}(j_E)$. The polarised abelian variety $(E, \mathcal{C})$, the Kummer variety W_E and the morphism π can be defined over k_0 since the latter is a field of definition for E and $\mathrm{Aut}_{\mathbb{C}}(E, \mathcal{C})$.

References

[1] G. Shimura. *Abelian Varieties with Complex Multiplication and Modular Functions* . Princeton University Press, 1998.