Weight polynomial modules over the $\mathfrak{gl}_2$ Yangian

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Abstract

We study representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_2)$ which admit a weight space decomposition with respect to a natural commutative subalgebra. We describe irreducible generic weight modules satisfying a polynomiality condition. We also determine the representation type of categories of modules with finite-dimensional weight spaces and classify indecomposable modules in the categories of tame representation type.

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1 Introduction

The *Yangian for* $\mathfrak{gl}_n$ is a unital associative algebra $\mathrm{Y}(\mathfrak{gl}_n)$ with countably many generators $t_{ij}^{(1)},\ t_{ij}^{(2)},\dots$ where $1 \le i,j \le n$, and the defining relations

$$(u-v)\,[t_{ij}(u), t_{kl}(v)] = t_{kj}(u)\,t_{il}(v) - t_{kj}(v)\,t_{il}(u), \tag{1.1}$$

where

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots \tag{1.2}$$

and u, v are formal variables. This algebra originally appeared in the works on the *quantum inverse scattering method*; see e.g. Takhtajan–Faddeev [16], Kulish–Sklyanin [12]. The term "Yangian" and generalizations of $\mathrm{Y}(\mathfrak{gl}_n)$ to an arbitrary simple Lie algebra were introduced by Drinfeld [7]. He then classified finite-dimensional irreducible modules over the Yangians in [8] using earlier results of Tarasov [17, 18] for the $\mathfrak{sl}_2$ case. An explicit construction of all such modules over $\mathrm{Y}(\mathfrak{sl}_2)$ is given in those papers by Tarasov and also in the work by Chari and Pressley [3]. Apart from this case, the structure of a general Yangian representation remains unknown. In the case of $\mathrm{Y}(\mathfrak{gl}_n)$ a description of "generic" modules was given in [13] via Gelfand–Tsetlin bases. A more general class of "tame" representations of $\mathrm{Y}(\mathfrak{gl}_n)$ was introduced and explicitly constructed by Nazarov and Tarasov [15]. An important role in these works is played by the *Drinfeld generators* [8]

$$a_i(u), \quad i = 1,\dots,n, \qquad b_i(u), \quad c_i(u), \quad i = 1,\dots,n-1 \tag{1.3}$$

of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ which are defined as certain *quantum minors* of the matrix $T(u) = \big(t_{ij}(u)\big)$. The coefficients of the series $a_i(u)$, $i = 1,\dots,n$ form a commutative subalgebra of $\mathrm{Y}(\mathfrak{gl}_n)$ which can be regarded as an analog of a Cartan subalgebra for a semisimple Lie algebra. We shall call a representation of $\mathrm{Y}(\mathfrak{gl}_n)$ *weight* if it is a direct sum of weight spaces with respect to this subalgebra. In particular, finite-dimensional weight modules are precisely the tame modules of [15]. We, however, prefer the weight terminology to avoid an ambiguity with the tame categories discussed below in Section 6.

In this paper we are concerned with weight representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_2)$. Recall that every irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_2)$-module contains a unique vector ξ annihilated by $t_{12}(u)$ and which is an eigenvector for the Drinfeld generators $a_1(u)$ and $a_2(u)$ defined by

$$a_1(u) = t_{11}(u)\,t_{22}(u-1) - t_{21}(u)\,t_{12}(u-1), \qquad a_2(u) = t_{22}(u); \tag{1.4}$$

see [17, 18] and [3]. Moreover, there exists an automorphism $t_{ij}(u) \mapsto c(u)\,t_{ij}(u)$ of $\mathrm{Y}(\mathfrak{gl}_2)$, where $c(u) \in 1 + u^{-1}\,\mathbb{C}[[u^{-1}]]$, such that the eigenvalues of ξ become polynomials in u^{-1} under the twisted action of the Yangian. This prompts the introduction of the class of *weight polynomial* modules over $\mathrm{Y}(\mathfrak{gl}_2)$, i.e., such weight modules where the operators $a_1(u)$ and $a_2(u)$ are polynomials. More precisely, by (1.4) it is natural to require that for some positive integer p the polynomials $a_1(u)$ and $a_2(u)$ have degrees $2p$ and p, respectively. Note that $a_1(u)$ is the *quantum determinant* of the matrix $T(u)$ [11], [12]. Its coefficients are algebraically independent generators of the center of $\mathrm{Y}(\mathfrak{gl}_2)$.

We can interpret the definition of weight polynomial modules using the algebra $\mathrm{Y}_p(\mathfrak{gl}_2)$ called the *Yangian of level* p; see Cherednik [4, 5]. It is defined as the quotient of $\mathrm{Y}(\mathfrak{gl}_2)$ by the ideal generated by the elements $t_{ij}^{(r)}$ with $r \geq p+1$. A weight polynomial module over $\mathrm{Y}(\mathfrak{gl}_2)$ is just a weight module over $\mathrm{Y}_p(\mathfrak{gl}_2)$ for some positive integer p.

For another interpretation consider the *Yangian for* $\mathfrak{sl}_2$ which is the subalgebra $\mathrm{Y}(\mathfrak{sl}_2)$ of $\mathrm{Y}(\mathfrak{gl}_2)$ generated by the coefficients of the series $e(u)$, $f(u)$ and $h(u)$ [8] defined by

$$\begin{aligned} e(u) &= t_{22}(u)^{-1} t_{12}(u), \\ f(u) &= t_{21}(u)\, t_{22}(u)^{-1}, \\ h(u) &= t_{11}(u)\, t_{22}(u)^{-1} - t_{21}(u)\, t_{22}(u)^{-1} t_{12}(u)\, t_{22}(u)^{-1}. \end{aligned} \tag{1.5}$$

Note that the series $h(u)$ can also be given by

$$h(u) = a_1(u)\, a_2(u)^{-1} a_2(u-1)^{-1} \tag{1.6}$$

so that the coefficients of $h(u)$ form a commutative subalgebra of $\mathrm{Y}(\mathfrak{sl}_2)$. Therefore, the restriction of a $\mathrm{Y}(\mathfrak{gl}_2)$-weight module to $\mathrm{Y}(\mathfrak{sl}_2)$ admits an eigenbasis for this subalgebra. We also point out that both the above interpretations extend to an arbitrary Yangian $\mathrm{Y}(\mathfrak{gl}_n)$.

In this paper we study *generic* weight polynomial modules over $\mathrm{Y}(\mathfrak{gl}_2)$. This imposes a certain integrability condition on the eigenvalues of $a_2(u)$ while those of $a_1(u)$ can be arbitrary; see Section 2 for the details. We consider the class of modules admitting a central character so that the coefficients of $a_1(u)$ act as scalars. This class contains all irreducible weight modules. Our main results are the complete descriptions of all irreducible and indecomposable tame weight polynomial modules from this class. In particular, we show that every irreducible module is a quotient of the corresponding universal weight polynomial module. We also describe the supports (weight diagrams) of the irreducible modules.

2 Generic weight modules over $\mathrm{Y}_p(\mathfrak{gl}_2)$

Let p be a positive integer. The *level p Yangian* $\mathrm{Y}_p(\mathfrak{gl}_2)$ for the Lie algebra $\mathfrak{gl}_2$ [5] can be defined as the algebra with generators $t_{ij}^{(1)}, \dots, t_{ij}^{(p)}$, $i, j = 1, 2$, subject to the relations

$$[T_{ij}(u), T_{kl}(v)] = \frac{1}{u-v}(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)), \tag{2.1}$$

where u, v are formal variables and

$$T_{ij}(u) = \delta_{ij}\, u^p + \sum_{k=1}^{p} t_{ij}^{(k)}\, u^{p-k} \in \mathrm{Y}_p(\mathfrak{gl}_2)[u]. \tag{2.2}$$

Explicitly, (2.1) reads

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{a=1}^{\min(r,s)} \big(t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)}\big), \tag{2.3}$$

where $t_{ij}^{(0)} = \delta_{ij}$ and $t_{ij}^{(r)} = 0$ for $r \geq p+1$. Note that the level 1 Yangian $\mathrm{Y}_1(\mathfrak{gl}_2)$ coincides with the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_2)$. We have the following analog of the Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{Y}_p(\mathfrak{gl}_2)$ [5]; see also [14].

Theorem 2.1. *Given an arbitrary linear ordering on the set of the generators $t_{ij}^{(k)}$, any element of the algebra $\mathrm{Y}_p(\mathfrak{gl}_2)$ is uniquely written as a linear combination of ordered monomials in the generators.*

Denote by $D(u)$ the *quantum determinant*

$$\begin{aligned} D(u) &= T_{11}(u)T_{22}(u-1) - T_{21}(u)T_{12}(u-1) \\ &= T_{11}(u-1)T_{22}(u) - T_{12}(u-1)T_{21}(u). \end{aligned} \tag{2.4}$$

It was shown in [4, 5] (see also [14] for a different proof) that the coefficients of the polynomial $D(u)$ are algebraically independent generators of the center of the algebra $\mathrm{Y}_p(\mathfrak{gl}_2)$.

Let V be a $\mathrm{Y}_p(\mathfrak{gl}_2)$-module. Given two monic polynomials $\beta = \beta(u)$ and $\gamma = \gamma(u)$ in u of degrees p and $2p$, respectively, introduce the corresponding *weight subspace* of V by

$$V_{\beta,\gamma} = \{\eta \in V \mid T_{22}(u)\eta = \beta(u)\eta, \quad D(u)\eta = \gamma(u)\eta\}. \tag{2.5}$$

Definition 2.2. *A $\mathrm{Y}_p(\mathfrak{gl}_2)$-module V is said to be* weight *if V is a direct sum of its weight subspaces:*

$$V = \bigoplus_{\beta,\gamma} V_{\beta,\gamma}. \tag{2.6}$$

As we pointed out in the Introduction, if V is irreducible then $\gamma(u)$ is the same for all nonzero weight subspaces. So if $\gamma(u)$ is given we shall simply write V_β instead of $V_{\beta,\gamma}$.

From now on we shall assume that all our modules over $\mathrm{Y}_p(\mathfrak{gl}_2)$ admit a central character $\gamma = \gamma(u)$. A standard argument [6] shows that any non-trivial submodule of a weight module V is also a weight module.

Write

$$\beta(u) = (u + \beta_1) \cdots (u + \beta_p). \tag{2.7}$$

The collection $(\beta_1, \ldots, \beta_p)$ uniquely determines $\beta(u)$. We shall also denote this collection by β if this does not lead to an ambiguity. We denote by $\operatorname{Supp} V$ the *support* of a weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module V, i.e., the set of all β such that the subspace V_β is nonzero.

We shall say that a weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module V is *generic* if $\beta_i - \beta_j \notin \mathbb{Z}$ for all $i \neq j$ and all $\beta = (\beta_1, \ldots, \beta_p) \in \operatorname{Supp} V$.

Lemma 2.3. *Let V be a generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module and let $\beta = (\beta_1, \ldots, \beta_p) \in \operatorname{Supp} V$. Then*

$$T_{21}(u) V_\beta \subseteq \sum_{i=1}^{p} V_{\beta+\delta_i} \qquad \textit{and} \qquad T_{12}(u) V_\beta \subseteq \sum_{i=1}^{p} V_{\beta-\delta_i} \tag{2.8}$$

where $\beta \pm \delta_i = (\beta_1, \ldots, \beta_i \pm 1, \ldots, \beta_p)$.

Proof. First we show that $T_{21}(-\beta_i) V_\beta \subseteq V_{\beta+\delta_i}$ for all $i = 1, \ldots, p$. Since

$$T_{22}(u-1) T_{21}(u) = T_{21}(u-1) T_{22}(u)$$

we have

$$T_{22}(-\beta_i - 1) T_{21}(-\beta_i)\, \eta = T_{21}(-\beta_i - 1) T_{22}(-\beta_i)\, \eta = 0$$

for all $\eta \in V_\beta$. Also,

$$\begin{aligned} T_{22}(-\beta_j) T_{21}(-\beta_i)\, \eta = {} & (\beta_i - \beta_j)^{-1} \big(T_{21}(-\beta_i) T_{22}(-\beta_j) - T_{21}(-\beta_j) T_{22}(-\beta_i) \big)\, \eta \\ & + T_{21}(-\beta_i) T_{22}(-\beta_j)\, \eta = 0 \end{aligned}$$

since $T_{22}(-\beta_k)\, \eta = 0$ for all $k = 1, \ldots, p$. Using the fact that $\beta_i - \beta_j \notin \mathbb{Z}$ we conclude that $T_{21}(-\beta_i) V_\beta \subseteq V_{\beta+\delta_i}$ for all $i = 1, \ldots, p$. Since $T_{21}(u)$ is a polynomial of degree $p-1$ in u and $\beta_i \neq \beta_j$ if $i \neq j$, we have that $T_{21}(u) V_\beta \subseteq \sum_{i=1}^{p} V_{\beta+\delta_i}$. The case of $T_{12}(u)$ is treated analogously using the identity $T_{22}(u) T_{12}(u-1) = T_{12}(u) T_{22}(u-1)$. $\square$

Corollary 2.4. *If V is indecomposable generic weight module over $\mathrm{Y}_p(\mathfrak{gl}_2)$ and $\beta \in \operatorname{Supp} V$ then $\operatorname{Supp} V \subseteq \beta + \mathbb{Z}^p$.*

Lemma 2.5. *If V is a generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module with central character $\gamma(u)$ then for any $\beta = (\beta_1, \ldots, \beta_p) \in \operatorname{Supp} V$ and any $\eta \in V_\beta$ we have*

$$T_{12}(-\beta_r) T_{21}(-\beta_s)\, \eta = T_{21}(-\beta_s) T_{12}(-\beta_r)\, \eta,$$

if $s \neq r$, and

$$T_{12}(-\beta_i - 1)T_{21}(-\beta_i)\,\eta = -\gamma(-\beta_i)\,\eta,$$
$$T_{21}(-\beta_i + 1)T_{12}(-\beta_i)\,\eta = -\gamma(-\beta_i + 1)\,\eta.$$

Proof. The first equality follows from the defining relations (1.1). The others follow from (2.4). □

Corollary 2.6. *Let V be a generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module with a central character $\gamma(u)$ and let $\beta = (\beta_1, \ldots, \beta_p) \in \operatorname{Supp} V$.*

(*i*) *If $\gamma(-\beta_i) \neq 0$ then $\operatorname{Ker} T_{21}(-\beta_i) \cap V_\beta = 0$.*

(*ii*) *If $\gamma(-\beta_i + 1) \neq 0$ then $\operatorname{Ker} T_{12}(-\beta_i) \cap V_\beta = 0$.*

(*iii*) *If V is indecomposable and $\gamma(-\beta_i + k) \neq 0$ for all $k \in \mathbb{Z}$ then*

$$\operatorname{Ker} T_{21}(-\psi_i) \cap V_\psi = \operatorname{Ker} T_{12}(-\psi_i) \cap V_\psi = 0$$

for all $\psi = (\psi_1, \ldots, \psi_p) \in \operatorname{Supp} V$.

3 Irreducible weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-modules

In this section we give a description of all irreducible generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-modules. Let $\beta = (\beta_1, \ldots, \beta_p) \in \mathbb{C}^p$, $\beta_i - \beta_j \notin \mathbb{Z}$ for $i \neq j$ and let $\gamma = (\gamma_1, \ldots, \gamma_{2p}) \in \mathbb{C}^{2p}$. Set

$$\begin{aligned} \beta(u) &= (u + \beta_1) \cdots (u + \beta_p), \\ \gamma(u) &= (u + \gamma_1) \cdots (u + \gamma_{2p}). \end{aligned} \tag{3.1}$$

Denote by $R_{\beta,\gamma}$ the category of generic weight modules V with central character $\gamma(u)$ and with $\operatorname{Supp} V \subseteq \beta + \mathbb{Z}^p$. Now we construct a universal module in $R_{\beta,\gamma}$. Let $I_{\beta,\gamma}$ be the left ideal of $\mathrm{Y}_p(\mathfrak{gl}_2)$ generated by the coefficients of the polynomials $T_{22}(u) - \beta(u)$ and $D(u) - \gamma(u)$, where $D(u)$ is defined in (2.4). Define the corresponding quotient module over $\mathrm{Y}_p(\mathfrak{gl}_2)$ by

$$M(\beta, \gamma) = \mathrm{Y}_p(\mathfrak{gl}_2)/I_{\beta,\gamma}.$$

It is clear from Theorem 2.1 that $I_{\beta,\gamma}$ is a proper ideal of $\mathrm{Y}_p(\mathfrak{gl}_2)$ and so $M(\beta, \gamma)$ is a non-trivial module. Therefore, the image of 1 in $M(\beta, \gamma)$ is nonzero. We shall denote it by ξ. The next proposition shows the universality of the module $M(\beta, \gamma)$.

Proposition 3.1. *Let V be a weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module with a central character γ generated by a nonzero $\eta \in V_\beta$. Then V is a homomorphic image of $M(\beta, \gamma)$.*

Proof. Indeed, there is a homomorphism $f : M(\beta,\gamma) \to V$ which maps ξ to η. Since η generates V the statement follows. $\square$

Given $(k) = (k_1,\dots,k_p) \in \mathbb{Z}^p$ define the corresponding vector of the module $M(\beta,\gamma)$ by

$$\begin{aligned}\xi^{(k)} = &\prod_{i,\ k_i>0} T_{21}(-\beta_i - k_i + 1)\cdots T_{21}(-\beta_i - 1)\, T_{21}(-\beta_i)\\ \times &\prod_{i,\ k_i<0} T_{12}(-\beta_i - k_i - 1)\cdots T_{12}(-\beta_i + 1)\, T_{12}(-\beta_i)\, \xi.\end{aligned}$$

Theorem 3.2. *The vectors $\xi^{(k)}, (k) \in \mathbb{Z}^p$ form a basis of $M(\beta,\gamma)$. Moreover, we have the formulas*

$$T_{22}(u)\, \xi^{(k)} = \prod_{i=1}^{p} (u + \beta_i + k_i)\, \xi^{(k)}, \tag{3.2}$$

$$\begin{aligned}T_{21}(u)\, \xi^{(k)} &= \sum_{i=1}^{p} A_i(k)\, \frac{(u+\beta_1+k_1)\cdots\wedge_i\cdots(u+\beta_p+k_p)}{(\beta_1-\beta_i+k_1-k_i)\cdots\wedge_i\cdots(\beta_p-\beta_i+k_p-k_i)}\, \xi^{(k+\delta_i)},\\ T_{12}(u)\, \xi^{(k)} &= \sum_{i=1}^{p} B_i(k)\, \frac{(u+\beta_1+k_1)\cdots\wedge_i\cdots(u+\beta_p+k_p)}{(\beta_1-\beta_i+k_1-k_i)\cdots\wedge_i\cdots(\beta_p-\beta_i+k_p-k_i)}\, \xi^{(k-\delta_i)},\end{aligned} \tag{3.3}$$

where

$$A_i(k) = \begin{cases} 1 & \text{if} \quad k_i \geq 0 \\ -\gamma(-\beta_i - k_i) & \text{if} \quad k_i < 0 \end{cases}$$

and

$$B_i(k) = \begin{cases} -\gamma(-\beta_i - k_i + 1) & \text{if} \quad k_i > 0 \\ 1 & \text{if} \quad k_i \leq 0. \end{cases}$$

The action of $T_{11}(u)$ is found from the relation

$$\Big(T_{11}(u)T_{22}(u-1) - T_{21}(u)T_{12}(u-1)\Big)\, \xi^{(k)} = \gamma(u)\, \xi^{(k)}. \tag{3.4}$$

Proof. We start by proving the formulas for the action of the generators of $\mathrm{Y}_p(\mathfrak{gl}_2)$. Relation (3.2) follows by induction from the defining relations (1.1). By Lemma 2.5 we have: if $k_i > 0$ then

$$\begin{aligned}T_{21}(-\beta_i - k_i)\, \xi^{(k)} &= \xi^{(k+\delta_i)},\\ T_{12}(-\beta_i - k_i)\, \xi^{(k)} &= -\gamma(-\beta_i - k_i + 1)\, \xi^{(k-\delta_i)};\end{aligned} \tag{3.5}$$

if $k_i < 0$ then

$$\begin{aligned}T_{12}(-\beta_i - k_i)\, \xi^{(k)} &= \xi^{(k-\delta_i)},\\ T_{21}(-\beta_i - k_i)\, \xi^{(k)} &= -\gamma(-\beta_i - k_i)\, \xi^{(k+\delta_i)};\end{aligned} \tag{3.6}$$

and if $k_i = 0$ then

$$\begin{aligned} T_{12}(-\beta_i)\,\xi^{(k)} &= \xi^{(k-\delta_i)}, \\ T_{21}(-\beta_i)\,\xi^{(k)} &= \xi^{(k+\delta_i)}. \end{aligned} \tag{3.7}$$

Applying the Lagrange interpolation formula we obtain the remaining formulas.

To show that the vectors $\xi^{(k)}$ form a basis of $M(\beta,\gamma)$, denote by $\mathcal{T}_\beta$ the subspace of $\mathrm{Y}_p(\mathfrak{gl}_2)$ spanned by the elements

$$\begin{aligned} \tau^{(k)} = &\prod_{i,\ k_i>0} T_{21}(-\beta_i - k_i + 1)\cdots T_{21}(-\beta_i - 1)\,T_{21}(-\beta_i) \\ &\times \prod_{i,\ k_i<0} T_{12}(-\beta_i - k_i - 1)\cdots T_{12}(-\beta_i + 1)\,T_{12}(-\beta_i), \end{aligned}$$

where (k) runs over $\mathbb{Z}^p$. It suffices to prove the vector space decomposition

$$\mathrm{Y}_p(\mathfrak{gl}_2) = \mathcal{T}_\beta \oplus I_{\beta,\gamma}. \tag{3.8}$$

Due to the formulas proved above, $\mathrm{Y}_p(\mathfrak{gl}_2) = \mathcal{T}_\beta + I_{\beta,\gamma}$. We now need to show that the vectors $\tau^{(k)}$ are linearly independent modulo the left ideal $I_{\beta,\gamma}$. By (3.2) and the genericity assumption, the elements $\tau^{(k)} \mod I_{\beta,\gamma}$ are eigenvectors for $T_{22}(u)$ with distinct eigenvalues. So the claim will follow if we demonstrate that each $\tau^{(k)}$ is nonzero modulo $I_{\beta,\gamma}$. Suppose first that $\gamma(u)$ is generic: $\gamma(-\beta_i - k) \neq 0$ for all $k \in \mathbb{Z}$ and all i. Then we deduce from (3.5)–(3.7) that $\tau^{(k)} \neq 0 \mod I_{\beta,\gamma}$ since $1 \neq 0 \mod I_{\beta,\gamma}$ which gives (3.8) for generic $\gamma(u)$.

Let now $\gamma(u)$ be arbitrary. Suppose that a nonzero element τ belongs to the intersection $\mathcal{T}_\beta \cap I_{\beta,\gamma}$. Then

$$\tau = \sum_{i=1}^{p} a_i\,(t_{22}^{(i)} - \beta^{(i)}) + \sum_{i=1}^{2p} b_i\,(D^{(i)} - \gamma^{(i)}), \tag{3.9}$$

where $D^{(i)}$, $\beta^{(i)}$ and $\gamma^{(i)}$ are the coefficients of the polynomials $D(u)$, $\beta(u)$ and $\gamma(u)$, respectively, while $a_i, b_i \in \mathrm{Y}_p(\mathfrak{gl}_2)$. Let $\widetilde{\gamma}(u)$ be generic. Then we can rewrite (3.9) as

$$\tau = \sum_{i=1}^{p} a_i\,(t_{22}^{(i)} - \beta^{(i)}) + \sum_{i=1}^{2p} b_i\,(D^{(i)} - \widetilde{\gamma}^{(i)}) + \sum_{i=1}^{2p} b_i\,(\widetilde{\gamma}^{(i)} - \gamma^{(i)}). \tag{3.10}$$

Consider the unique decompositions of the elements b_j in accordance with (3.8) where $\gamma(u)$ is taken to be $\widetilde{\gamma}(u)$:

$$b_j = \tau_j + \sum_{i=1}^{p} a_{ij}\,(t_{22}^{(i)} - \beta^{(i)}) + \sum_{i=1}^{2p} b_{ij}\,(D^{(i)} - \widetilde{\gamma}^{(i)}) \tag{3.11}$$

for some a_{ij}, $b_{ij} \in \mathrm{Y}_p(\mathfrak{gl}_2)$. Using the decomposition (3.8) for generic $\widetilde{\gamma}(u)$ we must have

$$\tau = \sum_{j=1}^{2p} \tau_j \, (\widetilde{\gamma}^{(j)} - \gamma^{(j)}). \tag{3.12}$$

for all such $\widetilde{\gamma}(u)$. This means that the $\mathcal{T}_\beta$-component of each element $b_j \, (\widetilde{\gamma}^{(j)} - \gamma^{(j)})$ is independent of $\widetilde{\gamma}(u)$. However, due to the formulas (3.2)–(3.4), this is only possible if all b_j are zero. Finally, the elements a_i must be zero too by the decomposition (3.9) with generic $\gamma(u)$. So, (3.8) holds for all $\gamma(u)$. □

Given two monic polynomials $\alpha = \alpha(u)$ and $\beta = \beta(u)$ of degree p define the corresponding *Verma module* $V(\alpha, \beta)$ as the quotient of $\mathrm{Y}_p(\mathfrak{gl}_2)$ by the left ideal generated by the coefficients of the polynomials $T_{11}(u) - \alpha(u)$, $T_{22}(u) - \beta(u)$ and $T_{12}(u)$; cf. [17, 18]. Then the same argument as above shows that $V(\alpha, \beta)$ has a basis $\{\xi^{(k)}\}$ parametrized by p-tuples of nonnegative integers $(k) = (k_1, \dots, k_p)$. The formulas of Theorem 3.2 hold for the basis vectors $\xi^{(k)}$, where $\gamma(u)$ should be taken to be $\alpha(u)\,\beta(u-1)$ which is the central character of $V(\alpha, \beta)$. In fact, $V(\alpha, \beta)$ is isomorphic to the quotient of the corresponding universal module $M(\beta, \gamma)$ by the submodule spanned by the vectors $\{\xi^{(k)}\}$ containing at least one negative component k_i.

Corollary 3.3. *With the notation* (3.1) *we have*

(*i*) *The module $M(\beta, \gamma)$ is a generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module with central character $\gamma(u)$, $\operatorname{Supp} M(\beta, \gamma) = \mathbb{Z}^p$ and all weight spaces are 1-dimensional.*

(*ii*) *The module $M(\beta, \gamma)$ has a unique maximal submodule and hence a unique irreducible quotient.*

Proof. Statement (*i*) follows immediately from Theorem 3.2. By Proposition 3.1 the sum of all proper submodules is again a proper submodule. Thus $M(\beta, \gamma)$ has a unique maximal submodule which implies (*ii*). □

We will denote the unique irreducible quotient of $M(\beta, \gamma)$ by $L(\beta, \gamma)$. It follows from Corollary 3.3 that all weight spaces of $L(\beta, \gamma)$ are 1-dimensional. Using Proposition 3.1 we can now describe all irreducible generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-modules.

Theorem 3.4. (*i*) *For any $\beta = (\beta_1, \dots, \beta_p) \in \mathbb{C}^p$, such that $\beta_i - \beta_j \notin \mathbb{Z}$ for all $i \neq j$, and for any $\gamma = (\gamma_1, \dots, \gamma_{2p}) \in \mathbb{C}^{2p}$ there exists an irreducible generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module $L(\beta, \gamma)$ with $L(\beta, \gamma)_\beta \neq 0$ and with central character γ. Moreover, $\dim L(\beta, \gamma)_\psi = 1$ for all $\psi \in \operatorname{Supp} L(\beta, \gamma)$.*

(*ii*) *Any irreducible weight module over $\mathrm{Y}_p(\mathfrak{gl}_2)$ with central character γ generated by a nonzero vector of generic weight β is isomorphic to $L(\beta, \gamma)$.*

4 Category $K_{\beta,\gamma}$

Fix $\beta = (\beta_1, \ldots, \beta_p) \in \mathbb{C}^p$ and $\gamma = (\gamma_1, \ldots, \gamma_{2p}) \in \mathbb{C}^{2p}$ such that $\beta_i - \beta_j \notin \mathbb{Z}$ for all $i, j = 1, \ldots, p$, $i \neq j$. Set

$$\beta(u) = \prod_{i=1}^{p}(u + \beta_i) \qquad \text{and} \qquad \gamma(u) = \prod_{i=1}^{2p}(u + \gamma_i). \tag{4.1}$$

Consider the following category $K_{\beta,\gamma}$: $\mathrm{Ob}\,(K_{\beta,\gamma}) = \mathbb{Z}^p$ and the morphisms are generated by

$$f_i(k) : (k) \mapsto (k + \delta_i) \qquad \text{and} \qquad e_i(k) : (k) \mapsto (k - \delta_i), \tag{4.2}$$

where $i = 1, \ldots, p$ and $(k) = (k_1, \ldots, k_p) \in \mathbb{Z}^p$ with the following relations:

$$\begin{aligned}
&f_j(k+\delta_i)\, f_i(k) = f_i(k+\delta_j)\, f_j(k),\\
&e_j(k-\delta_i)\, e_i(k) = e_i(k-\delta_j)\, e_j(k),\\
&e_i(k+\delta_j)\, f_j(k) = f_j(k-\delta_i)\, e_i(k) \qquad \text{for} \quad i \neq j,\\
&e_i(k+\delta_i)\, f_i(k) = -\gamma(-\beta_i - k_i)\, 1_{(k)},\\
&f_i(k-\delta_i)\, e_i(k) = -\gamma(-\beta_i - k_i + 1)\, 1_{(k)}.
\end{aligned}$$

It follows immediately from Lemmas 2.3 and 2.5 that any module in the category $R_{\beta,\gamma}$ can naturally be viewed as a module over the category $K_{\beta,\gamma}$. Hence there exists a monic functor $F : R_{\beta,\gamma} \to K_{\beta,\gamma}$−mod. Consider the cyclic subalgebra $C_{\beta,\gamma}(a) = \mathrm{Hom}_{K_{\beta,\gamma}}(a, a)$ for any $a \in \mathbb{Z}^p$. Clearly, $C_{\beta,\gamma}(a) \simeq \mathbb{C}$ for any $a \in \mathbb{Z}^p$ due to the defining relations of $K_{\beta,\gamma}$. For any $a = (k_1, \ldots, k_p) \in \mathbb{Z}^p$ we can construct a universal module $M(\beta, \gamma, a) \in K_{\beta,\gamma}$−mod. Consider $\mathbb{C}$ as a $C_{\beta,\gamma}(a)$-module with

$$\begin{aligned}
&e_i(k+\delta_i)\, f_i(k)\, 1 = -\gamma(-\beta_i - k_i),\\
&f_i(k-\delta_i)\, e_i(k)\, 1 = -\gamma(-\beta_i - k_i + 1).
\end{aligned}$$

Let $A_{\beta,\gamma,a}$ be an algebra of paths in $K_{\beta,\gamma}$ originating in a. Now construct a $\mathbb{Z}^p$-graded $K_{\beta,\gamma}$-module

$$M(\beta, \gamma, a) = A_{\beta,\gamma,a} \otimes_{C_{\beta,\gamma}(a)} \mathbb{C}.$$

Clearly, all graded components of $M(\beta, \gamma, a)$ are 1-dimensional and $M(\beta, \gamma, a)_a = 1_a \otimes \mathbb{C}$. A module $M(\beta, \gamma, a)$ contains a unique maximal $\mathbb{Z}^p$-graded submodule which intersects $M(\beta, \gamma, a)_a$ trivially and hence has a unique irreducible quotient $L(\beta, \gamma, a)$ with $L(\beta, \gamma, a)_a \simeq \mathbb{C}$ and $\dim L(\beta, \gamma, a)_b \leq 1$ for all $b \in \mathbb{Z}^p$. If V is another irreducible $K_{\beta,\gamma}$-module with $V_a \neq 0$ then there exists a non-trivial $C_{\beta,\gamma}(a)$-homomorphism from $\mathbb{C}$ to V_a which can be extended to an epimorphism from $M(\beta, \gamma, a)$ to V. Since V is irreducible we conclude that $V \simeq L(\beta, \gamma, a)$.

Obviously, we can view $M(\beta,\gamma)$ as a module over the category $K_{\beta,\gamma}$ with a natural action of the morphisms of $K_{\beta,\gamma}$ and $F(M(\beta,\gamma)) = M(\beta,\gamma,\beta)$. Thus a $K_{\beta,\gamma}$-module $M(\beta,\gamma,\beta)$ can be extended to a $\mathrm{Y}_p(\mathfrak{gl}_2)$-module $M(\beta,\gamma)$. Moreover, the functor F preserves the submodule structure of $M(\beta,\gamma)$. In particular, $F(L(\beta,\gamma)) = L(\beta,\gamma,\beta)$.

Proposition 4.1. *Categories $K_{\beta,\gamma}-$mod and $R_{\beta,\gamma}$ are equivalent.*

Proof. We already have a monic functor $F : R_{\beta,\gamma} \to K_{\beta,\gamma}-$mod. Suppose that $V \in K_{\beta,\gamma}-$mod. We want to show that V can be extended to a $\mathrm{Y}_p(\mathfrak{gl}_2)$-module. Fix $v \in V_{(k)} \setminus \{0\}$. Let $W \subseteq V$ be a submodule generated by v. Then $W_{(k)} = \mathbb{C}v$ and there is an epimorphism from $M(\beta,\gamma,a)$ to W, where $a = (k_1,\dots,k_p)$, which maps $1_a \otimes 1$ to v. Since $F(M(\beta + a,\gamma)) = M(\beta,\gamma,a)$, $\beta + a = (\beta_1 + a_1,\dots,\beta_p + a_p)$, then W can be extended to a corresponding quotient of $M(\beta + a,\gamma)$. We conclude that V can be extended to a $\mathrm{Y}_p(\mathfrak{gl}_2)$-module and will denote that module by $G(V)$. Clearly, G defines a functor from $K_{\beta,\gamma}-$mod to $R_{\beta,\gamma}$ (action on morphisms is obvious). One can easily see that the functors F and G define an equivalence between the categories $K_{\beta,\gamma}-$mod and $R_{\beta,\gamma}$. □

5 Support of irreducible generic weight modules

To complete the classification of irreducible modules we have to know when two irreducible modules $L(\beta,\gamma)$ and $L(\beta',\gamma)$ are isomorphic. For that we need to describe the support $\operatorname{Supp} L(\beta,\gamma)$.

We shall say that the weight subspaces $M(\beta,\gamma)_\psi$ and $M(\beta,\gamma)_{\psi+\delta_i}$ are *strongly isomorphic* if $\gamma(-\psi_i) \neq 0$ where $\psi = (\psi_1,\dots,\psi_p)$. This implies

$$f_i(\psi_1,\dots,\psi_p)\, M(\beta,\gamma)_\psi \neq 0 \quad \text{and} \quad e_i(\psi_1,\dots,\psi_i+1,\dots,\psi_p)\, M(\beta,\gamma)_{\psi+\delta_i} \neq 0.$$

The statement below follows immediately from the relations in $K_{\beta,\gamma}$ (cf. also Corollary 2.6).

Lemma 5.1. *If $M(\beta,\gamma)_\psi$ and $M(\beta,\gamma)_{\psi+\delta_i}$ are strongly isomorphic then $M(\beta,\gamma)_{\psi\pm\delta_j}$ and $M(\beta,\gamma)_{\psi+\delta_i\pm\delta_j}$ are strongly isomorphic for all $i,j = 1,\dots,p$, $i \neq j$. Moreover, if*

$$f_i(\psi_1,\dots,\psi_p)\, M(\beta,\gamma)_\psi = 0 \qquad \text{or} \qquad e_i(\psi_1,\dots,\psi_p)\, M(\beta,\gamma)_\psi = 0$$

then

$$\begin{aligned} &f_i(\psi_1,\dots,\psi_j \pm 1,\dots,\psi_p)\, M(\beta,\gamma)_{\psi\pm\delta_j} = 0 \qquad \text{or} \\ &e_i(\psi_1,\dots,\psi_j + 1,\dots,\psi_p)\, M(\beta,\gamma)_{\psi\pm\delta_j} = 0, \end{aligned}$$

respectively, for all $j \neq i$.

Let $a_i, b_i \in \mathbb{Z} \cup \{\pm\infty\}$, $a_i \leq b_i$, $i \in \{1, \dots, p\}$. Denote

$$P(a_1, \dots, a_p, b_1, \dots, b_p) = \{(x_1, \dots, x_p) \in \mathbb{Z}^p \mid a_i \leq x_i \leq b_i,\ i = 1, \dots, p\},$$

a parallelepiped in $\mathbb{Z}^p$. Note that some faces of the parallelepiped can be infinite in some directions. In particular, in the case $a_i = -\infty$, $b_i = \infty$ for all i, the parallelepiped coincides with $\mathbb{Z}^p$.

Theorem 5.2. *For any irreducible weight module $L(\beta, \gamma)$ over $\mathrm{Y}_p(\mathfrak{gl}_2)$ there exist elements $a_i, b_i \in \mathbb{Z} \cup \{\pm\infty\}$, $a_i \leq b_i$, $i \in \{1, \dots, p\}$ such that*

$$\operatorname{Supp} L(\beta, \gamma) = P(a_1, \dots, a_p, b_1, \dots, b_p).$$

Proof. Fix $i \in \{1, \dots, p\}$. If $\gamma(-\beta_i + k) \neq 0$ for all $k \in \mathbb{Z}$ then $(k_1, \dots, k_i + m, \dots, k_p) \in \operatorname{Supp} L(\beta, \gamma)$ as soon as $(k_1, \dots, k_p) \in \operatorname{Supp} L(\beta, \gamma)$. This follows immediately from Lemma 5.1. In this case we set $a_i = -\infty$ and $b_i = \infty$. Let now $\gamma(-\beta_i + k) = 0$ for some $k \in \mathbb{Z}$. Let $m \geq 0$ be the smallest integer (if exists) such that $\gamma(-\beta_i - m) = 0$ and let $n \leq 0$ be the largest integer (if exists) such that $\gamma(-\beta_i - n + 1) = 0$. It follows from Lemma 5.1 that

$$\operatorname{Supp} L(\beta, \gamma) \cap \{\beta + k\delta_i \mid k \in \mathbb{Z}\} = \{\beta + n\delta_i, \dots, \beta, \dots, \beta + m\delta_i\}.$$

If $\beta + s\delta_j \in \operatorname{Supp} L(\beta, \gamma)$, $j \neq i$ then

$$\operatorname{Supp} L(\beta, \gamma) \cap \{\beta + s\delta_j + k\delta_i \mid k \in \mathbb{Z}\} = \{\beta + s\delta_j + n\delta_i, \dots, \beta + s\delta_j, \dots, \beta + s\delta_j + m\delta_i\}.$$

In this case we set $a_i = \beta_i + n$ and $b_i = \beta_i + m$. The statement of the theorem now follows. $\square$

Example 5.3. *Suppose that $p = 2$. Then we have the following possibilities.*

(i) $\operatorname{Supp} L(\beta, \gamma) = \mathbb{Z}^2$ if and only if $\gamma(-\beta_i + k) \neq 0$ for all $i = 1, \dots, p$ and all $k \in \mathbb{Z}$.

(ii) Let $\gamma(-\beta_1 + k) \neq 0$ for all $k \in \mathbb{Z}$, $\gamma(-\beta_2 - m) = 0$, $m \geq 0$ and let $\gamma(-\beta_2 - n) \neq 0$ for all integer $n \leq m$. Then $\operatorname{Supp} L(\beta, \gamma) = \{\beta + k\delta_1 + n\delta_2 \mid k, n \in \mathbb{Z}, n \leq m\}$. Hence the support in this case is a half-plane.

(iii) Let $\gamma(-\beta_1 + k) \neq 0$ for all $k \in \mathbb{Z}$, $\gamma(-\beta_2 - m + 1) = 0$, $m \leq 0$ and let $\gamma(-\beta_2 - n) \neq 0$ for all integer $n \geq m$. Then $\operatorname{Supp} L(\beta, \gamma) = \{\beta + k\delta_1 + n\delta_2 \mid k, n \in \mathbb{Z}, n \geq m\}$.

(iv) Let $\gamma(-\beta_1 + k) \neq 0$ for all $k \in \mathbb{Z}$, $\gamma(-\beta_2 - m) = 0$, $m \geq 0$, $\gamma(-\beta_2 - n + 1) = 0$, $n \leq 0$ and let $\gamma(-\beta_2 - s) \neq 0$ for all integer $n \leq s \leq m$. Then $\operatorname{Supp} L(\beta, \gamma) = \{\beta + k\delta_1 + s\delta_2 \mid k, s \in \mathbb{Z}, n \leq s \leq m\}$. Hence the support in this case is a lane of width $m - n$.

(v) Let $\gamma(-\beta_i - m_i) = 0$ for some $m_i \geq 0$, $\gamma(-\beta_i - n_i + 1) = 0$ for some $n_i \leq 0$ and let $\gamma(-\beta_i - s_i) \neq 0$ for all $n_i \leq s_i \leq m_i$, $i = 1, 2$. Then $\operatorname{Supp} L(\beta, \gamma) = \{\beta + s_1\delta_1 + s_2\delta_2 \mid s_i \in \mathbb{Z}, n_i \leq s_i \leq m_i, i = 1, 2\}$.

(vi) Let $\gamma(-\beta_i - m_i) = 0$, $m_i \geq 0$ and let $\gamma(-\beta_i - k_i) \neq 0$ for all integer $k_i \leq m_i$, $i = 1, 2$. Then $\operatorname{Supp} L(\beta, \gamma) = \{\beta + k_1\delta_1 + k_2\delta_2 \mid k_i \in \mathbb{Z}, k_i \leq m_i\}$.

(vii) Let $\gamma(-\beta_i - m_i + 1) = 0$, $m_i \leq 0$ and let $\gamma(-\beta_i - k_i) \neq 0$ for all integer $k_i \geq m_i$, $i = 1, 2$. Then $\operatorname{Supp} L(\beta, \gamma) = \{\beta + k_1\delta_1 + k_2\delta_2 \mid k_i \in \mathbb{Z}, k_i \geq m_i\}$.

(viii) Let $\gamma(-\beta_1 - m_1) = 0$, $m_1 \geq 0$, $\gamma(-\beta_2 - m_2 + 1) = 0$, $m_2 \leq 0$, $\gamma(-\beta_i - k_i) \neq 0$ for all integer $k_1 \leq m_1$ and $k_2 \geq m_2$. Then $\operatorname{Supp} L(\beta, \gamma) = \{\beta + k_1\delta_1 + k_2\delta_2 \mid k_i \in \mathbb{Z}, k_1 \leq m_1, k_2 \geq m_2\}$.

(ix) Interchange β_1 and β_2 in (ii), (iii), (iv) and (viii).

6 Indecomposable generic weight modules

A full subcategory $\mathcal{S} \subseteq K_{\beta,\gamma}$ is called a *skeleton* of $K_{\beta,\gamma}$ provided the objects of $\mathcal{S}$ are pairwise non-isomorphic and any object of $K_{\beta,\gamma}$ is isomorphic to some object of $\mathcal{S}$. In this case the categories of $K_{\beta,\gamma}$−mod and $\mathcal{S}$−mod are equivalent.

For each $i \in \{1, \dots, p\}$ consider a set $I_i = \{k \in \mathbb{Z} \mid \gamma(-\beta_i - k) = 0\}$. Define a category $S_{\beta,\gamma}$ as a $\mathbb{C}$-category with the set of objects

$$S_0 = \{0, \dots, |I_1|\} \times \dots \times \{0, \dots, |I_p|\}$$

and with morphisms generated by

$$\alpha^k_{(i_1,\dots,i_p)} : (i_1, \dots, i_p) \mapsto (i_1, \dots, i_k + 1, \dots, i_p),$$
$$\beta^k_{(j_1,\dots,j_p)} : (j_1, \dots, j_p) \mapsto (j_1, \dots, j_k - 1, \dots, j_p),$$

where $k \in \{1, \dots, p\}$ is such that $I_k \neq \emptyset$, $i_k < |I_k|$, $j_k > 0$, subject to the relations:

$$\beta^k_{(i_1,\dots,i_k+1,\dots,i_p)}\alpha^k_{(i_1,\dots,i_p)} = \alpha^k_{(i_1,\dots,i_p)}\beta^k_{(i_1,\dots,i_k+1,\dots,i_p)} = 0$$

and

$$x^k_{(a_1,\dots,a_p)}y^r_{(b_1,\dots,b_p)} = y^r_{(c_1,\dots,c_p)}x^k_{(b_1,\dots,b_p)}$$

for all $k \neq r$ and all possible $x, y \in \{\alpha, \beta\}$, $a_i, b_i, c_i, 1 \leq i \leq p$ for which this equality makes sense.

It follows from the construction that $S_{\beta,\gamma}$ is the skeleton of the category $K_{\beta,\gamma}$. Note that the corresponding algebra is finite-dimensional. In particular, $S_{\beta,\gamma}$ is semisimple when $I_k = \emptyset$ for all $1 \le k \le p$, i.e. when $\gamma(-\beta_k + r) \ne 0$ for all $k \in \mathbb{Z}$ and all $i = 1, \dots, p$. Hence it is enough to describe all indecomposable modules over $S_{\beta,\gamma}$.

Fix $a \in S_0$ and define a simple $S_{\beta,\gamma}$-module S_a such that $S_a(b) = \delta_{a,b}\mathbb{C}$ for all $b \in S_0$ and all morphisms are trivial. Since $S_{\beta,\gamma}$ defines a finite-dimensional algebra we have the following

Proposition 6.1. *Any simple module over $S_{\beta,\gamma}$ is isomorphic to S_a for some $a \in S_0$.*

This is another confirmation of the fact that all weight spaces in any irreducible generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module are 1-dimensional. But this need not to be the case for indecomposable modules. We restrict ourselves to a full subcategory $R^f_{\beta,\gamma} \subseteq R_{\beta,\gamma}$ which consists of weight modules V with $\dim V_\psi < \infty$ for all $\psi \in \operatorname{Supp} V$.

We shall employ the following definition from [9].

Definition 6.2. (i) *A category $\mathcal{C}$ is called* wild *if there exists a $\mathcal{C} \times \mathbb{C}\langle x, y\rangle$- bimodule M, free and of finite rank as right $\mathbb{C}\langle x, y\rangle$-module, and such that the functor $M \otimes_{\mathbb{C}\langle x,y\rangle} \{\cdot\}$ preserves indecomposability and isomorphism classes.*

(ii) *$\mathcal{C}$ is called* tame *if for each dimension vector d there is a finite number of $\mathcal{C} \times \mathbb{C}[x]$-bimodules $C_1, \cdots, C_n$ such that each C_j is free and of finite rank as right $\mathbb{C}[x]$-module and such that every indecomposable $X \in \mathcal{C} - \mathrm{mod}$ with $\dim X(i) = d_i$ is isomorphic to $C_j \otimes_{\mathbb{C}[x]} S$ for some j and some simple $\mathbb{C}[x]$-module S.*

We will also say that a category is of *finite* type if it has only finitely many non-isomorphic indecomposable modules.

To establish the representation type of the category $R^f_{\beta,\gamma}$ we need to find the representation type of the category $S_{\beta,\gamma}{-}\mathrm{mod}^f$ due to the category equivalence, where $S_{\beta,\gamma}{-}\mathrm{mod}^f$ denotes a category of modules over the category $S_{\beta,\gamma}$ with finite-dimensional weight spaces. Denote $X_{\beta,\gamma} = \{k \in \{1, \dots, p\} \mid I_k \ne \emptyset\}$.

6.1 Indecomposable modules in the case $|X_{\beta,\gamma}| = 1$

In this section we describe all indecomposable modules over $S_{\beta,\gamma}$ in the case $|X_{\beta,\gamma}| = 1$. Let $X_{\beta,\gamma} = \{i\}$ and let $|I_i| = r > 0$. In this case the category $S_{\beta,\gamma}$ has the following quiver $\mathbf{A}$ with relations:

$$\begin{tikzcd} \overset{1}{\circ} \arrow[r, shift left, "a_1"] & \overset{2}{\circ} \arrow[l, shift left, "b_1"] \cdots \overset{r}{\circ} \arrow[r, shift left, "a_r"] & \overset{r+1}{\circ} \arrow[l, shift left, "b_r"] \end{tikzcd} \qquad a_i\, b_i = b_i\, a_i = 0$$

We denote by S_i, $i \in \{1, \dots, r+1\}$, the simple module corresponding to the point i. These modules correspond to all irreducible modules in $R^f_{\beta,\gamma}$ by Proposition 6.1.

Now describe remaining indecomposable modules for a quiver above. Fix integers $1 \leq k_1 < k_2 \leq r+1$ and a function $\xi_{k_1,k_2} : \{k_1, k_1+1, \dots, k_2\} \to \{0,1\}$. Define a module $M = M(k_1, k_2, \xi_{k_1,k_2})$ as follows: $M(i) = \mathbb{C}e_i$, $k_1 \leq i \leq k_2$, $M(j) = 0$ otherwise, $a_i e_i = e_{i+1}$, $b_i e_{i+1} = 0$ if $\xi_{k_1,k_2}(i) = 1$ and $a_i e_i = 0$, $b_i e_{i+1} = e_i$ if $\xi_{k_1,k_2}(i) = 0$ for all $1 \leq i < k_2$.

The proof of the following proposition is standard; see e.g. [10].

Proposition 6.3. *The modules S_i, $1 \leq i \leq r+1$ and $M(k_1, k_2, \xi_{k_1,k_2})$ with $1 \leq k_1 < k_2 \leq r+1$ and*

$$\xi_{k_1,k_2} : \{k_1, k_1+1, \dots, k_2\} \to \{0,1\},$$

exhaust all non-isomorphic indecomposable modules for $\mathbf{A}$.

6.2 Indecomposable modules in the case $|X| = 2$

In this section we describe the indecomposable modules for $S_{\beta,\gamma}$ when $|X| = 2$ and $|I_k| = 1$ for each $k \in X$. Then $S_{\beta,\gamma}$ is isomorphic to the following category $\mathbf{B}$ considered in [2, Section 6.1.2]:

$$\mathbf{B}: \quad \begin{tikzcd} 1 \circ \arrow[r, shift left, "a_1"] \arrow[d, shift right, "b_0"'] & \circ\, 2 \arrow[l, shift left, "b_1"] \arrow[d, shift right, "a_2"'] \\ 0 \circ \arrow[r, shift left, "b_3"] \arrow[u, shift right, "a_0"'] & \circ\, 3 \arrow[l, shift left, "a_3"] \arrow[u, shift right, "b_2"'] \end{tikzcd} \qquad \begin{array}{l} a_i b_i = b_i a_i = 0, \qquad i = 0, \dots, 3, \\ a_i a_j = b_l b_m \quad \text{for any} \quad i, j, l, m \in \{0,1,2,3\}, \\ \text{where possible} \end{array}$$

We will consider the objects $0,1,2,3$ as the elements of $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$. We have four simple modules S_i, $0 \leq i \leq 3$, with a support in a chosen point i by Proposition 6.1.

Fix an $0 \leq i \leq 3$ and define the $\mathbf{B}$-module M_i such that $M_i(j) = \mathbb{C}e_j$ for each $j = 0, \dots, 3$ and $a_i e_i = e_{i+1}$, $a_{i+1} e_{i+1} = e_{i+2}$, $b_{i-1} e_i = e_{i-2}$, $b_{i-2} e_{i-1} = e_{i-2}$ and $u_j e_k = 0$ for all other cases of $u \in \{a, b\}$ and $j, k = 0, \dots, 3$. Obviously, M_i is indecomposable module for any i.

Next we define a discrete family of indecomposable $\mathbf{B}$-modules. Let $n \in \mathbb{N}$, $n > 1$, and $j \in \mathbb{Z}_4$. Define a $\mathbf{B}$-module $M_{n,j,1}$ (resp., $M_{n,j,2}$) as follows. Consider n elements $e_1, \dots, e_n$. A $\mathbb{C}$-basis of the vector space $M_{n,j,1}(l)$ (resp., $M_{n,j,2}(l)$) is the set of e_k such that $j + k - 1 \equiv l (mod\, 4)$. The elements a_l and b_{l-1} act as follows:

$$a_l e_k = \begin{cases} e_{k+1}, & \text{if } l \text{ is even (resp., odd), } k < n \text{ and } j + k - 1 \equiv l (mod\, 4); \\ 0, & \text{otherwise.} \end{cases}$$

$$b_{l-1} e_k = \begin{cases} e_{k-1}, & \text{if } l \text{ is even (resp., odd), } k > 1 \text{ and } j + k - 1 \equiv l (mod\, 4); \\ 0, & \text{otherwise.} \end{cases}$$

All modules $M_{n,j,1}$ and $M_{n,j,2}$, $n > 1$, $0 \leq j \leq 3$ are non-isomorphic indecomposable $\mathbf{B}$-modules.

Finally, we construct a continuous family of indecomposable $\mathbf{B}$-modules. For each $\lambda \in \mathbb{C}$ and $d \in \mathbb{Z}$, $d > 0$ define the $\mathbf{B}$-modules $M_{d,\lambda,1}$ and $M_{d,\lambda,2}$ as follows. Set

$$\begin{aligned} M_{d,\lambda,1}(i) &= \mathbb{C}^d, \\ M_{d,\lambda,1}(a_0) &= M_{d,\lambda,1}(a_2) = M_{d,\lambda,1}(b_1) = \mathbf{I}_d, \\ M_{d,\lambda,1}(b_0) &= M_{d,\lambda,1}(b_2) = M_{d,\lambda,1}(a_1) = M_{d,\lambda,1}(a_3) = 0, \\ M_{d,\lambda,1}(b_3) &= J_{d,\lambda} \end{aligned}$$

and

$$\begin{aligned} M_{d,\lambda,2}(i) &= \mathbb{C}^d, \\ M_{d,\lambda,2}(b_0) &= M_{d,\lambda,2}(b_2) = M_{d,\lambda,2}(a_1) = \mathbf{I}_d, \\ M_{d,\lambda,2}(a_0) &= M_{d,\lambda,2}(a_2) = M_{d,\lambda,2}(b_1) = M_{d,\lambda,2}(b_3) = 0, \\ M_{d,\lambda,2}(a_3) &= J_{d,\lambda}, \end{aligned}$$

where $J_{d,\lambda}$ is the Jordan cell of dimension d with the eigenvalue λ.

All modules $M_{d,\lambda,k}$, $k = 1, 2$ are indecomposable and corresponding indecomposable modules in $R^f_{\beta,\gamma}$ have all weight spaces of dimension d.

The next result follows from Proposition 3.3.1 in [1] (cf. [2], Proposition 6).

Proposition 6.4. *The modules S_i, M_i, $M_{n,i,1}$, $M_{n,i,2}$, $M_{d,\lambda,1}$, $M_{d,\lambda,2}$ where $0 \leq i \leq 3$, d is a positive integer, $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and $n \geq 2$ is an integer, constitute an exhaustive list of pairwise non-isomorphic indecomposable $\mathbf{B}$-modules.*

6.3 Representation type

The main result in this section is the following theorem which describes the representation type of $R^f_{\beta,\gamma}$.

Theorem 6.5. *(i) If $|X_{\beta,\gamma}| = 0$ then $R^f_{\beta,\gamma}$ is a semisimple category with a unique indecomposable (=irreducible) module;*

(ii) If $|X_{\beta,\gamma}| = 1$ then $R^f_{\beta,\gamma}$ has finite representation type;

(iii) If $|X_{\beta,\gamma}| = 2$ then $R^f_{\beta,\gamma}$ has tame representation type if and only if $|I_k| = 1$ for all $k \in X$. Otherwise, $R^f_{\beta,\gamma}$ has wild representation type;

(iv) If $|X_{\beta,\gamma}| > 2$ then $R^f_{\beta,\gamma}$ has wild representation type.

Proof. In the case when $|X_{\beta,\gamma}| = 1$ all indecomposable modules for $S_{\beta,\gamma}$ are described in Proposition 6.3. Hence $R^f_{\beta,\gamma}$ has finite representation type. If $|X_{\beta,\gamma}| = 2$ and $|I_k| = 1$ for each $k \in X$ then all indecomposable modules for $S_{\beta,\gamma}$ are described in Proposition 6.4. It follows from the definition that $R^f_{\beta,\gamma}$ has tame representation type in this case. If $|I_k| > 1$ for at least one k then it is easy to construct a family of indecomposable modules that depends on two continuous parameters. Hence, in this case $R^f_{\beta,\gamma}$ has wild representation type. Suppose now that $|X_{\beta,\gamma}| > 2$. Then $S_{\beta,\gamma}$ contains a full subcategory of wild representation type considered in [1], Theorem 1. We immediately conclude that $R^f_{\beta,\gamma}$ has wild representation type. This completes the proof. $\square$

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