K^F-INVARIANTS IN IRREDUCIBLE REPRESENTATIONS OF G^F, WHEN $G = GL_n$

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Abstract. Using a general result of Lusztig, we give explicit formulas for the dimensions of K^F-invariants in irreducible representations of G^F, when $G = GL_n$, $F : G \to G$ is a Frobenius map, and K is an F-stable subgroup of finite index in G^θ for some involution $\theta : G \to G$ commuting with F. The proofs use some combinatorial facts about characters of symmetric groups.

Introduction

Let G be a connected reductive group defined over a finite field $\mathbb{F}_q$ of odd characteristic. Let F be the Frobenius morphism on G, whose fixed-point subgroup $G^F = G(\mathbb{F}_q)$ is finite. Let $\theta : G \to G$ be an involution of algebraic groups commuting with F, and K an F-stable subgroup of the fixed-point subgroup G^θ which contains $(G^\theta)^\circ$. The homogeneous variety G/K is a *symmetric space*, and the set of cosets G^F/K^F might reasonably be called a *finite symmetric space*.

The classification of irreducible representations of G^F (in characteristic 0) was completed by Lusztig in the mid-1980s (see [7] for a general statement, and [9] and [10] for more details). A roughly analogous problem for symmetric spaces is that of determining the dimension of the K^F-invariants in each irreducible representation of G^F; equivalently, calculating the multiplicities $\langle \chi, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle$ for every irreducible character χ of G^F. A large first step towards solving this problem was Lusztig's calculation, in [11], of $\langle \mathrm{tr}(-, R_T^\lambda), \mathrm{Ind}_{K^F}^{G^F}(1)\rangle$ for every Deligne-Lusztig virtual representation R_T^λ of G^F. In [12], Lusztig proceeded to solve the problem completely in the case when $G^F = (G')^{F^2}$, $K^F = (G')^F$, and G' has connected centre.

This paper is devoted to the solution of this problem when G is a general linear group (with either split or non-split F, so that G^F is either $GL_n(\mathbb{F}_q)$ or $U_n(\mathbb{F}_{q^2})$) and θ is arbitrary. (The solution for the case $G^F = GL_n(\mathbb{F}_q)$, $K^F = Sp_n(\mathbb{F}_q)$ was found by Bannai, Kawanaka, and Song in [1, §4].) For such G, the functions $\mathrm{tr}(-, R_T^\lambda)$ form a basis of the class functions, and the transition matrix from this basis to that of the irreducible characters has a simple form. So calculating $\langle \chi, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle$ for $\chi \in \widehat{G^F}$ from Lusztig's result is a straightforward (though not entirely easy) matter. Two justifications for presenting it in detail are the potential interest of the answers, and the pleasantness of the symmetric group combinatorics involved.

After recalling Lusztig's formula and the character theory of the finite general linear and unitary groups in §1, we traverse the various cases in §§2-4, which could be thought of as a theme and variations: the theme, or underlying pattern, is stated in its simplest form in §2.1 (the case already known from [1]), and successive

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subsections follow the same pattern with progressively more elaborate alterations. The key combinatorial results we need along the way are all proved in §5.

To give some idea of how the formulas in §§2-4 connect with previously known results, we here extract the answers for *unipotent* irreducible characters. For both $GL_n(\mathbb{F}_q)$ and $U_n(\mathbb{F}_{q^2})$, these are parametrized by partitions of n, say $\rho \mapsto \chi^\rho \in \widehat{G^F}$. (In our convention $\chi^{(n)}$ is the trivial character and $\chi^{(1^n)}$ is the Steinberg character.) Recall that a *signed tableau of shape* ρ is a signed Young diagram of shape ρ where signs alternate across rows, modulo permutations of rows of equal length. Its *signature* is the number of boxes of each sign. For all other combinatorial notation, see §1.2. We have:

$$(n \text{ even}) \qquad \langle \chi^\rho, \mathrm{Ind}_{Sp_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1)\rangle = \begin{cases} 1, & \text{if } \rho \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

(for the general $GL_n(\mathbb{F}_q)/Sp_n(\mathbb{F}_q)$ case, see Theorem 2.1.1);

$$(n \text{ even}) \qquad \langle \chi^\rho, \mathrm{Ind}_{Sp_n(\mathbb{F}_q)}^{U_n(\mathbb{F}_{q^2})}(1)\rangle = \begin{cases} 1, & \text{if } \rho \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

(for the general $U_n(\mathbb{F}_{q^2})/Sp_n(\mathbb{F}_q)$ case, see Theorem 2.2.1);

$$\langle \chi^\rho, \mathrm{Ind}_{GL_{n^+}(\mathbb{F}_q)\times GL_{n^-}(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1)\rangle = \begin{array}{l}\text{the number of signed tableaux of shape } \rho' \\ \text{and signature } (n^+, n^-)\end{array}$$

(for the general $GL_n(\mathbb{F}_q)/(GL_{n^+}(\mathbb{F}_q) \times GL_{n^-}(\mathbb{F}_q))$ case, see Theorem 3.1.1);

$$(n \text{ even}) \qquad \langle \chi^\rho, \mathrm{Ind}_{GL_{n/2}(\mathbb{F}_{q^2})}^{GL_n(\mathbb{F}_q)}(1)\rangle = \begin{array}{l}\text{the number of signed tableaux of shape } \rho', \\ \text{stable under changing all signs}\end{array}$$
$$= \begin{cases} 1, & \text{if } \rho \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

(for the general $GL_n(\mathbb{F}_q)/GL_{n/2}(\mathbb{F}_{q^2})$ case, see Theorem 3.2.1);

$$\langle \chi^\rho, \mathrm{Ind}_{U_{n^+}(\mathbb{F}_{q^2})\times U_{n^-}(\mathbb{F}_{q^2})}^{U_n(\mathbb{F}_{q^2})}(1)\rangle = \begin{array}{l}\text{the number of signed tableaux of shape } \rho' \\ \text{and signature } (n^+, n^-), \text{ stable under} \\ \text{inverting all rows}\end{array}$$

(for the general $U_n(\mathbb{F}_{q^2})/(U_{n^+}(\mathbb{F}_{q^2}) \times U_{n^-}(\mathbb{F}_{q^2}))$ case, see Theorem 3.3.1);

$$(n \text{ even}) \qquad \langle \chi^\rho, \mathrm{Ind}_{U_{n/2}(\mathbb{F}_{q^4})}^{U_n(\mathbb{F}_{q^2})}(1)\rangle = \begin{array}{l}\text{the number of signed tableaux of shape } \rho', \\ \text{stable under changing all signs} \\ \text{and inverting all rows}\end{array}$$
$$= \begin{cases} \prod_i (m_{2i}(\rho')+1), & \text{if } 2 \mid m_{2i+1}(\rho'),\ \forall i \\ 0, & \text{otherwise} \end{cases}$$

(for the general $U_n(\mathbb{F}_{q^2})/U_{n/2}(\mathbb{F}_{q^4})$ case, see Theorem 3.4.1);

$$(n \text{ odd}) \qquad \langle \chi^\rho, \mathrm{Ind}_{O_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1)\rangle = \langle \chi^\rho, \mathrm{Ind}_{SO_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1)\rangle$$
$$= \begin{array}{l}\text{the number of signed tableaux of shape } \rho, \\ \text{modulo changing all signs}\end{array}$$
$$= \frac{1}{2}\prod_i (m_i(\rho)+1)$$

(for the general $GL_n(\mathbb{F}_q)/O_n(\mathbb{F}_q)$ and $GL_n(\mathbb{F}_q)/SO_n(\mathbb{F}_q)$ cases, see Theorem 4.1.1);

$$(n \text{ even}) \qquad \langle \chi^\rho, \mathrm{Ind}_{O_n^\epsilon(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1)\rangle = \begin{cases} \lceil \frac{1}{2}\prod_i (m_i(\rho)+1)\rceil, & \text{if } \rho' \text{ is even, } \epsilon = + \\ \lfloor \frac{1}{2}\prod_i (m_i(\rho)+1)\rfloor, & \text{if } \rho' \text{ is even, } \epsilon = - \\ \frac{1}{2}\prod_i (m_i(\rho)+1), & \text{otherwise,} \end{cases}$$

$$\langle \chi^\rho, \mathrm{Ind}_{SO_n^\epsilon(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1)\rangle = \begin{cases} \lceil \frac{1}{2}\prod_i (m_i(\rho)+1)\rceil + 1, & \text{if } \rho' \text{ is even, } \epsilon = + \\ \lfloor \frac{1}{2}\prod_i (m_i(\rho)+1)\rfloor, & \text{if } \rho' \text{ is even, } \epsilon = - \\ \frac{1}{2}\prod_i (m_i(\rho)+1), & \text{otherwise} \end{cases}$$

(for the general $GL_n(\mathbb{F}_q)/O_n^\pm(\mathbb{F}_q)$ and $GL_n(\mathbb{F}_q)/SO_n^\pm(\mathbb{F}_q)$ cases, see Theorem 4.2.1);

$$\begin{aligned} (n \text{ odd}) \qquad \langle \chi^\rho, \mathrm{Ind}_{O_n(\mathbb{F}_q)}^{U_n(\mathbb{F}_{q^2})}(1)\rangle &= \langle \chi^\rho, \mathrm{Ind}_{SO_n(\mathbb{F}_q)}^{U_n(\mathbb{F}_{q^2})}(1)\rangle \\ &= \begin{array}{l}\text{the number of signed tableaux of shape } \rho, \\ \text{stable under inverting all rows,} \\ \text{modulo changing all signs}\end{array} \\ &= \begin{cases} \frac{1}{2}\prod_i (m_{2i+1}(\rho)+1), & \text{if } 2 \mid m_{2i}(\rho),\ \forall i \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

(for the general $U_n(\mathbb{F}_{q^2})/O_n(\mathbb{F}_q)$ and $U_n(\mathbb{F}_{q^2})/SO_n(\mathbb{F}_q)$ cases, see Theorem 4.3.1);

$$(n \text{ even}) \qquad \langle \chi^\rho, \mathrm{Ind}_{O_n^\epsilon(\mathbb{F}_q)}^{U_n(\mathbb{F}_{q^2})}(1)\rangle = \begin{cases} \lceil \frac{1}{2}\prod_i (m_{2i}(\rho)+1)\rceil, & \text{if } \rho' \text{ is even, } \epsilon = + \\ \lfloor \frac{1}{2}\prod_i (m_{2i}(\rho)+1)\rfloor, & \text{if } \rho' \text{ is even, } \epsilon = - \\ \frac{1}{2}\prod_i (m_{2i}(\rho)+1), & \begin{array}{l}\text{if } 2 \mid m_{2i+1}(\rho),\ \forall i, \\ \text{but } \rho' \text{ not even}\end{array} \\ 0, & \text{otherwise} \end{cases}$$

$$\langle \chi^\rho, \mathrm{Ind}_{SO_n^\epsilon(\mathbb{F}_q)}^{U_n(\mathbb{F}_{q^2})}(1)\rangle = \begin{cases} \lceil \frac{1}{2}\prod_i (m_{2i+1}(\rho)+1)\rceil + \lceil \frac{1}{2}\prod_i (m_{2i}(\rho)+1)\rceil, & \\ \qquad \text{if } \rho' \text{ is even, } \epsilon = + & \\ \lfloor \frac{1}{2}\prod_i (m_{2i+1}(\rho)+1)\rfloor + \lfloor \frac{1}{2}\prod_i (m_{2i}(\rho)+1)\rfloor, & \\ \qquad \text{if } \rho' \text{ is even, } \epsilon = - & \\ \frac{1}{2}\prod_i (m_{2i+1}(\rho)+1), & \\ \qquad \text{if } 2 \mid m_{2i}(\rho),\ \forall i, \text{ but } \rho' \text{ not even} & \\ \frac{1}{2}\prod_i (m_{2i}(\rho)+1), & \\ \qquad \text{if } 2 \mid m_{2i+1}(\rho),\ \forall i, \text{ but } \rho' \text{ not even} & \\ 0, \quad \text{otherwise} & \end{cases}$$

(for the general $U_n(\mathbb{F}_{q^2})/O_n^{\pm}(\mathbb{F}_q)$ and $U_n(\mathbb{F}_{q^2})/SO_n^{\pm}(\mathbb{F}_q)$ cases, see Theorem 4.4.1).

Readers experienced in the representation theory of real Lie groups will find these answers familiar. When $G^F \cong GL_n(\mathbb{F}_q)$ and K is split over $\mathbb{F}_q$, the connection is as follows. The unipotent irreducible characters are the constituents of $\mathrm{Ind}_{B^F}^{G^F}(1)$ where B is an F-stable Borel subgroup, and the Hecke algebra $\mathcal{H}(G^F, B^F)$ is the specialization at q of the abstract Hecke algebra of S_n. This means that the above multiplicities $\langle \chi^\rho, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle$ can be interpreted as the multiplicity of the irreducible representation of S_n labelled by ρ in the *coherent continuation* representation associated to the appropriate real form of $GL_n(\mathbb{C})$ and the trivial infinitesimal character. For example, our answer in the case of $GL_n(\mathbb{F}_q)/(GL_{n^+}(\mathbb{F}_q) \times GL_{n^-}(\mathbb{F}_q))$ could be deduced from the results in [2], and the fact that

$$\langle \chi^\rho, \mathrm{Ind}_{O_n^{(+)}(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1)\rangle = \sum_{\substack{n^+ + n^- = n \\ n^+ \geq n^-}} \langle \chi^{\rho'}, \mathrm{Ind}_{GL_{n^+}(\mathbb{F}_q) \times GL_{n^-}(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1)\rangle$$

is, in view of the results of [15], a manifestation of Vogan duality. Similarly, our results below for non-unipotent irreducible characters connect with representations of non-trivial infinitesimal character.

However, when K is not split over $\mathbb{F}_q$, or when $G^F \cong U_n(\mathbb{F}_{q^2})$ (in which case the Hecke algebra $\mathcal{H}(G^F, B^F)$ is a specialization with unequal parameters of the abstract Hecke algebra of type $B_{\lfloor \frac{n}{2} \rfloor}$, see [8, §10]), the significance of our results for real groups is unclear. In particular, I do not know an *a priori* explanation for the fact that

$$\langle \chi^\rho, \mathrm{Ind}_{O_n^{(+)}(\mathbb{F}_q)}^{U_n(\mathbb{F}_{q^2})}(1)\rangle = \sum_{\substack{n^+ + n^- = n \\ n^+ \geq n^-}} \langle \chi^{\rho'}, \mathrm{Ind}_{U_{n^+}(\mathbb{F}_{q^2}) \times U_{n^-}(\mathbb{F}_{q^2})}^{U_n(\mathbb{F}_{q^2})}(1)\rangle.$$

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1. Review of Known Results

In this section, we recall the theorem of Lusztig which underpins all our results, and the relevant parts of the character theory of the finite general linear and unitary groups in a convenient form. In between, we introduce some combinatorial notation to be used throughout the paper.

1.1. **Lusztig's Formula.** Let k be the algebraic closure of a finite field $\mathbb{F}_q$ of odd cardinality q. Let G be a connected reductive group over k defined over $\mathbb{F}_q$, with Frobenius map $F : G \to G$. Let $\theta : G \to G$ be an involution of algebraic groups commuting with F, and K an F-stable subgroup of the fixed-point subgroup G^θ which contains $(G^\theta)^\circ$. Fix a prime l not dividing q. All representations and characters of finite groups in this paper will be over $\overline{\mathbb{Q}_l}$.

A *pair* (T, λ) means an F-stable maximal torus T and a character $\lambda : T^F \to \overline{\mathbb{Q}_l}^\times$. We have a conjugation action of G^F on the set of pairs. In [3], Deligne and Lusztig attached to each pair a virtual representation R_T^λ of G^F, depending only on the G^F-orbit of (T, λ). (In general, "most" of the R_T^λ are irreducible up to sign, and every irreducible representation occurs in some R_T^λ; when $G = GL_n$ the situation is even better, as we will see below.) The main result of [11] is a formula for

$$\frac{1}{|K^F|} \sum_{k \in K^F} \mathrm{tr}(k, R_T^\lambda) = \langle \mathrm{tr}(-, R_T^\lambda), \mathrm{Ind}_{K^F}^{G^F}(1) \rangle.$$

After some trivial adjustments, it reads as follows. Define

$$\Theta_T = \{ f \in G \mid \theta(f^{-1} T f) = f^{-1} T f \}.$$

Then T acts on Θ_T by left multiplication and K acts by right multiplication. If B is a Borel subgroup containing T, the obvious map $T\backslash\Theta_T/K \to B\backslash G/K$ is a bijection (see [11, Proposition 1.3]), so $T\backslash\Theta_T/K$ is in bijection with the set of K-orbits on the flag variety. For any $f \in \Theta_T^F$, define $\epsilon_{T,f} : (T \cap fKf^{-1})^F \to \{\pm 1\}$ by

$$\epsilon_{T,f}(t) = (-1)^{\mathbb{F}_q\text{-rank}(Z_G((T \cap fKf^{-1})^\circ)) + \mathbb{F}_q\text{-rank}(Z_G^\circ(t) \cap Z_G((T \cap fKf^{-1})^\circ))}.$$

It follows from [11, Proposition 2.3] that $\epsilon_{T,f}$ is a group homomorphism which factors through $(T \cap fKf^{-1})^F / ((T \cap fKf^{-1})^\circ)^F$. Note that $(T \cap fKf^{-1})^\circ = (T \cap fK^\circ f^{-1})^\circ$, so that $\epsilon_{T,f}$ defined for $K \neq G^\theta$ is simply the restriction of $\epsilon_{T,f}$ defined for $K = G^\theta$. Finally, define

$$\Theta_{T,K,\lambda}^F = \{ f \in \Theta_T^F \mid \lambda|_{(T \cap fKf^{-1})^F} = \epsilon_{T,f} \},$$

a union of T^F–K^F double cosets. When K is determined by θ, we write $\Theta_{T,\lambda}^F$.

Theorem 1.1.1. (Lusztig, [11, Theorem 3.3])

$$\langle \mathrm{tr}(-, R_T^\lambda), \mathrm{Ind}_{K^F}^{G^F}(1) \rangle = \sum_{f \in T^F \backslash \Theta_{T,K,\lambda}^F / K^F} (-1)^{\mathbb{F}_q\text{-rank}(T) + \mathbb{F}_q\text{-rank}(Z_G((T \cap fKf^{-1})^\circ))}.$$

1.2. **Combinatorial Notation.** In general, our combinatorial notation always follows [14]. For instance, $\lambda \vdash n$ means that λ is a partition of n. The size of a partition λ is written $|\lambda|$ and its length $\ell(\lambda)$; it has parts $\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}$. The transpose partition is λ'. We define

$$n(\lambda) = \sum_i (i-1)\lambda_i = \sum_i \binom{\lambda'_i}{2}.$$

The multiplicity of i as a part of λ is written $m_i(\lambda)$. We say that λ is *even* if all its parts are even, or equivalently if $2|m_i(\lambda'),\ \forall i$.

By abuse of notation, we will write simply ν for an (unspecified) element of the symmetric group $S_{|\nu|}$ of cycle-type ν. The sign of such an element is written ϵ_ν, and the size of its centralizer $Z_{S_{|\nu|}}(\nu)$ is z_ν. We will have much to do with involutions in $Z_{S_{|\nu|}}(\nu)$. Note that if w is such an involution, then the cycles of the element ν (i.e. the parts of the partition ν) are of three types:

(1) those fixed pointwise by w;
(2) those fixed by w, not pointwise – these must be of even length, and w must "turn the cycle half-way around"; and
(3) those moved (i.e. not fixed) by w.

If w is fixed-point free, all cycles of ν fixed by w must be of type (2). We label the irreducible characters of $S_{|\nu|}$ as $\{\chi^\lambda \,|\, \lambda \vdash |\nu|\}$ as in [14, I.7], and write χ^λ_ν for the value of χ^λ at ν, so that $\chi^{(|\nu|)}$ is the trivial character and $\chi^{\lambda'}_\nu = \epsilon_\nu \chi^\lambda_\nu$.

We define the *signature* of an involution in S_n with signed fixed points to be (n^+, n^-), where $n^+ + n^- = n$ and

$$n^+ - n^- = |\{\text{fixed points signed } +\}| - |\{\text{fixed points signed } -\}|.$$

In this context and that of signed tableaux (for which see the introduction), "signature $(p^+, p^-) + \mathbb{Z}$" means "signature $(p^+ + s, p^- + s)$ for some $s \in \mathbb{Z}$".

As in [14, Chapter IV], we will need to consider the system of maps $\widehat{\mathbb{F}^\times_{q^e}} \to \widehat{\mathbb{F}^\times_{q^{e'}}}$ for $e \,|\, e'$ (the transpose of the norm map), and its limit $L = \operatorname{colim} \widehat{\mathbb{F}^\times_{q^e}}$. Let σ denote the q-th power map on both $k^\times$ and L, so that $(k^\times)^{\sigma^e} \cong \mathbb{F}^\times_{q^e}$, $L^{\sigma^e} \cong \widehat{\mathbb{F}^\times_{q^e}}$ for all $e \geq 1$. Write $\langle \cdot, \cdot \rangle^{\sigma^e} : (k^\times)^{\sigma^e} \times L^{\sigma^e} \to \overline{\mathbb{Q}_l}^\times$ for the canonical pairing. Let ι denote the inverse map on $k^\times$ and L, and write $\tilde{\sigma}$ for $\iota\sigma$, the $(-q)$-th power map. (Note that $\tilde{\sigma}^2 = \sigma^2$.) We also have a canonical pairing $\langle \cdot, \cdot \rangle^{\tilde{\sigma}^e} : (k^\times)^{\tilde{\sigma}^e} \times L^{\tilde{\sigma}^e} \to \overline{\mathbb{Q}_l}^\times$ (the same as $\langle \cdot, \cdot \rangle^{\sigma^e}$ if e is even).

We will fix some set of representatives for the orbits of the group $\langle \sigma \rangle$ generated by σ on L, and call it $\langle \sigma \rangle \backslash L$. Similarly define $\langle \tilde{\sigma} \rangle \backslash L$. For $\xi \in \langle \sigma \rangle \backslash L$, let $m_\xi = |\langle \sigma \rangle . \xi|$, in other words the smallest $e \geq 1$ such that $\sigma^e(\xi) = \xi$. Let $d_\xi = \langle -1, \xi \rangle^{\sigma^{m_\xi}}$, which equals 1 if $L^{\sigma^{m_\xi}}$ contains square roots of ξ, and -1 if it does not. Fix once and for all an element ζ of L^σ such that $d_\zeta = -1$. Then for any $\xi \in L$,

$$m_{\zeta\xi} = m_\xi, \text{ and } d_{\zeta\xi} = (-1)^{m_\xi} d_\xi.$$

Similarly define $\tilde{m}_\xi$, $\tilde{d}_\xi$ and $\tilde{\zeta}$, using $\tilde{\sigma}$ instead of σ, so that for any $\xi \in L$,

$$\tilde{m}_{\tilde{\zeta}\xi} = \tilde{m}_\xi, \text{ and } \tilde{d}_{\tilde{\zeta}\xi} = (-1)^{\tilde{m}_\xi} \tilde{d}_\xi.$$

1.3. **Character Theory of $GL_n(\mathbb{F}_q)$ and $U_n(\mathbb{F}_{q^2})$.** For the remainder of the paper, we specialize the context of §1.1 drastically, to the case when $G \cong GL_n$, for some positive integer n. More concretely, let V be a vector space over k of dimension n and let $G = GL(V)$. There are two kinds of $\mathbb{F}_q$-structures on G, split and non-split. A Frobenius map $F : G \to G$ is split if it is induced by some Frobenius map $F_V : V \to V$, in the sense that

$$F_V(gv) = F(g)F_V(v),\ \forall g \in G, v \in V.$$

Then $G^F \cong \mathrm{Aut}_{\mathbb{F}_q}(V^{F_V}) \cong GL_n(\mathbb{F}_q)$, the finite general linear group. If F is a non-split Frobenius map, there exists some outer involution θ' of G commuting with F, and for *any* such θ', $\theta' F$ is a split Frobenius map. In this case $G^F \cong U_n(\mathbb{F}_{q^2})$, the finite unitary group.

First consider the case when $F : G \to G$ is a split Frobenius map. Let $\mathcal{P}_n$ be the set of collections of partitions $\underline{\lambda} = (\lambda_\alpha)_{\alpha \in k^\times}$, almost all zero, such that $\sum_{\alpha \in k^\times} |\lambda_\alpha| = n$. Let $\mathcal{P}_n^\sigma$ be the subset of $\mathcal{P}_n$ consisting of all $\underline{\lambda}$ such that $\lambda_{\sigma(\alpha)} = \lambda_\alpha$ for all α. It is well known that there is a natural bijection between $\mathcal{P}_n^\sigma$ and the set of conjugacy classes in G^F. Dually, let $\widehat{\mathcal{P}}_n$ be the set of collections of partitions $\underline{\nu} = (\nu_\xi)_{\xi \in L}$, almost all zero, such that $\sum_{\xi \in L} |\nu_\xi| = n$. Let $\widehat{\mathcal{P}}_n^\sigma$ be the subset of $\widehat{\mathcal{P}}_n$ of all $\underline{\nu}$ such that $\nu_{\sigma(\xi)} = \nu_\xi$ for all ξ. Note that for $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$,

$$\sum_{\xi \in \langle\sigma\rangle \backslash L} m_\xi |\nu_\xi| = n.$$

We can define a bijection between $\widehat{\mathcal{P}}_n^\sigma$ and the set of G^F-orbits of pairs (T, λ) as above, so that if (T, λ) is in the orbit corresponding to $\underline{\nu}$:

(1) the eigenlines of T can be labelled

$$\{L_{(\xi,j,i)} \,|\, \xi \in \langle\sigma\rangle \backslash L,\ 1 \le j \le \ell(\nu_\xi),\ i \in \mathbb{Z}/m_\xi(\nu_\xi)_j\mathbb{Z}\}$$

so that under the resulting isomorphism

$$T \cong \prod_{\xi \in \langle\sigma\rangle \backslash L} \prod_{j=1}^{\ell(\nu_\xi)} \underbrace{k^\times \times \cdots \times k^\times}_{m_\xi(\nu_\xi)_j \text{ factors}},$$

F on T corresponds to cyclic permutation of each group of factors $k^\times$, composed with σ;

(2) consequently,

$$T^F \cong \prod_{\xi \in \langle\sigma\rangle \backslash L} \prod_{j=1}^{\ell(\nu_\xi)} (k^\times)^{\sigma^{m_\xi(\nu_\xi)_j}};$$

(3) under this isomorphism, λ corresponds to

$$\prod_{\xi \in \langle\sigma\rangle \backslash L} \prod_{j=1}^{\ell(\nu_\xi)} \langle \cdot, \xi \rangle^{\sigma^{m_\xi(\nu_\xi)_j}}.$$

For $\underline{\nu}, \underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma$, we write $|\underline{\nu}| = |\underline{\rho}|$ to mean that $|\nu_\xi| = |\rho_\xi|$ for all ξ.

For $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$, let $B_{\underline{\nu}} = \mathrm{tr}(-, R_T^\lambda)$ for (T, λ) in the corresponding G^F-orbit. As proved by Lusztig in [6], these coincide with the *basic characters* defined by Green

in [4]. (In particular, their values are computable, but this is not relevant here.) Green's main result on the character theory of $GL_n(\mathbb{F}_q)$ states that for any $\underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma$,

$$\chi^{\underline{\rho}} := (-1)^{n+\sum_{\xi\in\langle\sigma\rangle\backslash L}|\rho_\xi|} \sum_{\substack{\underline{\nu}\in\widehat{\mathcal{P}}_n^\sigma \\ |\underline{\nu}|=|\underline{\rho}|}} \left(\prod_{\xi\in\langle\sigma\rangle\backslash L} (z_{\nu_\xi})^{-1} \chi_{\nu_\xi}^{\rho_\xi} \right) B_{\underline{\nu}}$$

is an irreducible character of G^F, and all irreducible characters arise in this way for unique $\underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma$. (See also [14, Chapter IV] and [1, Theorem 1.2.10]. Note that Macdonald's parameters in $\widehat{\mathcal{P}}_n^\sigma$ differ from those of [1] by transposing all partitions; we are following the convention of [1].) In words, the transition matrix between the basic characters and the irreducible characters is formed from the character table of various symmetric groups. Inverting this matrix, we have that for any $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$,

$$B_{\underline{\nu}} = (-1)^{n+\sum_{\xi\in\langle\sigma\rangle\backslash L}|\nu_\xi|} \sum_{\substack{\underline{\rho}\in\widehat{\mathcal{P}}_n^\sigma \\ |\underline{\rho}|=|\underline{\nu}|}} \left(\prod_{\xi\in\langle\sigma\rangle\backslash L} \chi_{\nu_\xi}^{\rho_\xi} \right) \chi^{\underline{\rho}}. \tag{1.3.1}$$

There is an obvious action of L^σ on $\widehat{\mathcal{P}}_n^\sigma$, and in particular, for $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$, $\zeta.\underline{\nu}$ is well defined. For any $\eta \in L^\sigma$, $B_{\eta.\underline{\nu}}$ and $\chi^{\eta.\underline{\rho}}$ are the result of multiplying $B_{\underline{\nu}}$ and $\chi^{\underline{\rho}}$ by the one-dimensional character $\langle \det(\cdot), \eta\rangle^\sigma$ of G^F. The *unipotent* irreducible characters referred to in the introduction are those $\chi^{\underline{\rho}}$ for which $\rho_\xi = 0$ unless $\xi = 1$. (In the introduction we parametrized these by $\rho = \rho_1$.)

The case when $F : G \to G$ is a non-split Frobenius map is less well known, but very similar, in fact mostly identical once σ is replaced by $\tilde{\sigma}$, m_ξ by $\tilde{m}_\xi$, and so on. Define $\widehat{\mathcal{P}}_n^{\tilde{\sigma}}$ in the obvious way. Again, for any $\underline{\nu} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$,

$$\sum_{\xi\in\langle\tilde{\sigma}\rangle\backslash L} \tilde{m}_\xi |\nu_\xi| = n.$$

We can define a bijection between $\widehat{\mathcal{P}}_n^{\tilde{\sigma}}$ and the set of G^F-orbits of pairs (T, λ) as above, so that if (T, λ) is in the orbit corresponding to $\underline{\nu}$:

(1) the eigenlines of T can be labelled

$$\{L_{(\xi,j,i)} \,|\, \xi \in \langle\tilde{\sigma}\rangle\backslash L,\ 1 \le j \le \ell(\nu_\xi),\ i \in \mathbb{Z}/\tilde{m}_\xi(\nu_\xi)_j\mathbb{Z}\}$$

so that under the resulting isomorphism

$$T \cong \prod_{\xi\in\langle\tilde{\sigma}\rangle\backslash L} \prod_{j=1}^{\ell(\nu_\xi)} \underbrace{k^\times \times \cdots \times k^\times}_{\tilde{m}_\xi(\nu_\xi)_j \text{ factors}},$$

F on T corresponds to cyclic permutation of each group of factors $k^\times$, composed with $\tilde{\sigma}$;

(2) consequently,

$$T^F \cong \prod_{\xi\in\langle\tilde{\sigma}\rangle\backslash L} \prod_{j=1}^{\ell(\nu_\xi)} (k^\times)^{\tilde{\sigma}^{\tilde{m}_\xi(\nu_\xi)_j}};$$

(3) under this isomorphism, λ corresponds to

$$\prod_{\xi \in \langle \tilde{\sigma} \rangle \backslash L} \prod_{j=1}^{\ell(\nu_\xi)} \langle \cdot, \xi \rangle^{\tilde{\sigma}^{\tilde{m}_\xi (\nu_\xi)_j}}.$$

For $\underline{\nu}, \underline{\rho} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$, we write $|\underline{\nu}| = |\underline{\rho}|$ to mean that $|\nu_\xi| = |\rho_\xi|$ for all ξ.

For $\underline{\nu} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$, let $B_{\underline{\nu}} = \mathrm{tr}(-, R_T^\lambda)$ for (T, λ) in the corresponding G^F-orbit. The extension of Green's result to the non-split case was proved by Lusztig and Srinivasan in [13, Theorem 3.2]: in our notation, for any $\underline{\rho} \in \widehat{\mathcal{P}}_n^{\sigma}$,

$$\chi^{\underline{\rho}} := (-1)^{\lceil \frac{n}{2} \rceil + \sum_{\xi \in \langle \tilde{\sigma} \rangle \backslash L} \tilde{m}_\xi n(\rho'_\xi) + |\rho_\xi|} \sum_{\substack{\underline{\nu} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}} \\ |\underline{\nu}| = |\underline{\rho}|}} \left(\prod_{\xi \in \langle \tilde{\sigma} \rangle \backslash L} (z_{\nu_\xi})^{-1} \chi_{\nu_\xi}^{\rho_\xi} \right) B_{\underline{\nu}}$$

is an irreducible character of G^F, and all irreducible characters arise in this way for unique $\underline{\rho} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$. Inverting, we see that for any $\underline{\nu} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$,

$$B_{\underline{\nu}} = \sum_{\substack{\underline{\rho} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}} \\ |\underline{\rho}| = |\underline{\nu}|}} (-1)^{\lceil \frac{n}{2} \rceil + \sum_{\xi \in \langle \tilde{\sigma} \rangle \backslash L} \tilde{m}_\xi n(\rho'_\xi) + |\rho_\xi|} \left(\prod_{\xi \in \langle \tilde{\sigma} \rangle \backslash L} \chi_{\nu_\xi}^{\rho_\xi} \right) \chi^{\underline{\rho}}. \tag{1.3.2}$$

Again, the obvious action of $L^{\tilde{\sigma}}$ on $\widehat{\mathcal{P}}_n^{\tilde{\sigma}}$ corresponds to multiplication by one-dimensional characters, and the *unipotent* irreducible characters of the introduction are those $\chi^{\underline{\rho}}$ for which $\rho_\xi = 0$ unless $\xi = 1$. (In contrast to the case of $GL_n(\mathbb{F}_q)$, not all the unipotent characters are constituents of $\mathrm{Ind}_{B^F}^{G^F}(1)$ for an F-stable Borel subgroup B.)

2. Cases where $G/K = GL_n/Sp_n$

In this section, we suppose that V has a nondegenerate symplectic form $\langle \cdot, \cdot \rangle$ (so in particular n is even), and that $\theta : G \to G$ is the involution defined by

$$\langle \theta(g)v, v' \rangle = \langle v, g^{-1}v' \rangle, \ \forall g \in G, v, v' \in V.$$

Since $G^\theta = Sp(V, \langle \cdot, \cdot \rangle)$ is connected, K must be equal to it. So G/K is the symmetric space GL_n/Sp_n.

Let T be any maximal torus of G. Let $L_1, \cdots, L_n$ be some ordering of the eigenlines of T, so that $T \cong GL(L_1) \times \cdots \times GL(L_n)$. This ordering gives an identification of $W(T)$ with S_n. If $f \in \Theta_T$, then $f^{-1}Tf$ is a θ-stable maximal torus, so for a unique $w_f \in W(T)$, we have

$$(f^{-1}L_i)^\perp = \bigoplus_{i' \neq w_f(i)} f^{-1}L_{i'}, \text{for all } i,$$

and

$$\theta|_{f^{-1}Tf} = \mathrm{Ad}(f^{-1}) \circ w_f \circ \mathrm{Ad}(f) \circ \iota.$$

Since θ is an involution, w_f must be also; and since $\langle \cdot, \cdot \rangle$ is a symplectic form, every line is orthogonal to itself, so that w_f is fixed-point free. Note that

$$T \cap fKf^{-1} = \{t \in T \,|\, t|_{L_{w_f(i)}} = (t|_{L_i})^{-1}, \forall i\}$$

is connected, and $Z_G(T \cap fKf^{-1}) = T$.

Clearly the association $f \mapsto w_f$ induces a map

$$T\backslash\Theta_T/K \to \{\text{fixed-point free involutions in } W(T)\}.$$

Proposition 2.0.1. *This map is a bijection.*

Proof. This is very well known, especially when translated into the language of flags via the connection mentioned in §1.1. Surjectivity can be proved by an explicit construction, and injectivity is easy by induction. □

2.1. **The $GL_n(\mathbb{F}_q)/Sp_n(\mathbb{F}_q)$ Case.** In this subsection, let $F : G \to G$ be a split Frobenius map which commutes with θ. So F is induced by a Frobenius map F_V on V which respects $\langle \cdot, \cdot \rangle$. One has $G^F \cong GL_n(\mathbb{F}_q)$, $K^F \cong Sp_n(\mathbb{F}_q)$. The following result was obtained by a different (and simpler) method in [1, §4]:

Theorem 2.1.1. *For any $\underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma$,*

$$\langle \chi^{\underline{\rho}}, \mathrm{Ind}_{Sp_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) \rangle = \begin{cases} 1, & \text{if all } \rho_\xi \text{ are even} \\ 0, & \text{otherwise.} \end{cases}$$

By the results in §1.3, it is equivalent to prove that for any $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$,

$$\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1) \rangle = \prod_{\xi \in \langle\sigma\rangle\backslash L} \sum_{\substack{\rho_\xi \vdash |\nu_\xi| \\ \rho_\xi \text{ even}}} \chi_{\nu_\xi}^{\rho_\xi}. \tag{2.1.1}$$

Note that the sign in (1.3.1) disappears since n is even, and $|\rho|$ is even for even ρ.

We aim to deduce this from Lusztig's general formula for the left-hand side (Theorem 1.1.1 above). This entails successively analysing the components of the formula in our combinatorial terms, until we are reduced to a fact about class functions on the symmetric group, whose proof we defer to §5. This basic strategy will be repeated in every case; the main reason for including the present case, where the result is not new, is that it serves as the prototype for the following subsections.

Let T be an F-stable maximal torus, and $\lambda : T^F \to \overline{\mathbb{Q}_l}^\times$ a character, for which (T, λ) is in the G^F-orbit corresponding to $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$. Label the eigenlines of T as

$$\{L_{(\xi,j,i)} \,|\, \xi \in \langle\sigma\rangle\backslash L,\ 1 \leq j \leq \ell(\nu_\xi),\ i \in \mathbb{Z}/m_\xi(\nu_\xi)_j\mathbb{Z}\}$$

as in §1.3. Thus $W(T)$ is identified with the set of permutations of these triples (ξ, j, i). Let $w_F \in W(T)$ be the permutation induced by F_V, i.e. the disjoint product of the cycles $(\xi, j, i) \mapsto (\xi, j, i+1)$.

Lemma 2.1.2. *For $f \in \Theta_T$, $w_{F(f)} = w_F w_f w_F^{-1} \in W(T)$.*

Proof. Applying F_V to the equation

$$(f^{-1}L_{(\xi,j,i)})^\perp = \bigoplus_{(\xi',j',i')\neq w_f(\xi,j,i)} f^{-1}L_{(\xi',j',i')},$$

we find that

$$(F(f)^{-1}L_{w_F(\xi,j,i)})^\perp = \bigoplus_{(\xi',j',i')\neq w_f(\xi,j,i)} F(f)^{-1}L_{w_F(\xi',j',i')}.$$

The Lemma follows. □

Corollary 2.1.3. *The map $f \mapsto w_f$ induces a bijection*

$$T^F\backslash\Theta_T^F/K^F \xrightarrow{\sim} \{\textit{fixed-point free involutions in } Z_{W(T)}(w_F)\}.$$

Proof. One need only note that since T, K, and all $T \cap fKf^{-1}$ are connected, $T^F\backslash\Theta_T^F/K^F = (T\backslash\Theta_T/K)^F$. □

Now any fixed-point free involution w in $Z_{W(T)}(w_F)$ is of the form

$$(\xi, j, i) \mapsto (\bar{w}(\xi, j), i + i(w, \xi, j))$$

where $\bar{w}$ is an involution of the set of (ξ, j) preserving the quantity $m_\xi(\nu_\xi)_j$, and $i(w, \xi, j) \in \mathbb{Z}/m_\xi(\nu_\xi)_j\mathbb{Z}$ satisfies $i(w, \bar{w}(\xi, j)) = -i(w, \xi, j)$ and $i(w, \xi, j) = \frac{1}{2}m_\xi(\nu_\xi)_j$ if $\bar{w}(\xi, j) = (\xi, j)$. Let Z' be the set of fixed-point free involutions w in $Z_{W(T)}(w_F)$ for which $\bar{w}(\xi, j) = (\xi, j')$ (for some j') and $m_\xi \,|\, i(w, \xi, j)$ hold for all (ξ, j).

Lemma 2.1.4. *For $f \in \Theta_T^F$, $f \in \Theta_{T,\lambda}^F \Leftrightarrow w_f \in Z'$.*

Proof. Since $T \cap fKf^{-1}$ is connected, $\epsilon_{T,f} = 1$, so

$$\Theta_{T,\lambda}^F = \{f \in \Theta_T^F \,|\, \lambda|_{(T\cap fKf^{-1})^F} = 1\}.$$

Now

$$\begin{aligned}(T \cap fKf^{-1})^F &= \{t \in T^F \,|\, t|_{L_{w_f(\xi,j,i)}} = (t|_{L_{(\xi,j,i)}})^{-1}\} \\ &\cong \left\{(\alpha_{(\xi,j)}) \in \prod_{(\xi,j)} (k^\times)^{\sigma^{m_\xi(\nu_\xi)_j}} \,\middle|\, \sigma^{i(w_f,\xi,j)}(\alpha_{\bar{w}_f(\xi,j)}) = \alpha_{(\xi,j)}^{-1},\ \forall \xi, j\right\} \\ &= \left\{(\beta_{(\xi,j)}) \in \prod_{\substack{(\xi,j)\\ \bar{w}_f(\xi,j)=(\xi,j)}} (k^\times)^{\iota\sigma^{\frac{1}{2}m_\xi(\nu_\xi)_j}}\right\} \\ &\quad \times \left\{(\gamma_{(\xi,j)}) \in \prod_{\substack{(\xi,j)\\ \bar{w}_f(\xi,j)\neq(\xi,j)}} (k^\times)^{\sigma^{m_\xi(\nu_\xi)_j}} \,\middle|\, \gamma_{\bar{w}_f(\xi,j)} = \sigma^{-i(w_f,\xi,j)}(\gamma_{(\xi,j)}^{-1})\right\}\end{aligned}$$

and in this form $\lambda|_{(T\cap fKf^{-1})^F}$ is the function

$$\big((\beta_{(\xi,j)}), (\gamma_{(\xi,j)})\big) \mapsto \prod_{\substack{(\xi,j)\\ \bar{w}_f(\xi,j)=(\xi,j)}} \langle\beta_{(\xi,j)}, \xi\rangle^{\sigma^{m_\xi(\nu_\xi)_j}} \prod_{\substack{(\xi,j)\\ \bar{w}_f(\xi,j)\neq(\xi,j)}} \langle\gamma_{(\xi,j)}, \xi\rangle^{\sigma^{m_\xi(\nu_\xi)_j}}.$$

Hence $\lambda|_{(T\cap fKf^{-1})^F} = 1$ if and only if:

- whenever $\bar{w}_f(\xi,j)=(\xi,j)$, we have $\sigma^{\frac{1}{2}m_\xi(\nu_\xi)_j}(\xi)=\xi$; and
- whenever $\bar{w}_f(\xi,j)=(\xi',j')\neq(\xi,j)$, we have $\xi\sigma^{-i(w_f,\xi,j)}(\xi')^{-1}=1$.

By definition of $\langle\sigma\rangle\backslash L$ and m_ξ, this is equivalent to $w_f\in Z'$. □

Corollary 2.1.5. *The map $f\mapsto w_f$ induces a bijection $T^F\backslash\Theta^F_{T,\lambda}/K^F\xrightarrow{\sim}Z'$.*

As noted above, $Z_G((T\cap fKf^{-1})^\circ)=T$, so Lusztig's formula becomes

$$\langle B_{\underline{\nu}},\mathrm{Ind}_{K^F}^{G^F}(1)\rangle=|T^F\backslash\Theta^F_{T,\lambda}/K^F|=|Z'|.$$

For any $w\in Z'$, we define a permutation $\hat{w}$ of the set of triples (ξ,j,s) with $\xi\in\langle\sigma\rangle\backslash L$, $1\leq j\leq\ell(\nu_\xi)$, and $s\in\mathbb{Z}/(\nu_\xi)_j\mathbb{Z}$ by

$$\hat{w}(\xi,j,s)=(\bar{w}(\xi,j),s+\frac{i(w,\xi,j)}{m_\xi}).$$

It is clear that $w\mapsto\hat{w}$ defines a bijection between Z' and the set of fixed-point free involutions in $\prod_{\xi\in\langle\sigma\rangle\backslash L}Z_{S_{|\nu_\xi|}}(\nu_\xi)$. Now we apply the combinatorial fact (for which see §5):

$$|\{\text{fixed-point free involutions in } Z_{S_{|\nu|}}(\nu)\}|=\sum_{\substack{\lambda\vdash|\nu|\\ \lambda\text{ even}}}\chi^\lambda_\nu. \tag{2.1.2}$$

This gives Equation (2.1.1) and hence Theorem 2.1.1.

Remark 2.1.6. For the only time in this paper, let us consider what happens if $G=SL_n$ instead of GL_n. As is well known, every irreducible character of $SL_n(\mathbb{F}_q)$ occurs with multiplicity one in the restriction of a unique irreducible character of $GL_n(\mathbb{F}_q)$. From Theorem 2.1.1 we see that if not all ρ_ξ are even, the constituents of $\mathrm{Res}^{GL_n(\mathbb{F}_q)}_{SL_n(\mathbb{F}_q)}(\chi^{\underline{\rho}})$ are all orthogonal to $\mathrm{Ind}^{SL_n(\mathbb{F}_q)}_{Sp_n(\mathbb{F}_q)}(1)$; and if all ρ_ξ are even, all but one of these constituents are orthogonal to $\mathrm{Ind}^{SL_n(\mathbb{F}_q)}_{Sp_n(\mathbb{F}_q)}(1)$, and one, say $\chi^{\underline{\rho},\star}$, has

$$\langle\chi^{\underline{\rho},\star},\mathrm{Ind}^{SL_n(\mathbb{F}_q)}_{Sp_n(\mathbb{F}_q)}(1)\rangle=1.$$

The question remains to identify $\chi^{\underline{\rho},\star}$ in terms of a parametrization of the constituents of $\mathrm{Res}^{GL_n(\mathbb{F}_q)}_{SL_n(\mathbb{F}_q)}(\chi^{\underline{\rho}})$, say the parametrization given by [10, Proposition 5.1]. It seems likely that the answer will be such that the *almost characters* of $SL_n(\mathbb{F}_q)$ have inner product with $\mathrm{Ind}^{SL_n(\mathbb{F}_q)}_{Sp_n(\mathbb{F}_q)}(1)$ which is an algebraic integer. Indeed, this may be a general property of finite reductive symmetric spaces G^F/K^F in the setting of §1.1. (It holds when $G^F=(G')^F\times(G')^F$, $K^F=(G')^F$ by the orthonormality of almost characters; it also holds when $G^F=(G')^{F^2}$, $K^F=(G')^F$ by [12, Corollary 1.6].)

2.2. **The $U_n(\mathbb{F}_{q^2})/Sp_n(\mathbb{F}_q)$ Case.** Now we keep the assumptions from before §2.1, but take F to be instead a non-split Frobenius map which commutes with θ. So θF is a split Frobenius map as above, induced by a Frobenius map F_V on V which respects $\langle\cdot,\cdot\rangle$. One has $G^F\cong U_n(\mathbb{F}_{q^2})$, $K^F\cong Sp_n(\mathbb{F}_q)$. In this case the result is:

Theorem 2.2.1. *For any $\underline{\rho}\in\widehat{\mathcal{P}}^{\tilde{\sigma}}_n$,*

$$\langle\chi^{\underline{\rho}},\mathrm{Ind}^{U_n(\mathbb{F}_{q^2})}_{Sp_n(\mathbb{F}_q)}(1)\rangle=\begin{cases}1, & \text{if all } \rho_\xi \text{ are even}\\ 0, & \text{otherwise.}\end{cases}$$

We will prove the equivalent statement that for any $\underline{\nu} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$,

$$\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle = \prod_{\xi \in \langle\tilde{\sigma}\rangle\backslash L} \sum_{\substack{\rho_\xi \vdash |\nu_\xi| \\ \rho_\xi \text{ even}}} \chi_{\nu_\xi}^{\rho_\xi}. \tag{2.2.1}$$

Note that the sign in (1.3.2) disappears because $n(\rho') \equiv \frac{|\rho|}{2} \bmod 2$ for any even ρ, so that

$$\sum_{\xi \in \langle\tilde{\sigma}\rangle\backslash L} \tilde{m}_\xi n(\rho'_\xi) \equiv \sum_{\xi \in \langle\tilde{\sigma}\rangle\backslash L} \tilde{m}_\xi \frac{|\rho_\xi|}{2} \equiv \frac{n}{2} \bmod 2.$$

Let T be an F-stable maximal torus, and $\lambda : T^F \to \overline{\mathbb{Q}}^\times$ a character, for which (T, λ) is in the G^F-orbit corresponding to $\underline{\nu} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$. Label the eigenlines of T as

$$\{L_{(\xi,j,i)} \,|\, \xi \in \langle\tilde{\sigma}\rangle\backslash L,\ 1 \le j \le \ell(\nu_\xi),\ i \in \mathbb{Z}/\tilde{m}_\xi(\nu_\xi)_j\mathbb{Z}\}$$

as in §1.3. Thus $W(T)$ is identified with the set of permutations of these triples (ξ, j, i). Let $w_F \in W(T)$ be the disjoint product of the cycles $(\xi, j, i) \mapsto (\xi, j, i+1)$. It is no longer simply the permutation induced by F_V; instead

$$F_V(L_{(\xi,j,i)}) = \bigcap_{(\xi',j',i') \neq w_F(\xi,j,i)} L^\perp_{(\xi',j',i')}.$$

Lemma 2.2.2. *For $f \in \Theta_T$, $w_{F(f)} = w_F w_f w_F^{-1} \in W(T)$.*

Proof. Applying F_V to the equation

$$(f^{-1}L_{(\xi,j,i)})^\perp = \bigoplus_{(\xi',j',i') \neq w_f(\xi,j,i)} f^{-1}L_{(\xi',j',i')},$$

we find that

$$\begin{aligned}\left(\theta(F(f))^{-1} \bigcap_{\substack{(\xi'',j'',i'') \\ \neq w_F(\xi,j,i)}} L^\perp_{(\xi'',j'',i'')}\right)^\perp &= \bigoplus_{\substack{(\xi',j',i') \\ \neq w_f(\xi,j,i)}} \theta(F(f))^{-1} \bigcap_{\substack{(\xi''',j''',i''') \\ \neq w_F(\xi',j',i')}} L^\perp_{(\xi''',j''',i''')} \\ &= \theta(F(f))^{-1} L^\perp_{w_F w_f(\xi,j,i)},\end{aligned}$$

in other words that

$$F(f)^{-1} \bigoplus_{(\xi'',j'',i'') \neq w_F(\xi,j,i)} L_{(\xi'',j'',i'')} = \left(F(f)^{-1} L_{w_F w_f(\xi,j,i)}\right)^\perp.$$

The Lemma follows. □

Corollary 2.2.3. *The map $f \mapsto w_f$ induces a bijection*

$$T^F\backslash\Theta_T^F/K^F \xrightarrow{\sim} \{\textit{fixed-point free involutions in } Z_{W(T)}(w_F)\}.$$

Proof. This is deduced exactly as in the previous subsection. □

Define $Z' \subset Z_{W(T)}(w_F)$ in the same way as before, but with $\tilde{\sigma}$ instead of σ and $\tilde{m}_\xi$ instead of m_ξ.

Lemma 2.2.4. *For $f \in \Theta_T^F$, $f \in \Theta_{T,\lambda}^F \Leftrightarrow w_f \in Z'$.*

Proof. The proof is exactly analogous to that of Lemma 2.1.4. □

Reasoning exactly as in the previous subsection (with $\tilde{\sigma}$ instead of σ and $\tilde{m}_\xi$ instead of m_ξ) we get Equation (2.2.1) and hence Theorem 2.2.1.

Remark 2.2.5. One can also give a proof of Theorem 2.2.1 along the lines of the proof of Theorem 2.1.1 in [1, §4]. Indeed, all the results of [1] have analogues for the symmetric space $U_n(\mathbb{F}_{q^2})/Sp_n(\mathbb{F}_q)$, as is explained in [5].

3. Cases where $G/K = GL_n/(GL_{n^+} \times GL_{n^-})$

In this section, we suppose that θ is an *inner* involution, namely conjugation with respect to $s \in G$ such that $s^2 = 1$. Let V^+ be the $(+1)$-eigenspace and V^- the (-1)-eigenspace of s on V, so that $V = V^+ \oplus V^-$. Let $n^+ = \dim V^+$, $n^- = \dim V^-$, so that $n = n^+ + n^-$. Since $G^\theta = GL(V^+) \times GL(V^-)$ is connected, K must be equal to it. So G/K is the symmetric space $GL_n/(GL_{n^+} \times GL_{n^-})$.

Let T be any maximal torus of G. Let $L_1, \cdots, L_n$ be some ordering of the eigenlines of T. If $f \in \Theta_T$, then $f^{-1}Tf$ is a θ-stable maximal torus, so there is some $w_f \in W(T) = S_n$ such that

$$s(f^{-1}L_i) = f^{-1}L_{w_f(i)}$$

and

$$\theta|_{f^{-1}Tf} = \mathrm{Ad}(f^{-1}) \circ w_f \circ \mathrm{Ad}(f).$$

Since θ is an involution, w_f must be also. Moreover, if $w_f(i) = i$, then $f^{-1}L_i$ is stable under s, so must lie in either V^+ or V^-. Let

$$\epsilon_f : \{\text{fixed points of } w_f\} \to \{+, -\}$$

be the resulting association of a sign to each fixed point. Note that

$$T \cap fKf^{-1} = \{t \in T \,|\, t|_{L_{w_f(i)}} = t|_{L_i}, \forall i\}$$

is connected, and

$$Z_G(T \cap fKf^{-1}) = \prod_{w_f(i)=i} GL(L_i) \times \prod_{\{w_f(i)\neq i\}} GL(L_i \oplus L_{w_f(i)}).$$

Clearly the association $f \mapsto (w_f, \epsilon_f)$ induces a map

$$T\backslash\Theta_T/K \to \{\text{involutions in } W(T) \text{ with signed fixed points, signature } (n^+, n^-)\}.$$

Proposition 3.0.1. *This map is a bijection.*

Proof. As with Proposition 2.0.1, this is well known when translated in terms of K-orbits on the flag variety, and easy to prove (see for instance [16]). □

3.1. **The $GL_n(\mathbb{F}_q)/(GL_{n^+}(\mathbb{F}_q) \times GL_{n^-}(\mathbb{F}_q))$ Case.** In this subsection, assume that $F : G \to G$ is a split Frobenius map such that $F(s) = s$. So F is induced by a Frobenius map F_V on V which stabilizes V^+ and V^-. One has $G^F \cong GL_n(\mathbb{F}_q)$, $K^F \cong GL_{n^+}(\mathbb{F}_q) \times GL_{n^-}(\mathbb{F}_q)$. The result is:

Theorem 3.1.1. *For any $\underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma$,*

$$\langle \chi^{\underline{\rho}}, \mathrm{Ind}_{GL_{n^+}(\mathbb{F}_q)\times GL_{n^-}(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1)\rangle = \begin{cases} a(n^+, n^-, \rho'_1), & \text{if } \rho_\xi = \rho_{\xi^{-1}}, \forall \xi \\ & \text{and } \rho'_{-1} \text{ is even} \\ 0, & \text{otherwise,} \end{cases}$$

where $a(p^+, p^-, \lambda')$ denotes the number (possibly zero) of signed tableaux of shape λ' and signature $(p^+, p^-) + \mathbb{Z}$.

By (1.3.1), it is equivalent to say that for any $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$,

(3.1.1)

$$\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle = \Big(\sum_{\rho_1 \vdash |\nu_1|} a(n^+, n^-, \rho_1')\chi_{\nu_1}^{\rho_1}\Big)\Big(\sum_{\substack{\rho_{-1}\vdash|\nu_{-1}|\\ \rho'_{-1}\text{ even}}} \chi_{\nu_{-1}}^{\rho_{-1}}\Big) \times \prod_{\substack{\xi\in\langle\sigma\rangle\backslash L\\ \xi^{-1}\in\langle\sigma\rangle.\xi\\ \xi\neq 1,-1}} \Big((-1)^{|\nu_\xi|}\sum_{\rho_\xi\vdash|\nu_\xi|}\chi_{\nu_\xi}^{\rho_\xi}\Big) \prod_{\substack{\{\xi_1\neq\xi_2\}\in\langle\sigma\rangle\backslash L\\ \xi_1^{-1}\in\langle\sigma\rangle.\xi_2}} \delta_{\nu_{\xi_1}\nu_{\xi_2}} z_{\nu_{\xi_1}}.$$

Here the sign $(-1)^{n+\sum_{\xi\in\langle\sigma\rangle\backslash L}|\nu_\xi|} = (-1)^{\sum_{\xi\in\langle\sigma\rangle\backslash L}(m_\xi+1)|\nu_\xi|}$ in (1.3.1) is simplified by noting that $m_1 = m_{-1} = 1$, m_ξ is even for $\xi \neq 1, -1$ such that $\xi^{-1} \in \langle\sigma\rangle.\xi$, and ν_{ξ_1} and ν_{ξ_2} must be equal whenever $\xi_1^{-1} \in \langle\sigma\rangle.\xi_2$, $\xi_1 \neq \xi_2$ in order for the right-hand side to be nonzero.

Let (T, λ) be a pair in the G^F-orbit corresponding to $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$. Define $\{L_{(\xi,j,i)}\}$ and w_F exactly as in §2.1.

Lemma 3.1.2. *For $f \in \Theta_T$, $w_{F(f)} = w_F w_f w_F^{-1}$, and $\epsilon_{F(f)} = \epsilon_f \circ w_F^{-1}$.*

Proof. Applying F_V to the equation

$$s(f^{-1}L_{(\xi,j,i)}) = f^{-1}L_{w_f(\xi,j,i)},$$

we find that

$$s(F(f)^{-1}L_{w_F(\xi,j,i)}) = F(f)^{-1}L_{w_F w_f(\xi,j,i)},$$

whence the first statement. If $f^{-1}L_{(\xi,j,i)}$ lies in V^+ or V^-, then $F_V(f^{-1}L_{(\xi,j,i)}) = F(f)^{-1}L_{w_F(\xi,j,i)}$ lies in the same one, whence the second statement. □

Corollary 3.1.3. *The map $f \mapsto (w_f, \epsilon_f)$ induces a bijection*

$$T^F\backslash\Theta_T^F/K^F \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{involutions in } Z_{W(T)}(w_F) \textit{ with signed fixed points,}\\ \textit{signs stable under } w_F\textit{, of signature } (n^+, n^-) \end{array} \right\}.$$

Proof. Again, this follows directly from Proposition 3.0.1 and the Lemma, since T, K, and all $T \cap fKf^{-1}$ are connected. □

Given an involution w in $Z_{W(T)}(w_F)$, we can define as before an involution $\bar{w}$ of the set of (ξ, j) preserving $m_\xi(\nu_\xi)_j$, and the quantities $i(w, \xi, j) \in \mathbb{Z}/m_\xi(\nu_\xi)_j\mathbb{Z}$ satisfying $i(w, \bar{w}(\xi, j)) = -i(w, \xi, j)$. (Since w is not assumed to be fixed-point free, it is possible that $\bar{w}(\xi, j) = (\xi, j)$ and $i(w, \xi, j) = 0$.) For $\xi \in \langle\sigma\rangle\backslash L$, let $\xi^\vee$ be the chosen representative in the $\langle\sigma\rangle$-orbit of ξ^{-1}. Define $i_0(\xi) \in \mathbb{Z}/m_\xi\mathbb{Z}$ by

$$\xi^\vee = \sigma^{i_0(\xi)}(\xi^{-1}).$$

Clearly $i_0(\xi^\vee) = -i_0(\xi)$. Let Z' be the set of involutions w in $Z_{W(T)}(w_F)$ such that $\bar{w}(\xi, j) = (\xi^\vee, j')$ (for some j') and $i(w, \xi, j) \equiv i_0(\xi) \bmod m_\xi$ hold for all (ξ, j), and $\bar{w}(-1, j) = (-1, j)$ implies $i(w, -1, j) = \frac{1}{2}(\nu_{-1})_j$.

Lemma 3.1.4. *For $f \in \Theta_T^F$, $f \in \Theta_{T,\lambda}^F \Leftrightarrow w_f \in Z'$.*

Proof. Since $T \cap fKf^{-1}$ is connected, $\epsilon_{T,f} = 1$, so

$$\Theta_{T,\lambda}^F = \{f \in \Theta_T^F \mid \lambda|_{(T\cap fKf^{-1})^F} = 1\}.$$

Now

$$(T \cap fKf^{-1})^F \cong \left\{ (\alpha_{(\xi,j)}) \in \prod_{(\xi,j)} (k^\times)^{\sigma^{m_\xi(\nu_\xi)_j}} \,\middle|\, \sigma^{i(w_f,\xi,j)}(\alpha_{\bar{w}_f(\xi,j)}) = \alpha_{(\xi,j)},\ \forall \xi, j \right\}$$

$$= \left\{ (\alpha_{(\xi,j)}) \in \prod_{\substack{(\xi,j) \\ \bar{w}_f(\xi,j)=(\xi,j) \\ i(w_f,\xi,j)=0}} (k^\times)^{\sigma^{m_\xi(\nu_\xi)_j}} \right\}$$

$$\times \left\{ (\beta_{(\xi,j)}) \in \prod_{\substack{(\xi,j) \\ \bar{w}_f(\xi,j)=(\xi,j) \\ i(w_f,\xi,j)=\frac{1}{2}m_\xi(\nu_\xi)_j}} (k^\times)^{\sigma^{\frac{1}{2}m_\xi(\nu_\xi)_j}} \right\}$$

$$\times \left\{ (\gamma_{(\xi,j)}) \in \prod_{\substack{(\xi,j) \\ \bar{w}_f(\xi,j)\neq(\xi,j)}} (k^\times)^{\sigma^{m_\xi(\nu_\xi)_j}} \,\middle|\, \gamma_{\bar{w}_f(\xi,j)} = \sigma^{-i(w_f,\xi,j)}(\gamma_{(\xi,j)}) \right\}$$

and in this form $\lambda|_{(T\cap fKf^{-1})^F}$ is the function

$$\big((\alpha_{(\xi,j)}),(\beta_{(\xi,j)}),(\gamma_{(\xi,j)})\big) \mapsto \prod_{\substack{(\xi,j) \\ \bar{w}_f(\xi,j)=(\xi,j) \\ i(w_f,\xi,j)=0}} \langle \alpha_{(\xi,j)}, \xi \rangle^{\sigma^{m_\xi(\nu_\xi)_j}}$$

$$\times \prod_{\substack{(\xi,j) \\ \bar{w}_f(\xi,j)=(\xi,j) \\ i(w_f,\xi,j)=\frac{1}{2}m_\xi(\nu_\xi)_j}} \langle \beta_{(\xi,j)}, \xi \rangle^{\sigma^{m_\xi(\nu_\xi)_j}}$$

$$\times \prod_{\substack{(\xi,j) \\ \bar{w}_f(\xi,j)\neq(\xi,j)}} \langle \gamma_{(\xi,j)}, \xi \rangle^{\sigma^{m_\xi(\nu_\xi)_j}}.$$

Hence $\lambda|_{(T\cap fKf^{-1})^F} = 1$ if and only if:

- whenever $\bar{w}_f(\xi,j) = (\xi,j)$ and $i(w_f,\xi,j) = 0$, we have $\xi = 1$;
- whenever $\bar{w}_f(\xi,j) = (\xi,j)$ and $i(w_f,\xi,j) = \frac{1}{2}m_\xi(\nu_\xi)_j$, $\sigma^{\frac{1}{2}m_\xi(\nu_\xi)_j}(\xi) = \xi^{-1}$; and
- whenever $\bar{w}_f(\xi,j) = (\xi',j') \neq (\xi,j)$, $\xi\sigma^{-i(w_f,\xi,j)}(\xi') = 1$.

By definition of m_ξ, $\xi^\vee$, and $i_0(\xi)$, this is equivalent to $w_f \in Z'$. □

Corollary 3.1.5. *The map $f \mapsto (w_f, \epsilon_f)$ induces a bijection*

$$T^F \backslash \Theta^F_{T,\lambda} / K^F \xrightarrow{\sim} \left\{ \begin{array}{c} w \in Z' \text{ with signed fixed points,} \\ \text{signs stable under } w_F, \text{ signature } (n^+, n^-) \end{array} \right\}.$$

Now the $\mathbb{F}_q$-rank of T is $\sum_{\xi\in\langle\sigma\rangle\backslash L}\ell(\nu_\xi) = |\{(\xi,j)\}|$, and that of

$$\begin{aligned} Z_G((T\cap fKf^{-1})^\circ) = &\prod_{\substack{(\xi,j,i)\\ \bar w_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=0}} GL(L_{(\xi,j,i)})\\ &\times \prod_{\substack{\{(\xi,j,i),(\xi,j,i+\frac12 m_\xi(\nu_\xi)_j)\}\\ \bar w_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=\frac12 m_\xi(\nu_\xi)_j}} GL(L_{(\xi,j,i)}\oplus L_{(\xi,j,i+\frac12 m_\xi(\nu_\xi)_j)})\\ &\times \prod_{\substack{\{(\xi,j,i),(\xi',j',i')\}\\ \bar w_f(\xi,j)=(\xi',j')\neq(\xi,j)\\ i'=i+i(w_f,\xi,j)}} GL(L_{(\xi,j,i)}\oplus L_{(\xi',j',i')}) \end{aligned}$$

is

$$|\{(\xi,j)\}| + |\{(\xi,j)\,|\,\bar w_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=\frac12 m_\xi(\nu_\xi)_j\}|.$$

So Lusztig's formula becomes

$$\begin{aligned} \langle B_{\underline\nu}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle &= \sum_{f\in T^F\backslash\Theta_{T,\lambda}^F/K^F} (-1)^{|\{(\xi,j)\,|\,\bar w_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=\frac12 m_\xi(\nu_\xi)_j\}|}\\ &= \sum_{\substack{w\in Z' \text{ with signed fixed points,}\\ \text{signs stable under } w_F,\text{ signature } (n^+,n^-)}} (-1)^{|\{(\xi,j)\,|\,\bar w(\xi,j)=(\xi,j), i(w,\xi,j)=\frac12 m_\xi(\nu_\xi)_j\}|}. \end{aligned}$$

Suppose $w\in Z'$. Note that if $\xi^{-1}\in\langle\sigma\rangle.\xi$, $\xi\neq 1,-1$, we have $\xi^\vee=\xi$ and $i_0(\xi)=\frac{m_\xi}{2}$. If $\bar w(\xi,j)=(\xi,j)$ for such ξ, we have both $i(w,\xi,j)\equiv\frac{m_\xi}{2} \bmod m_\xi$ and $i(w,\xi,j)=-i(w,\xi,j)$, whence $i(w,\xi,j)=\frac12 m_\xi(\nu_\xi)_j$ and $(\nu_\xi)_j$ is odd. We define a permutation $\hat w$ of the set of triples (ξ,j,s) (as in §2.1) by

$$\hat w(\xi,j,s) = (\bar w(\xi,j), s+\frac{i(w,\xi,j)-i_0(\xi,j)}{m_\xi}),$$

where we choose some preimages $i_0(\xi,j)\in\mathbb{Z}/m_\xi(\nu_\xi)_j\mathbb{Z}$ of $i_0(\xi)\in\mathbb{Z}/m_\xi\mathbb{Z}$, equal to 0 if $\xi=1$ or -1, equal to $\frac12 m_\xi(\nu_\xi)_j$ if $\xi^{-1}\in\langle\sigma\rangle.\xi$, $\xi\neq1,-1$, $\bar w(\xi,j)=(\xi,j)$, and otherwise arbitrarily subject to the constraint $i_0(\bar w(\xi,j))=-i_0(\xi,j)$. It is clear that $w\mapsto\hat w$ defines a bijection between Z' and the set of involutions $\hat w$ of the set of triples (ξ,j,s), commuting with the disjoint product of the cycles $(\xi,j,s)\mapsto(\xi,j,s+1)$, such that $\hat w(\xi,j,s)=(\xi^\vee,j',s')$ (for some j',s') holds always, $\hat w$ has no fixed points of the form $(-1,j,s)$, and $\xi^{-1}\in\langle\sigma\rangle.\xi$, $\xi\neq1,-1$, $\hat w(\xi,j,s)=(\xi,j,s')$ implies that

$(\nu_\xi)_j$ is odd. Hence

$$\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle = \sum_{\substack{\text{involutions } w_1 \in Z_{S_{|\nu_1|}}(\nu_1)\\ \text{with signed fixed points,}\\ \text{signs stable under } \nu_1,\\ \text{signature } (n^+,n^-)+\mathbb{Z}}} (-1)^{|\{\text{cycles of } \nu_1 \text{ fixed by } w_1, \text{ not pointwise}\}|}$$
$$\times \sum_{\substack{\text{fixed-point free}\\ \text{involutions } w_{-1} \in Z_{S_{|\nu_{-1}|}}(\nu_{-1})}} (-1)^{|\{\text{cycles of } \nu_{-1} \text{ fixed by } w_{-1}\}|}$$
$$\times \prod_{\substack{\xi\in\langle\sigma\rangle\backslash L\\ \xi^{-1}\in\langle\sigma\rangle.\xi\\ \xi\neq 1,-1}} \left(\sum_{\substack{\text{involutions } w_\xi \in Z_{S_{|\nu_\xi|}}(\nu_\xi)\\ \text{fixing no even cycles of } \nu_\xi}} (-1)^{|\{\text{cycles of } \nu_\xi \text{ fixed by } w_\xi\}|} \right)$$
$$\times \prod_{\substack{\{\xi_1\neq\xi_2\}\in\langle\sigma\rangle\backslash L\\ \xi_1^{-1}\in\langle\sigma\rangle.\xi_2}} \left| \left\{ \begin{array}{c} \text{bijections } \{1,\cdots,|\nu_{\xi_1}|\} \to \{1,\cdots,|\nu_{\xi_2}|\}\\ \text{intertwining } \nu_{\xi_1} \text{ and } \nu_{\xi_2} \end{array} \right\} \right|.$$

Now we apply the following combinatorial facts (see §5):

(3.1.2)
$$\sum_{\substack{\text{involutions } w \in Z_{S_{|\nu|}}(\nu)\\ \text{with signed fixed points,}\\ \text{signs stable under } \nu,\\ \text{signature } (p^+,p^-)+\mathbb{Z}}} (-1)^{|\{\text{cycles of } \nu \text{ fixed by } w, \text{ not pointwise}\}|} = \sum_{\lambda\vdash|\nu|} a(p^+,p^-,\lambda')\chi_\nu^\lambda,$$

(3.1.3)
$$\sum_{\substack{\text{fixed-point free}\\ \text{involutions } w \in Z_{S_{|\nu|}}(\nu)}} (-1)^{|\{\text{cycles of } \nu \text{ fixed by } w\}|} = \sum_{\substack{\lambda\vdash|\nu|\\ \lambda' \text{ even}}} \chi_\nu^\lambda, \text{ and}$$

(3.1.4)
$$\sum_{\substack{\text{involutions } w \in Z_{S_{|\nu|}}(\nu)\\ \text{fixing no even cycles of } \nu}} (-1)^{|\{\text{cycles of } \nu \text{ fixed by } w\}|} = (-1)^{|\nu|} \sum_{\lambda\vdash|\nu|} \chi_\nu^\lambda.$$

These give Equation (3.1.1) and hence Theorem 3.1.1.

3.2. **The $GL_n(\mathbb{F}_q)/GL_{n/2}(\mathbb{F}_{q^2})$ Case.** Now keep the assumptions from before §3.1, but take F to be a split Frobenius map such that $F(s) = -s$. (Then F still commutes with θ.) So F is induced by a Frobenius map F_V on V which interchanges V^+ and V^-, whence n is even and $n^+ = n^- = \frac{n}{2}$. One has $G^F \cong GL_n(\mathbb{F}_q)$, $K^F \cong GL_{n/2}(\mathbb{F}_{q^2})$. The result is:

Theorem 3.2.1. *For any $\underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma$,*

$$\langle \chi^{\underline{\rho}}, \mathrm{Ind}_{GL_{n/2}(\mathbb{F}_{q^2})}^{GL_n(\mathbb{F}_q)}(1)\rangle = \begin{cases} & \text{if } \rho_\xi = \rho_{\xi^{-1}}, \forall \xi,\\ 1, & \rho_1 \text{ is even,}\\ & \text{and } \rho'_{-1} \text{ is even}\\ 0, & \text{otherwise.} \end{cases}$$

By (1.3.1), it is equivalent to say that for any $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$,

(3.2.1)

$$\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle = \Big(\sum_{\substack{\rho_1 \vdash |\nu_1| \\ \rho_1 \text{ even}}} \chi_{\nu_1}^{\rho_1}\Big)\Big(\sum_{\substack{\rho_{-1} \vdash |\nu_{-1}| \\ \rho'_{-1} \text{ even}}} \chi_{\nu_{-1}}^{\rho_{-1}}\Big) \times \prod_{\substack{\xi \in \langle\sigma\rangle \backslash L \\ \xi^{-1} \in \langle\sigma\rangle.\xi \\ \xi \neq 1, -1}} \Big((-1)^{|\nu_\xi|} \sum_{\rho_\xi \vdash |\nu_\xi|} \chi_{\nu_\xi}^{\rho_\xi}\Big) \prod_{\substack{\{\xi_1 \neq \xi_2\} \in \langle\sigma\rangle \backslash L \\ \xi_1^{-1} \in \langle\sigma\rangle.\xi_2}} \delta_{\nu_{\xi_1}\nu_{\xi_2}} z_{\nu_{\xi_1}}.$$

(For the signs here, see the comments after Equation (3.1.1).)

Let (T, λ) be a pair in the G^F-orbit corresponding to $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$. Define $\{L_{(\xi,j,i)}\}$ and w_F exactly as in §2.1.

Lemma 3.2.2. *For $f \in \Theta_T$, $w_{F(f)} = w_F w_f w_F^{-1}$, and $\epsilon_{F(f)} = -\epsilon_f \circ w_F^{-1}$.*

Proof. This is an obvious modification of Lemma 3.1.2. □

Corollary 3.2.3. *The map $f \mapsto (w_f, \epsilon_f)$ induces a bijection*

$$T^F \backslash \Theta_T^F / K^F \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{involutions in } Z_{W(T)}(w_F) \textit{ with signed fixed points,} \\ \textit{signs reversed by } w_F \end{array} \right\}.$$

Now Lemma 3.1.4 holds again here, with the same Z' and exactly the same proof. So arguing as in §3.1, we get

$$\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle = \sum_{\substack{w \in Z' \text{ with signed fixed points,} \\ \text{signs reversed by } w_F}} (-1)^{|\{(\xi,j) \,|\, \bar{w}(\xi,j) = (\xi,j), i(w,\xi,j) = \frac{1}{2} m_\xi(\nu_\xi)_j\}|}.$$

The rest of the proof is also the same as in §3.1, except that Equation (3.1.2) is replaced by:

$$\sum_{\substack{\text{involutions } w \in Z_{S_{|\nu|}}(\nu) \\ \text{with signed fixed points,} \\ \text{signs reversed by } \nu}} (-1)^{|\{\text{cycles of } \nu \text{ fixed by } w, \text{ not pointwise}\}|} = \sum_{\substack{\lambda \vdash |\nu| \\ \lambda \text{ even}}} \chi_\nu^\lambda. \tag{3.2.2}$$

This too will be proved in §5.

3.3. **The $U_n(\mathbb{F}_{q^2})/(U_{n^+}(\mathbb{F}_{q^2}) \times U_{n^-}(\mathbb{F}_{q^2}))$ Case.** Still under the general assumptions of this section, let $F : G \to G$ be a non-split Frobenius map for which $F(s) = s$. Replacing s by a G^F-conjugate if necessary, we may assume that there is some non-degenerate symmetric form $\langle \cdot, \cdot \rangle$ on V, for which V^+ and V^- are orthogonal, and so that the associated outer involution $\theta' : G \to G$ commutes with F. Then $\theta' F$ is the split Frobenius map induced by some $F_V : V \to V$ which respects $\langle \cdot, \cdot \rangle$ and fixes V^+ and V^-. One has $G^F \cong U_n(\mathbb{F}_{q^2})$, $K^F \cong U_{n^+}(\mathbb{F}_{q^2}) \times U_{n^-}(\mathbb{F}_{q^2})$.

Theorem 3.3.1. *For any $\underline{\rho} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$,*

$$\langle \chi^{\underline{\rho}}, \mathrm{Ind}_{U_{n^+}(\mathbb{F}_{q^2}) \times U_{n^-}(\mathbb{F}_{q^2})}^{U_n(\mathbb{F}_{q^2})}(1)\rangle = \left\{ \begin{array}{ll} \bar{a}(n^+, n^-, \rho'_1), & \textit{if } \rho_\xi = \rho_{\xi^{-1}}, \forall \xi \\ & \textit{and } \rho'_{-1} \textit{ is even} \\ 0, & \textit{otherwise,} \end{array} \right.$$

where $\bar{a}(p^+, p^-, \lambda')$ denotes the number (possibly zero) of signed tableaux of shape λ' and signature $(p^+, p^-) + \mathbb{Z}$ which are fixed under inverting all rows.

By (1.3.2), it is equivalent to say that for any $\underline{\nu} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$,

(3.3.1)

$$\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle = \Big(\sum_{\rho_1 \vdash |\nu_1|} (-1)^{n(\rho_1)} \bar{a}(n^+, n^-, \rho_1') \chi_{\nu_1}^{\rho_1}\Big)\Big((-1)^{\frac{|\nu_{-1}|}{2}} \sum_{\substack{\rho_{-1} \vdash |\nu_{-1}| \\ \rho_{-1}' \text{ even}}} \chi_{\nu_{-1}}^{\rho_{-1}}\Big)$$
$$\times \prod_{\substack{\xi \in \langle\tilde{\sigma}\rangle\backslash L \\ \xi^{-1} \in \langle\tilde{\sigma}\rangle.\xi \\ \xi \neq 1,-1 \\ 4 | \tilde{m}_\xi}} \Big((-1)^{|\nu_\xi|} \sum_{\rho_\xi \vdash |\nu_\xi|} \chi_{\nu_\xi}^{\rho_\xi}\Big) \prod_{\substack{\xi \in \langle\tilde{\sigma}\rangle\backslash L \\ \xi^{-1} \in \langle\tilde{\sigma}\rangle.\xi \\ \xi \neq 1,-1 \\ 4 \nmid \tilde{m}_\xi}} \Big(\sum_{\rho_\xi \vdash |\nu_\xi|} \chi_{\nu_\xi}^{\rho_\xi}\Big)$$
$$\times \prod_{\substack{\{\xi_1 \neq \xi_2\} \in \langle\tilde{\sigma}\rangle\backslash L \\ \xi_1^{-1} \in \langle\tilde{\sigma}\rangle.\xi_2 \\ 2 | \tilde{m}_{\xi_1}}} \delta_{\nu_{\xi_1}\nu_{\xi_2}} z_{\nu_{\xi_1}} \prod_{\substack{\{\xi_1 \neq \xi_2\} \in \langle\tilde{\sigma}\rangle\backslash L \\ \xi_1^{-1} \in \langle\tilde{\sigma}\rangle.\xi_2 \\ 2 \nmid \tilde{m}_{\xi_1}}} (-1)^{|\nu_{\xi_1}|} \delta_{\nu_{\xi_1}\nu_{\xi_2}} z_{\nu_{\xi_1}}.$$

Here the sign $(-1)^{\lceil \frac{n}{2} \rceil + \sum_{\xi \in \langle\tilde{\sigma}\rangle\backslash L} \tilde{m}_\xi n(\rho_\xi') + |\rho_\xi|}$ of (1.3.2) is simplified as follows. If $\rho_\xi = \rho_{\xi^{-1}}$ for all ξ and ρ_{-1}' is even, then

$$n = \sum_{\xi \in \langle\tilde{\sigma}\rangle\backslash L} \tilde{m}_\xi |\rho_\xi| \equiv |\rho_1| \bmod 2,$$

and the sign can be replaced by

$$(-1)^{\lfloor \frac{|\rho_1|}{2} \rfloor + n(\rho_1')} (-1)^{\frac{|\rho_{-1}|}{2} + n(\rho_{-1}')} \prod_{\substack{\xi \in \langle\tilde{\sigma}\rangle\backslash L \\ \xi^{-1} \in \langle\tilde{\sigma}\rangle.\xi \\ \xi \neq 1,-1}} (-1)^{\frac{\tilde{m}_\xi}{2}(|\rho_\xi| + 2n(\rho_\xi')) + |\rho_\xi|}$$
$$\times \prod_{\substack{\{\xi_1 \neq \xi_2\} \in \langle\tilde{\sigma}\rangle\backslash L \\ \xi_1^{-1} \in \langle\tilde{\sigma}\rangle.\xi_2}} (-1)^{\tilde{m}_{\xi_1}(|\rho_{\xi_1}| + 2n(\rho_{\xi_1}')) + 2|\rho_{\xi_1}|}.$$

Then we observe that if ρ_{-1}' is even, $n(\rho_{-1}')$ is even; and if all even parts of ρ_1' occur with even multiplicity (which is necessary for $\bar{a}(n^+, n^-, \rho_1') \neq 0$), then the Young diagram of ρ_1' (excluding the top left corner if $|\rho_1|$ is odd) can be tiled by 2×1 dominoes, from which we see that $\lfloor \frac{|\rho_1|}{2} \rfloor + n(\rho_1') + n(\rho_1)$ is even.

Let (T, λ) be a pair in the G^F-orbit corresponding to $\underline{\nu} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$. Define $\{L_{(\xi,j,i)}\}$ and w_F exactly as in §2.2.

Lemma 3.3.2. *For $f \in \Theta_T$, $w_{F(f)} = w_F w_f w_F^{-1}$, and $\epsilon_{F(f)} = \epsilon_f \circ w_F^{-1}$.*

Proof. Applying F_V to the equation $s(f^{-1} L_{(\xi,j,i)}) = f^{-1} L_{w_f(\xi,j,i)}$, we get

$$s\left(\theta'(F(f))^{-1} \bigcap_{\substack{(\xi',j',i') \\ \neq w_F(\xi,j,i)}} L_{(\xi',j',i')}^{\perp}\right) = \theta'(F(f))^{-1} \bigcap_{\substack{(\xi'',j'',i'') \\ \neq w_F w_f(\xi,j,i)}} L_{(\xi'',j'',i'')}^{\perp}.$$

Taking orthogonals with respect to $\langle \cdot, \cdot \rangle$, this becomes

$$s\left(F(f)^{-1} \bigoplus_{\substack{(\xi',j',i') \\ \neq w_F(\xi,j,i)}} L_{(\xi',j',i')}\right) = F(f)^{-1} \bigoplus_{\substack{(\xi'',j'',i'') \\ \neq w_F w_f(\xi,j,i)}} L_{(\xi'',j'',i'')},$$

in other words

$$\bigoplus_{\substack{(\xi',j',i')\\ \neq w_F(\xi,j,i)}} F(f)^{-1}L_{w_{F(f)}(\xi',j',i')} = \bigoplus_{\substack{(\xi'',j'',i'')\\ \neq w_F w_f(\xi,j,i)}} F(f)^{-1}L_{(\xi'',j'',i'')}.$$

The first statement follows. If, say, $f^{-1}L_{(\xi,j,i)} \subseteq V^+$, then applying F_V and taking orthogonals, we find that

$$F(f)^{-1} \bigoplus_{\substack{(\xi',j',i')\\ \neq w_F(\xi,j,i)}} L_{(\xi',j',i')} \supseteq V^-,$$

whence $F(f)^{-1}L_{w_F(\xi,j,i)} \subseteq V^+$. The second statement follows. □

Corollary 3.3.3. *The map $f \mapsto (w_f, \epsilon_f)$ induces a bijection*

$$T^F\backslash\Theta_T^F/K^F \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{involutions in } Z_{W(T)}(w_F) \textit{ with signed fixed points,} \\ \textit{signs stable under } w_F\textit{, of signature } (n^+,n^-) \end{array} \right\}.$$

Define $Z' \subset Z_{W(T)}(w_F)$ in the same way as in §3.1, but with $\tilde{\sigma}$ instead of σ and $\tilde{m}_\xi$ instead of m_ξ.

Lemma 3.3.4. *For $f \in \Theta_T^F$, $f \in \Theta_{T,\lambda}^F \Leftrightarrow w_f \in Z'$.*

Proof. The proof is exactly analogous to that of Lemma 3.1.4. □

Corollary 3.3.5. *The map $f \mapsto (w_f, \epsilon_f)$ induces a bijection*

$$T^F\backslash\Theta_{T,\lambda}^F/K^F \xrightarrow{\sim} \left\{ \begin{array}{c} w \in Z' \textit{ with signed fixed points,} \\ \textit{signs stable under } w_F\textit{, signature } (n^+,n^-) \end{array} \right\}.$$

The important point of difference from §3.1 is the $\mathbb{F}_q$-ranks involved. The $\mathbb{F}_q$-rank of T is now $|\{(\xi,j)\,|\,2|\tilde{m}_\xi(\nu_\xi)_j\}|$, and that of

$$\begin{aligned} Z_G((T \cap fKf^{-1})^\circ) = &\prod_{\substack{(\xi,j,i)\\ \bar{w}_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=0}} GL(L_{(\xi,j,i)}) \\ &\times \prod_{\substack{\{(\xi,j,i),(\xi,j,i+\frac{1}{2}\tilde{m}_\xi(\nu_\xi)_j)\}\\ \bar{w}_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=\frac{1}{2}\tilde{m}_\xi(\nu_\xi)_j}} GL(L_{(\xi,j,i)} \oplus L_{(\xi,j,i+\frac{1}{2}\tilde{m}_\xi(\nu_\xi)_j)}) \\ &\times \prod_{\substack{\{(\xi,j,i),(\xi',j',i')\}\\ \bar{w}_f(\xi,j)=(\xi',j')\neq(\xi,j)\\ i'=i+i(w_f,\xi,j)}} GL(L_{(\xi,j,i)} \oplus L_{(\xi',j',i')}) \end{aligned}$$

is

$$\begin{aligned} |\{(\xi,j)\,|\,2|\tilde{m}_\xi(\nu_\xi)_j\}| + |\{(\xi,j)\,|\,\bar{w}_f(\xi,j) = (\xi,j), i(w_f,\xi,j) = \frac{1}{2}\tilde{m}_\xi(\nu_\xi)_j, 4|\tilde{m}_\xi(\nu_\xi)_j\}| \\ + |\{\{(\xi,j) \neq \bar{w}_f(\xi,j)\}\,|\,2 \nmid \tilde{m}_\xi(\nu_\xi)_j\}|. \end{aligned}$$

(The GL_2 factors corresponding to (ξ,j) such that $\bar{w}_f(\xi,j) = (\xi,j)$, $i(w_f,\xi,j) = \frac{1}{2}\tilde{m}_\xi(\nu_\xi)_j$ are split precisely when $4|\tilde{m}_\xi(\nu_\xi)_j$, and those corresponding to $\{(\xi,j) \neq$

$\bar{w}_f(\xi,j)\}$ are split precisely when $2|\tilde{m}_\xi(\nu_\xi)_j$.) So Lusztig's formula becomes

$$\begin{aligned}
&\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle \\
&= \sum_{f\in T^F\backslash\Theta_{T,\lambda}^F/K^F} (-1)^{\substack{|\{(\xi,j)\,|\,\bar{w}_f(\xi,j)=(\xi,j),i(w_f,\xi,j)=\frac{1}{2}\tilde{m}_\xi(\nu_\xi)_j,4|\tilde{m}_\xi(\nu_\xi)_j\}| \\ +|\{\{(\xi,j)\neq\bar{w}_f(\xi,j)\}\,|\,2\nmid\tilde{m}_\xi(\nu_\xi)_j\}|}} \\
&= \sum_{\substack{w\in Z' \text{ with signed fixed points,} \\ \text{signs stable under } w_F\text{, signature } (n^+,n^-)}} (-1)^{\substack{|\{(\xi,j)\,|\,\bar{w}(\xi,j)=(\xi,j),i(w,\xi,j)=\frac{1}{2}\tilde{m}_\xi(\nu_\xi)_j,4|\tilde{m}_\xi(\nu_\xi)_j\}| \\ +\frac{1}{2}|\{(\xi,j)\,|\,\bar{w}(\xi,j)\neq(\xi,j),2\nmid\tilde{m}_\xi(\nu_\xi)_j\}|}} .
\end{aligned}$$

Using the same reasoning as in §3.1, we can transform this expression to get:

$$\begin{aligned}
&\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle \\
&= \sum_{\substack{\text{involutions } w_1\in Z_{S_{|\nu_1|}}(\nu_1) \\ \text{with signed fixed points,} \\ \text{signs stable under } \nu_1, \\ \text{signature } (n^+,n^-)+\mathbb{Z}}} (-1)^{\substack{|\{\text{cycles of } \nu_1 \text{ divisible by 4, fixed by } w_1\text{, not pointwise}\}| \\ +\frac{1}{2}|\{\text{odd cycles of } \nu_1 \text{ moved by } w_1\}|}} \\
&\times \sum_{\substack{\text{fixed-point free} \\ \text{involutions } w_{-1}\in Z_{S_{|\nu_{-1}|}}(\nu_{-1})}} (-1)^{\substack{|\{\text{cycles of } \nu_{-1} \text{ divisible by 4, fixed by } w_{-1}\}| \\ +\frac{1}{2}|\{\text{odd cycles of } \nu_{-1}\}|}} \\
&\times \prod_{\substack{\xi\in\langle\tilde{\sigma}\rangle\backslash L \\ \xi^{-1}\in\langle\tilde{\sigma}\rangle.\xi \\ \xi\neq 1,-1 \\ 4|\tilde{m}_\xi}} \left(\sum_{\substack{\text{involutions } w_\xi\in Z_{S_{|\nu_\xi|}}(\nu_\xi) \\ \text{fixing no even cycles of } \nu_\xi}} (-1)^{|\{\text{cycles of } \nu_\xi \text{ fixed by } w_\xi\}|} \right) \\
&\times \prod_{\substack{\xi\in\langle\tilde{\sigma}\rangle\backslash L \\ \xi^{-1}\in\langle\tilde{\sigma}\rangle.\xi \\ \xi\neq 1,-1 \\ 4\nmid\tilde{m}_\xi}} \left| \left\{ \begin{array}{c} \text{involutions } w_\xi\in Z_{S_{|\nu_\xi|}}(\nu_\xi) \\ \text{fixing no even cycles of } \nu_\xi \end{array} \right\} \right| \\
&\times \prod_{\substack{\{\xi_1\neq\xi_2\}\in\langle\tilde{\sigma}\rangle\backslash L \\ \xi_1^{-1}\in\langle\tilde{\sigma}\rangle.\xi_2 \\ 2|\tilde{m}_{\xi_1}}} \left| \left\{ \begin{array}{c} \text{bijections } \{1,\cdots,|\nu_{\xi_1}|\}\to\{1,\cdots,|\nu_{\xi_2}|\} \\ \text{intertwining } \nu_{\xi_1} \text{ and } \nu_{\xi_2} \end{array} \right\} \right| \\
&\times \prod_{\substack{\{\xi_1\neq\xi_2\}\in\langle\tilde{\sigma}\rangle\backslash L \\ \xi_1^{-1}\in\langle\tilde{\sigma}\rangle.\xi_2 \\ 2\nmid\tilde{m}_{\xi_1}}} (-1)^{|\{\text{odd cycles of } \nu_{\xi_1}\}|}\,|\{\text{bijections } \cdots\}| .
\end{aligned}$$

So in addition to (3.1.4), we need the following combinatorial facts:

$$\sum_{\substack{\text{involutions } w \in Z_{S_{|\nu|}}(\nu) \\ \text{with signed fixed points,} \\ \text{signs stable under } \nu, \\ \text{signature } (p^+,p^-)+\mathbb{Z}}} (-1)^{\substack{|\{\text{cycles of } \nu \text{ divisible by 4, fixed by } w, \text{ not pointwise}\}| \\ +\frac{1}{2}|\{\text{odd cycles of } \nu \text{ moved by } w\}|}} \tag{3.3.2}$$
$$= \sum_{\lambda \vdash |\nu|} (-1)^{n(\lambda)} \bar{a}(p^+,p^-,\lambda')\chi^\lambda_\nu,$$

$$\sum_{\substack{\text{fixed-point free} \\ \text{involutions } w \in Z_{S_{|\nu|}}(\nu)}} (-1)^{\substack{|\{\text{cycles of } \nu \text{ divisible by 4, fixed by } w\}| \\ +\frac{1}{2}|\{\text{odd cycles of } \nu\}|}} \tag{3.3.3}$$
$$= (-1)^{\frac{|\nu|}{2}} \sum_{\substack{\lambda \vdash |\nu| \\ \lambda' \text{ even}}} \chi^\lambda_\nu, \text{ and}$$

$$\left| \left\{ \begin{array}{c} \text{involutions } w \in Z_{S_{|\nu|}}(\nu) \\ \text{fixing no even cycles of } \nu \end{array} \right\} \right| = \sum_{\lambda \vdash |\nu|} \chi^\lambda_\nu. \tag{3.3.4}$$

These will be proved in §5.

3.4. **The $U_n(\mathbb{F}_{q^2})/U_{n/2}(\mathbb{F}_{q^4})$ Case.** The final case to consider in this section is when $F : G \to G$ is a non-split Frobenius map for which $F(s) = -s$. Replacing s by a G^F-conjugate if necessary, we may assume that there is a form $\langle \cdot, \cdot \rangle$ on V and an involution $\theta' : G \to G$ with the same properties as in §3.3. Then $\theta' F$ is the split Frobenius map induced by some $F_V : V \to V$ which respects $\langle \cdot, \cdot \rangle$ and interchanges V^+ and V^-. In particular, n is even, and $n^+ = n^- = \frac{n}{2}$. One has $G^F \cong U_n(\mathbb{F}_{q^2})$, $K^F \cong U_{n/2}(\mathbb{F}_{q^4})$. The result is:

Theorem 3.4.1. *For any $\underline{\rho} \in \widehat{\mathcal{P}}^{\tilde{\sigma}}_n$,*

$$\langle \chi^{\underline{\rho}}, \mathrm{Ind}_{U_{n/2}(\mathbb{F}_{q^4})}^{U_n(\mathbb{F}_{q^2})}(1) \rangle = \begin{cases} \prod_i (m_{2i}(\rho'_1)+1), & \text{if } \rho_\xi = \rho_{\xi^{-1}}, \forall \xi, \\ & 2|m_{2i+1}(\rho'_1), \forall i, \\ & \text{and } \rho'_{-1} \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$$

By (1.3.2), it is equivalent to say that for any $\underline{\nu} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$,

(3.4.1)

$$\begin{aligned}\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle = \Big(&\sum_{\substack{\rho_1 \vdash |\nu_1| \\ 2|m_{2i+1}(\rho_1')}} (-1)^{n(\rho_1)} \Big(\prod_i (m_{2i}(\rho_1')+1)\Big) \chi_{\nu_1}^{\rho_1} \Big) \\ &\times \Big((-1)^{\frac{|\nu_{-1}|}{2}} \sum_{\substack{\rho_{-1} \vdash |\nu_{-1}| \\ \rho_{-1}' \text{ even}}} \chi_{\nu_{-1}}^{\rho_{-1}}\Big) \\ &\times \prod_{\substack{\xi \in \langle\tilde{\sigma}\rangle \backslash L \\ \xi^{-1} \in \langle\tilde{\sigma}\rangle.\xi \\ \xi \neq 1,-1 \\ 4|\tilde{m}_\xi}} \Big((-1)^{|\nu_\xi|} \sum_{\rho_\xi \vdash |\nu_\xi|} \chi_{\nu_\xi}^{\rho_\xi}\Big) \prod_{\substack{\xi \in \langle\tilde{\sigma}\rangle \backslash L \\ \xi^{-1} \in \langle\tilde{\sigma}\rangle.\xi \\ \xi \neq 1,-1 \\ 4\nmid \tilde{m}_\xi}} \Big(\sum_{\rho_\xi \vdash |\nu_\xi|} \chi_{\nu_\xi}^{\rho_\xi}\Big) \\ &\times \prod_{\substack{\{\xi_1 \neq \xi_2\} \in \langle\tilde{\sigma}\rangle \backslash L \\ \xi_1^{-1} \in \langle\tilde{\sigma}\rangle.\xi_2 \\ 2|\tilde{m}_{\xi_1}}} \delta_{\nu_{\xi_1}\nu_{\xi_2}} z_{\nu_{\xi_1}} \prod_{\substack{\{\xi_1 \neq \xi_2\} \in \langle\tilde{\sigma}\rangle \backslash L \\ \xi_1^{-1} \in \langle\tilde{\sigma}\rangle.\xi_2 \\ 2\nmid \tilde{m}_{\xi_1}}} (-1)^{|\nu_{\xi_1}|} \delta_{\nu_{\xi_1}\nu_{\xi_2}} z_{\nu_{\xi_1}}.\end{aligned}$$

For the signs here, see the comments after (3.3.1).

Let (T, λ), $\{L_{(\xi,j,i)}\}$ and w_F be as in §3.3.

Lemma 3.4.2. *For $f \in \Theta_T$, $w_{F(f)} = w_F w_f w_F^{-1}$, and $\epsilon_{F(f)} = -\epsilon_f \circ w_F^{-1}$.*

Proof. This is an obvious modification of Lemma 3.3.2. □

Corollary 3.4.3. *The map $f \mapsto (w_f, \epsilon_f)$ induces a bijection*

$$T^F \backslash \Theta_T^F / K^F \xrightarrow{\sim} \left\{ \begin{array}{c} \text{involutions in } Z_{W(T)}(w_F) \text{ with signed fixed points,} \\ \text{signs reversed by } w_F \end{array} \right\}.$$

Now Lemma 3.3.4 holds again here, with the same Z' and exactly the same proof. So arguing as in §3.3, we get

$$\begin{aligned}&\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle \\ &= \sum_{\substack{w \in Z' \text{ with signed fixed points,} \\ \text{signs reversed by } w_F}} (-1)^{\substack{|\{(\xi,j) \,|\, \bar{w}(\xi,j)=(\xi,j), i(w,\xi,j)=\frac{1}{2}\tilde{m}_\xi(\nu_\xi)_j, 4|\tilde{m}_\xi(\nu_\xi)_j\}| \\ +\frac{1}{2}|\{(\xi,j) \,|\, \bar{w}(\xi,j)\neq(\xi,j), 2\nmid \tilde{m}_\xi(\nu_\xi)_j\}|}}.\end{aligned}$$

The rest of the proof is also the same as in §3.3, except that Equation (3.3.2) is replaced by:

$$\begin{aligned}&\sum_{\substack{\text{involutions } w \in Z_{S_{|\nu|}}(\nu) \\ \text{with signed fixed points,} \\ \text{signs reversed by } \nu}} (-1)^{\substack{|\{\text{cycles of } \nu \text{ divisible by 4, fixed by } w, \text{ not pointwise}\}| \\ +\frac{1}{2}|\{\text{odd cycles of } \nu \text{ moved by } w\}|}} \\ &\qquad = \sum_{\substack{\lambda \vdash |\nu| \\ 2|m_{2i+1}(\lambda')}} (-1)^{n(\lambda)} \Big(\prod_i (m_{2i}(\lambda')+1)\Big) \chi_\nu^\lambda.\end{aligned} \tag{3.4.2}$$

This will be proved in §5.

4. Cases where $G/K = GL_n/O_n$ or GL_n/SO_n

In this section, we suppose that V has a nondegenerate symmetric form $\langle \cdot, \cdot \rangle$, and that $\theta : G \to G$ is the involution defined by

$$\langle \theta(g)v, v' \rangle = \langle v, g^{-1}v' \rangle, \ \forall g \in G, v, v' \in V.$$

Since $G^\theta = O(V, \langle \cdot, \cdot \rangle)$ has two components, there are two possibilities for K. In order to treat them simultaneously, we will write K for G^θ and K° for $(G^\theta)^\circ = SO(V, \langle \cdot, \cdot \rangle)$. So $G/K \cong GL_n/O_n$ and $G/K^\circ \cong GL_n/SO_n$.

Let T be a maximal torus of G with eigenlines $L_1, \dots, L_n$. If $f \in \Theta_T$, then $f^{-1}Tf$ is a θ-stable maximal torus, so there is some $w_f \in W(T) = S_n$ such that

$$(f^{-1}L_i)^\perp = \bigoplus_{i' \neq w_f(i)} f^{-1}L_{i'}, \text{for all } i,$$

and

$$\theta|_{f^{-1}Tf} = \mathrm{Ad}(f^{-1}) \circ w_f \circ \mathrm{Ad}(f) \circ \iota.$$

Since θ is an involution, w_f must be also; in contrast to §2, w_f need not be fixed-point free, so

$$T \cap fKf^{-1} = \{t \in T \,|\, t|_{L_i} = \pm 1 \text{ if } w_f(i) = i,\ t|_{L_{w_f(i)}} = (t|_{L_i})^{-1} \text{ if } w_f(i) \neq i\}$$

is not necessarily connected.

Clearly the association $f \mapsto w_f$ induces a map

$$T \backslash \Theta_T / K \to \{\text{involutions in } W(T)\}.$$

Proposition 4.0.1. *This map is a bijection. Moreover, if $w \in W(T)$ is an involution, the corresponding T–K double coset breaks into two T–K° double cosets if w is fixed-point free, and is a single T–K° double coset otherwise.*

Proof. As with Propositions 2.0.1 and 3.0.1, this is better known as a statement about K-orbits on the flag variety (see [15, §6]). It is easy to prove. □

4.1. **The $GL_n(\mathbb{F}_q)/O_n(\mathbb{F}_q)$ and $GL_n(\mathbb{F}_q)/SO_n(\mathbb{F}_q)$ Cases (n odd).** In this subsection, suppose that n is odd and let $F : G \to G$ be a split Frobenius map which commutes with θ. So F is induced by a Frobenius map F_V on V which respects $\langle \cdot, \cdot \rangle$, such that $\langle \cdot, \cdot \rangle$ has Witt index $\lfloor \frac{n}{2} \rfloor$ on V^{F_V}. One has $G^F \cong GL_n(\mathbb{F}_q)$, $K^F \cong O_n(\mathbb{F}_q)$, $(K^\circ)^F \cong SO_n(\mathbb{F}_q)$. The result is:

Theorem 4.1.1. *For any $\underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma$,*

$$\langle \chi^{\underline{\rho}}, \mathrm{Ind}_{O_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) \rangle = \begin{cases} \frac{1}{2} \displaystyle\prod_{\substack{\xi \in \langle \sigma \rangle \backslash L \\ d_\xi = 1}} \Big(\prod_i (m_i(\rho_\xi) + 1) \Big), & \text{if } d_\xi = -1 \Rightarrow \rho'_\xi \text{ is even} \\ 0, & \text{otherwise, and} \end{cases}$$

$$\langle \chi^{\underline{\rho}}, \mathrm{Ind}_{SO_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) \rangle = \langle \chi^{\underline{\rho}}, \mathrm{Ind}_{O_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) \rangle + \langle \chi^{\zeta \cdot \underline{\rho}}, \mathrm{Ind}_{O_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) \rangle.$$

By (1.3.1), it is equivalent to say that for any $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$,

$$\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle = -\frac{1}{2} \prod_{\substack{\xi \in \langle\sigma\rangle\backslash L \\ d_\xi = 1}} ((-1)^{|\nu_\xi|} \sum_{\rho_\xi \vdash |\nu_\xi|} (\prod_i (m_i(\rho_\xi)+1)) \chi_{\nu_\xi}^{\rho_\xi})$$

$$\times \prod_{\substack{\xi \in \langle\sigma\rangle\backslash L \\ d_\xi = -1}} (\sum_{\substack{\rho_\xi \vdash |\nu_\xi| \\ \rho'_\xi \text{ even}}} \chi_{\nu_\xi}^{\rho_\xi}), \text{ and} \tag{4.1.1}$$

$$\langle B_{\underline{\nu}}, \mathrm{Ind}_{(K^\circ)^F}^{G^F}(1)\rangle = \langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle + \langle B_{\zeta.\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle.$$

Here the sign in (1.3.1) has been distributed in an obvious way.

Let (T, λ) be a pair in the G^F-orbit corresponding to $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$. Define $\{L_{(\xi,j,i)}\}$ and w_F exactly as in §2.1.

Lemma 4.1.2. *For $f \in \Theta_T$, $w_{F(f)} = w_F w_f w_F^{-1}$.*

Proof. The proof is identical to that of Lemma 2.1.2. □

Corollary 4.1.3. *The map $f \mapsto w_f$ induces a surjection*

$$T^F\backslash\Theta_T^F/K^F \twoheadrightarrow \{\textit{involutions in } Z_{W(T)}(w_F)\}.$$

Proof. By Proposition 4.0.1 and Lemma 4.1.2, we may identify the set of involutions in $Z_{W(T)}(w_F)$ with $(T\backslash\Theta_T/K)^F$. Under this identification, an involution is in the image of $T^F\backslash\Theta_T^F/K^F$ precisely when the corresponding F-stable K-orbit on $T\backslash\Theta_T$ contains an F-fixed point (by connectedness of T). Since n is odd, this K-orbit is a single K°-orbit, so this is automatic. □

For w an involution in $Z_{W(T)}(w_F)$, we define $\bar{w}$ and $i(w, \xi, j)$ as in §2.1 and §3.1.

Lemma 4.1.4. *For $f \in \Theta_T^F$, the number of T^F–K^F double cosets in TfK is*

$$2^{|\{(\xi,j)\,|\,\bar{w}_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=0\}|-1},$$

and each T^F–K^F double coset is a single T^F–$(K^\circ)^F$ double coset.

Proof. Since the image of the Lang map on K is K°, the number of T^F–K^F double cosets in TfK is the same as the number of orbits of $T \cap fKf^{-1}$ on $T \cap fK^\circ f^{-1}$ for the action $t.t' = tt'F(t)^{-1}$. Since

$$T \cap fKf^{-1} = \left\{ (\alpha_{(\xi,j,i)}) \in \prod_{\substack{(\xi,j,i) \\ w_f(\xi,j,i)=(\xi,j,i)}} \pm 1 \right\}$$

$$\times \left\{ (\beta_{(\xi,j,i)}) \in \prod_{\substack{(\xi,j,i) \\ w_f(\xi,j,i)\neq(\xi,j,i)}} k^\times \;\Big|\; \beta_{w_f(\xi,j,i)} = \beta_{(\xi,j,i)}^{-1} \right\},$$

this is the same as the number of orbits of

$$\left\{ (\epsilon_{(\xi,j,i)}) \in \prod_{\substack{(\xi,j) \\ \bar{w}_f(\xi,j)=(\xi,j) \\ i(w_f,\xi,j)=0}} (\pm 1)^{m_\xi(\nu_\xi)_j} \right\}$$

on

$$\left\{ (\epsilon'_{(\xi,j,i)}) \in \prod_{\substack{(\xi,j)\\ \bar{w}_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=0}} (\pm 1)^{m_\xi(\nu_\xi)_j} \;\middle|\; \prod_{(\xi,j,i)} \epsilon'_{(\xi,j,i)} = 1 \right\}$$

for the action $(\epsilon_{(\xi,j,i)}).(\epsilon'_{(\xi,j,i)}) = (\epsilon_{(\xi,j,i)})(\epsilon_{(\xi,j,i-1)})(\epsilon'_{(\xi,j,i)})$. Since n is odd, there must be some (ξ, j) for which $\bar{w}_f(\xi, j) = (\xi, j)$, $i(w_f, \xi, j) = 0$, $2 \nmid m_\xi(\nu_\xi)_j$. The first statement follows. Modifying the above analysis for K°, we get the second statement. □

Now let Z' be the set of involutions w in $Z_{W(T)}(w_F)$ for which $\bar{w}(\xi, j) = (\xi, j')$ (for some j') and $m_\xi \,|\, i(w, \xi, j)$ hold for all (ξ, j), and $\bar{w}(\xi, j) = (\xi, j)$, $i(w, \xi, j) = 0$, $d_\xi = -1$ implies $2 \,|\, (\nu_\xi)_j$. Define Z'' in the same way but with the last condition changed to: $\bar{w}(\xi, j) = (\xi, j)$, $i(w, \xi, j) = 0$, $d_{\zeta\xi} = -1$ implies $2 \,|\, (\nu_\xi)_j$.

Lemma 4.1.5. *For $f \in \Theta_T^F$, $f \in \Theta_{T,K,\lambda}^F \Leftrightarrow w_f \in Z'$, and $f \in \Theta_{T,K^\circ,\lambda}^F \Leftrightarrow w_f \in Z' \cup Z''$.*

Proof. We first note that $\epsilon_{T,f} = 1$, as may be seen directly (using formulas for $\mathbb{F}_q$-rank such as those below) or deduced by the method of [11, Lemma 11.3]. So

$$\begin{aligned}\Theta_{T,K,\lambda}^F &= \{f \in \Theta_T^F \,|\, \lambda|_{(T\cap fKf^{-1})^F} = 1\}\\ &\subseteq \Theta_{T,K^\circ,\lambda}^F = \{f \in \Theta_T^F \,|\, \lambda|_{(T\cap fK^\circ f^{-1})^F} = 1\}.\end{aligned}$$

Now

$$\begin{aligned}(T \cap fKf^{-1})^F &\cong \left\{ (\alpha_{(\xi,j)}) \in \prod_{(\xi,j)} (k^\times)^{\sigma^{m_\xi(\nu_\xi)_j}} \;\middle|\; \sigma^{i(w_f,\xi,j)}(\alpha_{\bar{w}_f(\xi,j)}) = \alpha_{(\xi,j)}^{-1},\ \forall \xi, j \right\}\\ &= \left\{ (\alpha_{(\xi,j)}) \in \prod_{\substack{(\xi,j)\\ \bar{w}_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=0}} (\pm 1) \right\}\\ &\quad \times \left\{ (\beta_{(\xi,j)}) \in \prod_{\substack{(\xi,j)\\ \bar{w}_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=\frac{1}{2}m_\xi(\nu_\xi)_j}} (k^\times)^{\iota\sigma^{\frac{1}{2}m_\xi(\nu_\xi)_j}} \right\}\\ &\quad \times \left\{ (\gamma_{(\xi,j)}) \in \prod_{\substack{(\xi,j)\\ \bar{w}_f(\xi,j)\neq(\xi,j)}} (k^\times)^{\sigma^{m_\xi(\nu_\xi)_j}} \;\middle|\; \gamma_{\bar{w}_f(\xi,j)} = \sigma^{-i(w_f,\xi,j)}(\gamma_{(\xi,j)}^{-1}) \right\}\end{aligned}$$

and in this form $\lambda|_{(T\cap fKf^{-1})^F}$ is the function

$$\big((\alpha_{(\xi,j)}),(\beta_{(\xi,j)}),(\gamma_{(\xi,j)})\big)\mapsto \prod_{\substack{(\xi,j)\\ \bar{w}_f(\xi,j)=(\xi,j)\\ i(w,\xi,j)=0}}\langle\alpha_{(\xi,j)},\xi\rangle^{\sigma^{m_\xi(\nu_\xi)_j}} \times \prod_{\substack{(\xi,j)\\ \bar{w}_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=\frac{1}{2}m_\xi(\nu_\xi)_j}}\langle\beta_{(\xi,j)},\xi\rangle^{\sigma^{m_\xi(\nu_\xi)_j}} \times \prod_{\substack{(\xi,j)\\ \bar{w}_f(\xi,j)\neq(\xi,j)}}\langle\gamma_{(\xi,j)},\xi\rangle^{\sigma^{m_\xi(\nu_\xi)_j}}.$$

Hence $\lambda|_{(T\cap fKf^{-1})^F}=1$ if and only if:

- whenever $\bar{w}_f(\xi,j)=(\xi,j)$ and $i(w_f,\xi,j)=0$, we have $\langle -1,\xi\rangle^{\sigma^{m_\xi(\nu_\xi)_j}}=1$;
- whenever $\bar{w}_f(\xi,j)=(\xi,j)$ and $i(w_f,\xi,j)=\frac{1}{2}m_\xi(\nu_\xi)_j$, $\sigma^{\frac{1}{2}m_\xi(\nu_\xi)_j}(\xi)=\xi$; and
- whenever $\bar{w}_f(\xi,j)=(\xi',j')\neq(\xi,j)$, $\xi\sigma^{-i(w_f,\xi,j)}(\xi')^{-1}=1$.

By definition of m_ξ and d_ξ, this is equivalent to $w_f\in Z'$. $(T\cap fK^\circ f^{-1})^F$ is the same as $(T\cap fKf^{-1})^F$ except that the condition

$$\prod_{(\xi,j)}\alpha_{(\xi,j)}^{m_\xi(\nu_\xi)_j}=1$$

is imposed. So the second statement follows similarly, using the fact that

$$d_{\zeta\xi}=-1\Leftrightarrow d_\xi=-(-1)^{m_\xi}.$$

□

Corollary 4.1.6. *The map $f\mapsto w_f$ induces surjections $T^F\backslash\Theta^F_{T,K,\lambda}/K^F\twoheadrightarrow Z'$ and $T^F\backslash\Theta^F_{T,K^\circ,\lambda}/(K^\circ)^F\twoheadrightarrow Z'\cup Z''$.*

Now the $\mathbb{F}_q$-rank of T is $|\{(\xi,j)\}|$, and that of

$$Z_G((T\cap fKf^{-1})^\circ)=Z_G((T\cap fK^\circ f^{-1})^\circ)$$
$$=GL\left(\bigoplus_{\substack{(\xi,j,i)\\ \bar{w}_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=0}}L_{(\xi,j,i)}\right)\times\prod_{\substack{(\xi,j,i)\\ \bar{w}_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=\frac{1}{2}m_\xi(\nu_\xi)_j}}GL(L_{(\xi,j,i)})\times\prod_{\substack{(\xi,j,i)\\ \bar{w}_f(\xi,j)\neq(\xi,j)}}GL(L_{(\xi,j,i)})$$

is

$$|\{(\xi,j)\}|+\sum_{\substack{(\xi,j,i)\\ \bar{w}_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=0}}(m_\xi(\nu_\xi)_j-1)$$
$$\equiv|\{(\xi,j)\}|+n+|\{(\xi,j)\,|\,\bar{w}_f(\xi,j)=(\xi,j),i(w_f,\xi,j)=0\}|\bmod 2.$$

So Lusztig's formula gives

$$\begin{aligned}\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle &= -\sum_{f\in T^F\backslash\Theta_{T,K,\lambda}^F/K^F} (-1)^{|\{(\xi,j)\,|\,\bar{w}_f(\xi,j)=(\xi,j),i(w_f,\xi,j)=0\}|}\\ &= -\sum_{w\in Z'}(-1)^{|\{(\xi,j)\,|\,\bar{w}(\xi,j)=(\xi,j),i(w,\xi,j)=0\}|}2^{|\{(\xi,j)\,|\,\bar{w}(\xi,j)=(\xi,j),i(w,\xi,j)=0\}|-1}\\ &= -\frac{1}{2}\sum_{w\in Z'}(-2)^{|\{(\xi,j)\,|\,\bar{w}(\xi,j)=(\xi,j),i(w,\xi,j)=0\}|},\ \text{and}\\ \langle B_{\underline{\nu}}, \mathrm{Ind}_{(K^\circ)^F}^{G^F}(1)\rangle &= -\frac{1}{2}\sum_{w\in Z'}(-2)^{|\{(\xi,j)\,|\,\bar{w}(\xi,j)=(\xi,j),i(w,\xi,j)=0\}|}\\ &\qquad -\frac{1}{2}\sum_{w\in Z''}(-2)^{|\{(\xi,j)\,|\,\bar{w}(\xi,j)=(\xi,j),i(w,\xi,j)=0\}|},\end{aligned}$$

since the fact that n is odd implies that Z' and Z'' are disjoint.

For any $w \in Z'\cup Z''$, we define a permutation $\hat{w}$ as in §2.1. It is clear that $w \mapsto \hat{w}$ defines bijections between Z' and the set of involutions $\hat{w}$ in $\prod_{\xi\in\langle\sigma\rangle\backslash L} Z_{S_{|\nu_\xi|}}(\nu_\xi)$ such that $\hat{w}(\xi,j,s) = (\xi,j,s)$ and $d_\xi = -1$ implies $2\,|\,(\nu_\xi)_j$, and between Z'' and the set of involutions $\hat{w}$ such that $\hat{w}(\xi,j,s) = (\xi,j,s)$ and $d_{\zeta\xi} = -1$ implies $2\,|\,(\nu_\xi)_j$. Hence

$$\begin{aligned}&\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle\\ &= -\frac{1}{2}\prod_{\xi\in\langle\sigma\rangle\backslash L}\left(\sum_{\substack{\text{involutions } w_\xi\in Z_{S_{|\nu_\xi|}}(\nu_\xi),\\ \text{fixing no odd cycles of } \nu_\xi \text{ if } d_\xi=-1}}(-2)^{|\{\text{cycles of } \nu_\xi \text{ fixed by } w_\xi \text{ pointwise}\}|}\right)\end{aligned}$$

and

$$\langle B_{\underline{\nu}}, \mathrm{Ind}_{(K^\circ)^F}^{G^F}(1)\rangle = \langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle + \langle B_{\zeta.\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle.$$

So to prove (4.1.1), the combinatorial facts we need are:

$$\begin{aligned}&\sum_{\text{involutions } w\in Z_{S_{|\nu|}}(\nu)}(-2)^{|\{\text{cycles of } \nu \text{ fixed by } w \text{ pointwise}\}|}\\ &\qquad = (-1)^{|\nu|}\sum_{\lambda\vdash|\nu|}(\prod_i(m_i(\lambda)+1))\chi_\nu^\lambda,\ \text{and}\end{aligned}\tag{4.1.2}$$

$$\sum_{\substack{\text{involutions } w\in Z_{S_{|\nu|}}(\nu)\\ \text{fixing no odd cycles of } \nu}}(-2)^{|\{\text{cycles of } \nu \text{ fixed by } w \text{ pointwise}\}|} = \sum_{\substack{\lambda\vdash|\nu|\\ \lambda' \text{ even}}}\chi_\nu^\lambda.\tag{4.1.3}$$

These will be proved in §5.

4.2. **The $GL_n(\mathbb{F}_q)/O_n^\pm(\mathbb{F}_q)$ and $GL_n(\mathbb{F}_q)/SO_n^\pm(\mathbb{F}_q)$ Cases (n even).** In this subsection, suppose that n is even and $F : G \to G$ is a split Frobenius map which commutes with θ. So F is induced by a Frobenius map F_V on V which respects $\langle\cdot,\cdot\rangle$, and once again $G^F \cong GL_n(\mathbb{F}_q)$. The Witt index of $\langle\cdot,\cdot\rangle$ on V^{F_V} is either $\frac{n}{2}$ or $\frac{n}{2}-1$, and accordingly either $K^F \cong O_n^+(\mathbb{F}_q)$, $(K^\circ)^F \cong SO_n^+(\mathbb{F}_q)$ or $K^F \cong O_n^-(\mathbb{F}_q)$, $(K^\circ)^F \cong SO_n^-(\mathbb{F}_q)$. Let ϵ be 1 in the first case and -1 in the second case.

Theorem 4.2.1. *For any* $\underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma$,

$$\langle \chi^{\underline{\rho}}, \mathrm{Ind}_{O_n^\epsilon(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1)\rangle = \begin{cases} \frac{1}{2}\prod\limits_{\substack{\xi\in\langle\sigma\rangle\backslash L\\ d_\xi=1}}(\prod\limits_i(m_i(\rho_\xi)+1)), & \text{if } d_\xi=-1\Rightarrow \rho'_\xi \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$
$$+ \begin{cases} \frac{1}{2}\epsilon, & \text{if all } \rho'_\xi \text{ are even} \\ 0, & \text{otherwise, and} \end{cases}$$
$$\langle \chi^{\underline{\rho}}, \mathrm{Ind}_{SO_n^\epsilon(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1)\rangle = \langle \chi^{\underline{\rho}}, \mathrm{Ind}_{O_n^\epsilon(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1)\rangle + \langle \chi^{\zeta\cdot\underline{\rho}}, \mathrm{Ind}_{O_n^\epsilon(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1)\rangle.$$

By (1.3.1), it is equivalent to say that for any $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$,

$$\begin{aligned}\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle = \frac{1}{2}\prod_{\substack{\xi\in\langle\sigma\rangle\backslash L\\ d_\xi=1}}((-1)^{|\nu_\xi|}\sum_{\rho_\xi\vdash|\nu_\xi|}(\prod_i(m_i(\rho_\xi)+1))\chi_{\nu_\xi}^{\rho_\xi}) \\ \times\prod_{\substack{\xi\in\langle\sigma\rangle\backslash L\\ d_\xi=-1}}(\sum_{\substack{\rho_\xi\vdash|\nu_\xi|\\ \rho'_\xi \text{ even}}}\chi_{\nu_\xi}^{\rho_\xi}) \\ + \frac{1}{2}\epsilon\prod_{\xi\in\langle\sigma\rangle\backslash L}(\sum_{\substack{\rho_\xi\vdash|\nu_\xi|\\ \rho'_\xi \text{ even}}}\chi_{\nu_\xi}^{\rho_\xi}), \text{ and}\end{aligned} \tag{4.2.1}$$

$$\langle B_{\underline{\nu}}, \mathrm{Ind}_{(K^\circ)^F}^{G^F}(1)\rangle = \langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle + \langle B_{\zeta\cdot\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle.$$

Let (T,λ) be a pair in the G^F-orbit corresponding to $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$. Define $\{L_{(\xi,j,i)}\}$ and w_F exactly as in §2.1.

Lemma 4.2.2. *For* $f \in \Theta_T$, $w_{F(f)} = w_F w_f w_F^{-1}$.

Proof. As in the previous subsection, the proof of Lemma 2.1.2 works. □

Corollary 4.2.3. *The map* $f \mapsto w_f$ *induces a map*

$$T^F\backslash\Theta_T^F/K^F \to \{\text{involutions in } Z_{W(T)}(w_F)\}.$$

If $\mathrm{sign}(w_F)=\epsilon$, *this map is surjective. If* $\mathrm{sign}(w_F)=-\epsilon$, *the image consists of all involutions with at least one fixed point.*

Proof. As in the proof of Corollary 4.1.3, an involution is in the image of $T^F\backslash\Theta_T^F/K^F$ precisely when the corresponding F-stable K-orbit on $T\backslash\Theta_T$ contains an F-fixed point. If the involution has a fixed point, this K-orbit is a single K°-orbit, so this is automatic. Suppose the involution $w \in Z_{W(T)}(w_F)$ is fixed-point free. It is in the image of $T^F\backslash\Theta_T^F/K^F$ precisely when there exists a decomposition of V into lines $\{L'_i \mid 1\le i\le n\}$ such that

(1) $(L'_i)^\perp = \oplus_{i'\neq w(i)} L'_{i'}$, and
(2) $F_V(L'_i) = L'_{w_F(i)}$

(here we have identified $W(T)$ with S_n in some way). We must prove that this happens if and only if $\mathrm{sign}(w_F)=\epsilon$. Since ϵ is multiplicative with respect to F_V-stable orthogonal direct sums, we may assume that $\langle w\rangle\times\langle w_F\rangle$ acts transitively on $\{1,\cdots,n\}$. So if $\mathrm{sign}(w_F)=1$, then $\langle w_F\rangle$ has two orbits on $\{1,\cdots,n\}$ which w interchanges; in this case the existence of $\{L'_i\}$ as above is clearly equivalent to the existence of a decomposition $V = V_1\oplus V_2$ into F_V-stable Lagrangian subspaces,

which indeed happens if and only if $\epsilon = 1$. On the other hand, if $\mathrm{sign}(w_F) = -1$, then w_F is an n-cycle and $w = w_F^{n/2}$; in this case the existence of $\{L_i'\}$ as above is easily seen to be equivalent to the existence of a Lagrangian subspace V_1 of V such that $\dim V_1 \cap F_V(V_1) = \frac{n}{2} - 1$, which indeed happens if and only if $\epsilon = -1$. □

For w an involution in $Z_{W(T)}(w_F)$, we define $\bar{w}$ and $i(w, \xi, j)$ as usual.

Lemma 4.2.4. *For $f \in \Theta_T^F$, the number of T^F–K^F double cosets in TfK is*

$$\begin{cases} 1, & \text{if } \bar{w}_f(\xi, j) = (\xi, j),\ i(w_f, \xi, j) = 0 \\ & \text{never occurs} \\ 2^{|\{(\xi,j)\,|\,\bar{w}_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=0\}|-1}, & \text{otherwise,} \end{cases}$$

and the number of T^F–$(K^\circ)^F$ double cosets in TfK° is

$$\begin{cases} 2^{|\{(\xi,j)\,|\,\bar{w}_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=0\}|}, & \text{if } \bar{w}_f(\xi, j) = (\xi, j),\ i(w_f, \xi, j) = 0 \\ & \Rightarrow 2 \,|\, m_\xi(\nu_\xi)_j \\ 2^{|\{(\xi,j)\,|\,\bar{w}_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=0\}|-1}, & \text{otherwise.} \end{cases}$$

Proof. The method of proof of Lemma 4.1.4 applies again here. □

Now define Z' and Z'' in exactly the same way as in §4.1.

Lemma 4.2.5. *For $f \in \Theta_T^F$, $f \in \Theta_{T,K,\lambda}^F \Leftrightarrow w_f \in Z'$, and $f \in \Theta_{T,K^\circ,\lambda}^F \Leftrightarrow w_f \in Z' \cup Z''$.*

Proof. The proof is exactly the same as that of Lemma 4.1.5. □

Corollary 4.2.6. (1) *The map $f \mapsto w_f$ induces maps $T^F \backslash \Theta_{T,K,\lambda}^F / K^F \to Z'$ and $T^F \backslash \Theta_{T,K^\circ,\lambda}^F / (K^\circ)^F \to Z' \cup Z''$. If $\mathrm{sign}(w_F) = \epsilon$, these maps are surjective; if $\mathrm{sign}(w_F) = -\epsilon$, their images are $Z' \setminus Z'''$ and $(Z' \cup Z'') \setminus Z'''$, where Z''' is the set of fixed-point free involutions in Z'.*

(2) *If w is in the image of $T^F \backslash \Theta_{T,K,\lambda}^F / K^F \to Z'$, there are*

$$\begin{cases} 1, & \text{if } w \in Z''' \\ 2^{|\{(\xi,j)\,|\,\bar{w}(\xi,j)=(\xi,j), i(w,\xi,j)=0\}|-1}, & \text{otherwise} \end{cases}$$

T^F–K^F double cosets in the preimage of w.

(3) *If w is in the image of $T^F \backslash \Theta_{T,K^\circ,\lambda}^F / (K^\circ)^F \to Z' \cup Z''$, there are*

$$\begin{cases} 2, & \text{if } w \in Z''' \\ 2^{|\{(\xi,j)\,|\,\bar{w}(\xi,j)=(\xi,j), i(w,\xi,j)=0\}|}, & \text{if } w \in (Z' \cap Z'') \setminus Z''' \\ 2^{|\{(\xi,j)\,|\,\bar{w}(\xi,j)=(\xi,j), i(w,\xi,j)=0\}|-1}, & \text{otherwise} \end{cases}$$

T^F–$(K^\circ)^F$ double cosets in the preimage of w.

Proof. This follows by combining Proposition 4.0.1, Corollary 4.2.3, Lemma 4.2.4 and Lemma 4.1.5. Note that $Z' \cap Z''$ is precisely the set of $w \in Z' \cup Z''$ such that $\bar{w}(\xi, j) = (\xi, j)$, $i(w, \xi, j) = 0 \Rightarrow 2 \,|\, m_\xi(\nu_\xi)_j$. □

Now as in §4.1, the $\mathbb{F}_q$-rank of T is $|\{(\xi, j)\}|$, and that of $Z_G((T \cap fKf^{-1})^\circ) = Z_G((T \cap fK^\circ f^{-1})^\circ)$ is congruent to

$$|\{(\xi, j)\}| + n + |\{(\xi, j)\,|\,\bar{w}_f(\xi, j) = (\xi, j), i(w_f, \xi, j) = 0\}| \bmod 2.$$

So Lusztig's formula gives

$$\begin{aligned}
\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle &= \sum_{f \in T^F \backslash \Theta_{T,K,\lambda}^F / K^F} (-1)^{|\{(\xi,j)\,|\,\bar{w}_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=0\}|} \\
&= \sum_{w \in Z'} (-1)^{|\{(\xi,j)\,|\,\bar{w}(\xi,j)=(\xi,j), i(w,\xi,j)=0\}|} \\
&\qquad \times \begin{cases} 1, & \text{if } w \in Z''' \\ 2^{|\{(\xi,j)\,|\,\bar{w}(\xi,j)=(\xi,j), i(w,\xi,j)=0\}|-1}, & \text{otherwise} \end{cases} \\
&\qquad - \frac{1}{2}(1-\mathrm{sign}(w_F)\epsilon)|Z'''| \\
&= \frac{1}{2} \sum_{w \in Z'} (-2)^{|\{(\xi,j)\,|\,\bar{w}(\xi,j)=(\xi,j), i(w,\xi,j)=0\}|} \\
&\qquad + \frac{1}{2}(-1)^{|\{(\xi,j)\,|\,2|m_\xi(\nu_\xi)j\}|}\epsilon|Z'''|, \text{ and} \\
\langle B_{\underline{\nu}}, \mathrm{Ind}_{(K^\circ)^F}^{G^F}(1)\rangle &= \sum_{w \in Z' \cup Z''} (-1)^{|\{(\xi,j)\,|\,\bar{w}(\xi,j)=(\xi,j), i(w,\xi,j)=0\}|} \\
&\qquad \times \begin{cases} 2, & \text{if } w \in Z''' \\ 2^{|\{(\xi,j)\,|\,\bar{w}(\xi,j)=(\xi,j), i(w,\xi,j)=0\}|}, & \text{if } w \in (Z' \cap Z'') \setminus Z''' \\ 2^{|\{(\xi,j)\,|\,\bar{w}(\xi,j)=(\xi,j), i(w,\xi,j)=0\}|-1}, & \text{otherwise} \end{cases} \\
&\qquad - \frac{1}{2}(1-\mathrm{sign}(w_F)\epsilon)2|Z'''| \\
&= \frac{1}{2} \sum_{w \in Z'} (-2)^{|\{(\xi,j)\,|\,\bar{w}(\xi,j)=(\xi,j), i(w,\xi,j)=0\}|} \\
&\qquad + \frac{1}{2} \sum_{w \in Z''} (-2)^{|\{(\xi,j)\,|\,\bar{w}(\xi,j)=(\xi,j), i(w,\xi,j)=0\}|} \\
&\qquad\qquad + (-1)^{|\{(\xi,j)\,|\,2|m_\xi(\nu_\xi)j\}|}\epsilon|Z'''|.
\end{aligned}$$

We transform these expressions as in the previous subsection to obtain:

$$\begin{aligned}
&\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle \\
&= \frac{1}{2} \prod_{\xi \in \langle\sigma\rangle \backslash L} \left(\sum_{\substack{\text{involutions } w_\xi \in Z_{S_{|\nu_\xi|}}(\nu_\xi), \\ \text{fixing no odd cycles of } \nu_\xi \text{ if } d_\xi = -1}} (-2)^{|\{\text{cycles of } \nu_\xi \text{ fixed by } w_\xi \text{ pointwise}\}|} \right) \\
&\quad + \frac{1}{2}\epsilon \prod_{\xi \in \langle\sigma\rangle \backslash L} \epsilon_{\nu_\xi} \left| \left\{ \begin{array}{c} \text{fixed-point free} \\ \text{involutions in } Z_{S_{|\nu_\xi|}}(\nu_\xi) \end{array} \right\} \right|,
\end{aligned}$$

and

$$\langle B_{\underline{\nu}}, \mathrm{Ind}_{(K^\circ)^F}^{G^F}(1)\rangle = \langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle + \langle B_{\zeta.\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle.$$

Here we have used the fact that if there exists a fixed-point free involution in $Z_{S_{|\nu_\xi|}}(\nu_\xi)$, then $|\{\text{odd cycles of } \nu_\xi\}|$ is even, so

$$(-1)^{|\{\text{cycles of } \nu_\xi\}|} = (-1)^{|\{\text{even cycles of } \nu_\xi\}|} = \epsilon_{\nu_\xi}.$$

So along with (4.1.2) and (4.1.3), we need (2.1.2) multiplied on both sides by ϵ_ν.

4.3. **The $U_n(\mathbb{F}_{q^2})/O_n(\mathbb{F}_q)$ and $U_n(\mathbb{F}_{q^2})/SO_n(\mathbb{F}_q)$ Cases (n odd).** In this subsection, suppose that n is odd and take $F : G \to G$ to be a non-split Frobenius map commuting with θ. Thus θF is a split Frobenius map induced by F_V as in §4.1. One has $G^F \cong U_n(\mathbb{F}_{q^2})$, $K^F \cong O_n(\mathbb{F}_q)$, $(K^\circ)^F \cong SO_n(\mathbb{F}_q)$. The result is:

Theorem 4.3.1. *For any $\underline{\rho} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$,*

$$\langle \chi^{\underline{\rho}}, \mathrm{Ind}_{O_n(\mathbb{F}_q)}^{U_n(\mathbb{F}_{q^2})}(1)\rangle = \begin{cases} \frac{1}{2}\prod\limits_{\substack{\xi\in\langle\tilde{\sigma}\rangle\backslash L\\ \tilde{d}_\xi=1\\ 2\nmid\tilde{m}_\xi}}(\prod\limits_i(m_{2i+1}(\rho_\xi)+1))\prod\limits_{\substack{\xi\in\langle\tilde{\sigma}\rangle\backslash L\\ \tilde{d}_\xi=-1\\ 2\nmid\tilde{m}_\xi}}(\prod\limits_i(m_{2i}(\rho_\xi)+1)) \\ \qquad\times\prod\limits_{\substack{\xi\in\langle\tilde{\sigma}\rangle\backslash L\\ \tilde{d}_\xi=1\\ 2|\tilde{m}_\xi}}(\prod\limits_i(m_i(\rho_\xi)+1)), \\ \qquad \text{if } \tilde{d}_\xi=1,\ 2\nmid\tilde{m}_\xi \Rightarrow 2|m_{2i}(\rho_\xi),\ \forall i, \\ \qquad \tilde{d}_\xi=-1,\ 2\nmid\tilde{m}_\xi \Rightarrow 2|m_{2i+1}(\rho_\xi),\ \forall i, \\ \qquad \text{and } \tilde{d}_\xi=-1,\ 2|\tilde{m}_\xi \Rightarrow \rho'_\xi \text{ is even} \\ 0, \quad \text{otherwise, and} \end{cases}$$

$$\langle \chi^{\underline{\rho}}, \mathrm{Ind}_{SO_n(\mathbb{F}_q)}^{U_n(\mathbb{F}_{q^2})}(1)\rangle = \langle \chi^{\underline{\rho}}, \mathrm{Ind}_{O_n(\mathbb{F}_q)}^{U_n(\mathbb{F}_{q^2})}(1)\rangle + \langle \chi^{\tilde{\zeta}\cdot\underline{\rho}}, \mathrm{Ind}_{O_n(\mathbb{F}_q)}^{U_n(\mathbb{F}_{q^2})}(1)\rangle.$$

By (1.3.2), it is equivalent to say that for any $\underline{\nu} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$,

(4.3.1)

$$\begin{aligned}\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle = \frac{1}{2}(-1)^{\lfloor\frac{n}{2}\rfloor}&\prod_{\substack{\xi\in\langle\tilde{\sigma}\rangle\backslash L\\ \tilde{d}_\xi=1\\ 2\nmid\tilde{m}_\xi}}\left(\sum_{\substack{\rho_\xi\vdash|\nu_\xi|\\ 2|m_{2i}(\rho_\xi)}}(-1)^{n(\rho'_\xi)}(\prod_i(m_{2i+1}(\rho_\xi)+1))\chi_{\nu_\xi}^{\rho_\xi}\right)\\ &\times\prod_{\substack{\xi\in\langle\tilde{\sigma}\rangle\backslash L\\ \tilde{d}_\xi=-1\\ 2\nmid\tilde{m}_\xi}}\left(\sum_{\substack{\rho_\xi\vdash|\nu_\xi|\\ 2|m_{2i+1}(\rho_\xi)}}(-1)^{n(\rho'_\xi)}(\prod_i(m_{2i}(\rho_\xi)+1))\chi_{\nu_\xi}^{\rho_\xi}\right)\\ &\times\prod_{\substack{\xi\in\langle\tilde{\sigma}\rangle\backslash L\\ \tilde{d}_\xi=1\\ 2|\tilde{m}_\xi}}(-1)^{|\nu_\xi|}\left(\sum_{\rho_\xi\vdash|\nu_\xi|}(\prod_i(m_i(\rho_\xi)+1))\chi_{\nu_\xi}^{\rho_\xi}\right)\\ &\times\prod_{\substack{\xi\in\langle\tilde{\sigma}\rangle\backslash L\\ \tilde{d}_\xi=-1\\ 2|\tilde{m}_\xi}}\left(\sum_{\substack{\rho_\xi\vdash|\nu_\xi|\\ \rho'_\xi \text{ even}}}\chi_{\nu_\xi}^{\rho_\xi}\right), \text{ and}\end{aligned}$$

$$\langle B_{\underline{\nu}}, \mathrm{Ind}_{(K^\circ)^F}^{G^F}(1)\rangle = \langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle + \langle B_{\tilde{\zeta}\cdot\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle.$$

Here the sign

$$(-1)^{\lceil \frac{n}{2} \rceil + \sum_{\xi \in \langle \tilde{\sigma} \rangle \backslash L} \tilde{m}_\xi n(\rho'_\xi) + |\rho_\xi|} = (-1)^{\lfloor \frac{n}{2} \rfloor + \sum_{\xi \in \langle \tilde{\sigma} \rangle \backslash L} \tilde{m}_\xi (n(\rho'_\xi) + |\rho_\xi|) + |\rho_\xi|}$$

of (1.3.2) has been distributed in an obvious way.

The proof of these statements is mostly very similar to that of (4.1.1). Let (T, λ) be a pair in the G^F-orbit corresponding to $\underline{\nu} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$. Define $\{L_{(\xi,j,i)}\}$ and w_F exactly as in §2.2.

Lemma 4.3.2. *For $f \in \Theta_T$, $w_{F(f)} = w_F w_f w_F^{-1}$.*

Proof. The proof is identical to that of Lemma 2.2.2. □

Corollary 4.3.3. *The map $f \mapsto w_f$ induces a surjection*

$$T^F \backslash \Theta_T^F / K^F \twoheadrightarrow \{\textit{involutions in } Z_{W(T)}(w_F)\}.$$

Proof. This is deduced in the same way as Corollary 4.1.3. □

For w an involution in $Z_{W(T)}(w_F)$, define $\bar{w}$ and $i(w, \xi, j)$ exactly as in §2.1, but with $\tilde{\sigma}$ instead of σ, $\tilde{m}_\xi$ instead of m_ξ.

Lemma 4.3.4. *For $f \in \Theta_T^F$, the number of T^F–K^F double cosets in TfK is*

$$2^{|\{(\xi,j) \,|\, \bar{w}_f(\xi,j) = (\xi,j), i(w_f,\xi,j) = 0\}| - 1},$$

and each T^F–K^F double coset is a single T^F–$(K^\circ)^F$ double coset.

Proof. The proof is the same as that of 4.1.4. □

Now define Z' and Z'' as in §4.1, but with $\tilde{\sigma}$ instead of σ, $\tilde{m}_\xi$ instead of m_ξ, and $\tilde{d}_\xi$ instead of d_ξ.

Lemma 4.3.5. *For $f \in \Theta_T^F$, $f \in \Theta_{T,K,\lambda}^F \Leftrightarrow w_f \in Z'$, and $f \in \Theta_{T,K^\circ,\lambda}^F \Leftrightarrow w_f \in Z' \cup Z''$.*

Proof. We have

$$(T \cap fKf^{-1})^F \cong \left\{ (\alpha_{(\xi,j)}) \in \prod_{\substack{(\xi,j) \\ \bar{w}_f(\xi,j) = (\xi,j) \\ i(w_f,\xi,j) = 0}} (\pm 1) \right\}$$

$$\times \left\{ (\beta_{(\xi,j)}) \in \prod_{\substack{(\xi,j) \\ \bar{w}_f(\xi,j) = (\xi,j) \\ i(w_f,\xi,j) = \frac{1}{2}\tilde{m}_\xi (\nu_\xi)_j}} (k^\times)^{\iota \tilde{\sigma}^{\frac{1}{2}\tilde{m}_\xi (\nu_\xi)_j}} \right\}$$

$$\times \left\{ (\gamma_{(\xi,j)}) \in \prod_{\substack{(\xi,j) \\ \bar{w}_f(\xi,j) \neq (\xi,j)}} (k^\times)^{\sigma^{\tilde{m}_\xi (\nu_\xi)_j}} \,\Big|\, \gamma_{\bar{w}_f(\xi,j)} = \sigma^{-i(w_f,\xi,j)} (\gamma_{(\xi,j)}^{-1}) \right\}$$

and $\lambda|_{(T\cap fKf^{-1})^F}$ has the usual form (cf. the proof of Lemma 4.1.5). Now the $\mathbb{F}_q$-rank of

$$Z_G((T\cap fKf^{-1})^\circ) = Z_G((T\cap fK^\circ f^{-1})^\circ)$$
$$= GL\left(\bigoplus_{\substack{(\xi,j,i)\\ \bar w_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=0}} L_{(\xi,j,i)}\right)$$
$$\times \prod_{\substack{(\xi,j,i)\\ \bar w_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=\frac12\tilde m_\xi(\nu_\xi)_j}} GL(L_{(\xi,j,i)}) \times \prod_{\substack{(\xi,j,i)\\ \bar w_f(\xi,j)\neq(\xi,j)}} GL(L_{(\xi,j,i)})$$

is

$$\lfloor\frac12 \sum_{\substack{(\xi,j)\\ \bar w_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=0}} \tilde m_\xi(\nu_\xi)_j\rfloor + |\{(\xi,j)\,|\,\bar w_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=\frac12\tilde m_\xi(\nu_\xi)_j\}|$$
$$+ |\{(\xi,j)\,|\,\bar w_f(\xi,j)\neq(\xi,j), 2|\tilde m_\xi(\nu_\xi)_j\}|.$$

If $t\in(T\cap fKf^{-1})^F$ corresponds to $\big((\alpha_{(\xi,j)}),(\beta_{(\xi,j)}),(\gamma_{(\xi,j)})\big)$, then

$$Z_G^\circ(t)\cap Z_G((T\cap fKf^{-1})^\circ)$$
$$= GL\left(\bigoplus_{\substack{(\xi,j,i)\\ \bar w_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=0\\ \alpha_{(\xi,j)}=1}} L_{(\xi,j,i)}\right) \times GL\left(\bigoplus_{\substack{(\xi,j,i)\\ \bar w_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=0\\ \alpha_{(\xi,j)}=-1}} L_{(\xi,j,i)}\right)$$
$$\times \prod_{\substack{(\xi,j,i)\\ \bar w_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=\frac12\tilde m_\xi(\nu_\xi)_j}} GL(L_{(\xi,j,i)}) \times \prod_{\substack{(\xi,j,i)\\ \bar w_f(\xi,j)\neq(\xi,j)}} GL(L_{(\xi,j,i)})$$

has $\mathbb{F}_q$-rank which differs from that of $Z_G((T\cap fKf^{-1})^\circ)$ by

$$\lfloor\frac12 \sum_{\substack{(\xi,j)\\ \bar w_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=0\\ \alpha_{(\xi,j)}=1}} \tilde m_\xi(\nu_\xi)_j\rfloor + \lfloor\frac12 \sum_{\substack{(\xi,j)\\ \bar w_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=0\\ \alpha_{(\xi,j)}=-1}} \tilde m_\xi(\nu_\xi)_j\rfloor - \lfloor\frac12 \sum_{\substack{(\xi,j)\\ \bar w_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=0}} \tilde m_\xi(\nu_\xi)_j\rfloor.$$

Since n is odd,

$$\sum_{\substack{(\xi,j)\\ \bar w_f(\xi,j)=(\xi,j)\\ i(w_f,\xi,j)=0}} \tilde m_\xi(\nu_\xi)_j$$

is odd, so this difference is zero. Thus $\epsilon_{T,f}=1$. The rest of the proof follows that of Lemma 4.1.5. □

Corollary 4.3.6. *The map $f \mapsto w_f$ induces surjections $T^F \backslash \Theta^F_{T,K,\lambda}/K^F \twoheadrightarrow Z'$ and $T^F \backslash \Theta^F_{T,K^\circ,\lambda}/(K^\circ)^F \twoheadrightarrow Z' \cup Z''$.*

Now the $\mathbb{F}_q$-rank of T is $|\{(\xi,j) \,|\, 2|\tilde{m}_\xi(\nu_\xi)_j\}|$, and that of $Z_G((T \cap fKf^{-1})^\circ)$ is given above, whence

$$\begin{aligned}
&\mathbb{F}_q\text{-rank}(T) + \mathbb{F}_q\text{-rank}(Z_G((T \cap fKf^{-1})^\circ)) \\
&\qquad \equiv \lfloor \frac{1}{2} \sum_{\substack{(\xi,j) \\ \bar{w}_f(\xi,j)=(\xi,j) \\ i(w_f,\xi,j)=0}} \tilde{m}_\xi(\nu_\xi)_j \rfloor \\
&\qquad\quad + |\{(\xi,j) \,|\, \bar{w}_f(\xi,j) = (\xi,j), i(w_f,\xi,j) = 0, 2|\tilde{m}_\xi(\nu_\xi)_j\}| \\
&\qquad \equiv \lfloor \frac{n}{2} \rfloor + |\{(\xi,j) \,|\, \bar{w}_f(\xi,j) = (\xi,j), i(w_f,\xi,j) = 0, 2|\tilde{m}_\xi(\nu_\xi)_j\}| \\
&\qquad\quad + |\{(\xi,j) \,|\, \bar{w}_f(\xi,j) = (\xi,j), i(w_f,\xi,j) = \frac{1}{2}\tilde{m}_\xi(\nu_\xi)_j, 4 \nmid \tilde{m}_\xi(\nu_\xi)_j\}| \\
&\qquad\quad + \frac{1}{2}|\{(\xi,j) \,|\, \bar{w}_f(\xi,j) \neq (\xi,j), 2 \nmid \tilde{m}_\xi(\nu_\xi)_j\}| \bmod 2.
\end{aligned}$$

So Lusztig's formula gives (compare §4.1):

$$\begin{aligned}
&\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1) \rangle \\
&\qquad = \frac{1}{2}(-1)^{\lfloor \frac{n}{2} \rfloor} \sum_{w \in Z'} (-1)^{\substack{|\{(\xi,j) \,|\, \bar{w}_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=0, 2|\tilde{m}_\xi(\nu_\xi)_j\}| \\ +|\{(\xi,j) \,|\, \bar{w}_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=\frac{1}{2}\tilde{m}_\xi(\nu_\xi)_j, 4 \nmid \tilde{m}_\xi(\nu_\xi)_j\}| \\ +\frac{1}{2}|\{(\xi,j) \,|\, \bar{w}_f(\xi,j) \neq (\xi,j), 2 \nmid \tilde{m}_\xi(\nu_\xi)_j\}|}} \\
&\qquad\qquad \times 2^{|\{(\xi,j) \,|\, \bar{w}(\xi,j)=(\xi,j), i(w,\xi,j)=0\}|}, \text{ and} \\
&\langle B_{\underline{\nu}}, \mathrm{Ind}_{(K^\circ)^F}^{G^F}(1) \rangle \\
&\qquad = \frac{1}{2}(-1)^{\lfloor \frac{n}{2} \rfloor} \sum_{w \in Z'} (-1)^{\substack{|\{(\xi,j) \,|\, \bar{w}_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=0, 2|\tilde{m}_\xi(\nu_\xi)_j\}| \\ +|\{(\xi,j) \,|\, \bar{w}_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=\frac{1}{2}\tilde{m}_\xi(\nu_\xi)_j, 4 \nmid \tilde{m}_\xi(\nu_\xi)_j\}| \\ +\frac{1}{2}|\{(\xi,j) \,|\, \bar{w}_f(\xi,j) \neq (\xi,j), 2 \nmid \tilde{m}_\xi(\nu_\xi)_j\}|}} \\
&\qquad\qquad \times 2^{|\{(\xi,j) \,|\, \bar{w}(\xi,j)=(\xi,j), i(w,\xi,j)=0\}|} \\
&\qquad + \frac{1}{2}(-1)^{\lfloor \frac{n}{2} \rfloor} \sum_{w \in Z''} (-1)^{\substack{|\{(\xi,j) \,|\, \bar{w}_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=0, 2|\tilde{m}_\xi(\nu_\xi)_j\}| \\ +|\{(\xi,j) \,|\, \bar{w}_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=\frac{1}{2}\tilde{m}_\xi(\nu_\xi)_j, 4 \nmid \tilde{m}_\xi(\nu_\xi)_j\}| \\ +\frac{1}{2}|\{(\xi,j) \,|\, \bar{w}_f(\xi,j) \neq (\xi,j), 2 \nmid \tilde{m}_\xi(\nu_\xi)_j\}|}} \\
&\qquad\qquad \times 2^{|\{(\xi,j) \,|\, \bar{w}(\xi,j)=(\xi,j), i(w,\xi,j)=0\}|},
\end{aligned}$$

since Z' and Z'' are disjoint by the oddness of n.

As in §4.1, we transform these expressions to get

$$\begin{aligned}
&\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle \\
&= \frac{1}{2}(-1)^{\lfloor \frac{n}{2} \rfloor} \\
&\times \prod_{\substack{\xi \in \langle \tilde{\sigma}\rangle \backslash L \\ 2 \nmid \tilde{m}_\xi}} \left(\sum_{\substack{\text{involutions } w_\xi \in Z_{S_{|\nu_\xi|}}(\nu_\xi), \\ \text{fixing no odd cycles of } \nu_\xi \text{ if } \tilde{d}_\xi = -1}} \right. \\
&\qquad (-1)^{\substack{|\{\text{even cycles of } \nu_\xi \text{ fixed by } w_\xi \text{ pointwise}\}| \\ +|\{\text{cycles of } \nu_\xi \text{ of length} \equiv 2 \bmod 4 \text{ fixed by } w_\xi, \text{ not pointwise}\}| \\ +\frac{1}{2}|\{\text{odd cycles of } \nu_\xi \text{ moved by } w_\xi\}|}} \\
&\qquad \left. \times\, 2^{|\{\text{cycles of } \nu_\xi \text{ fixed by } w_\xi \text{ pointwise}\}|} \right) \\
&\times \prod_{\substack{\xi \in \langle \tilde{\sigma}\rangle \backslash L \\ 2 \mid \tilde{m}_\xi}} \left(\sum_{\substack{\text{involutions } w_\xi \in Z_{S_{|\nu_\xi|}}(\nu_\xi) \\ \text{fixing no odd cycles of } \nu_\xi \text{ if } \tilde{d}_\xi = -1}} (-2)^{|\{\text{cycles of } \nu_\xi \text{ fixed by } w_\xi \text{ pointwise}\}|} \right),
\end{aligned}$$

and

$$\langle B_{\underline{\nu}}, \mathrm{Ind}_{(K^\circ)^F}^{G^F}(1)\rangle = \langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle + \langle B_{\tilde{\zeta}.\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle.$$

So in addition to (4.1.2) and (4.1.3), the combinatorial facts we need are:

(4.3.2)

$$\begin{aligned}
&\sum_{\text{involutions } w \in Z_{S_{|\nu|}}(\nu)} (-1)^{\substack{|\{\text{even cycles of } \nu \text{ fixed by } w \text{ pointwise}\}| \\ +|\{\text{cycles of } \nu \text{ of length} \equiv 2 \bmod 4 \text{ fixed by } w, \text{ not pointwise}\}| \\ +\frac{1}{2}|\{\text{odd cycles of } \nu \text{ moved by } w\}|}} \\
&\qquad \times 2^{|\{\text{cycles of } \nu \text{ fixed by } w \text{ pointwise}\}|} \\
&\qquad = \sum_{\substack{\lambda \vdash |\nu| \\ 2 \mid m_{2i}(\lambda)}} (-1)^{n(\lambda')} \Big(\prod_i (m_{2i+1}(\lambda)+1)\Big) \chi_\nu^\lambda,
\end{aligned}$$

and

(4.3.3)

$$\sum_{\substack{\text{involutions } w \in Z_{S_{|\nu|}}(\nu)\\ \text{fixing no odd cycles of } \nu}} (-1)^{\begin{array}{l}|\{\text{cycles of } \nu \text{ fixed by } w \text{ pointwise}\}|\\ +|\{\text{cycles of } \nu \text{ of length} \equiv 2 \bmod 4 \text{ fixed by } w, \text{ not pointwise}\}|\\ +\frac{1}{2}|\{\text{odd cycles of } \nu \text{ moved by } w\}|\end{array}} \times 2^{|\{\text{cycles of } \nu \text{ fixed by } w \text{ pointwise}\}|} = \sum_{\substack{\lambda \vdash |\nu|\\ 2|m_{2i+1}(\lambda)}} (-1)^{n(\lambda')} (\prod_i (m_{2i}(\lambda)+1)) \chi^\lambda_\nu.$$

These will be proved in §5.

4.4. **The $U_n(\mathbb{F}_{q^2})/O^\pm_n(\mathbb{F}_q)$ and $U_n(\mathbb{F}_{q^2})/SO^\pm_n(\mathbb{F}_q)$ Cases (n even).** Finally, we suppose that n is even and $F : G \to G$ is non-split. So θF is induced by F_V as in §4.2, and we have the same dichotomy as to the Witt index of $\langle \cdot, \cdot \rangle$ on V^{F_V}. Define $\epsilon \in \{\pm 1\}$ as in §4.2, so that $G^F \cong U_n(\mathbb{F}_{q^2})$, $K^F \cong O^\epsilon_n(\mathbb{F}_q)$, $(K^\circ)^F \cong SO^\epsilon_n(\mathbb{F}_q)$.

Theorem 4.4.1. *For any $\underline{\rho} \in \widehat{\mathcal{P}}^{\tilde{\sigma}}_n$,*

$$\langle \chi^{\underline{\rho}}, \mathrm{Ind}_{O^\epsilon_n(\mathbb{F}_q)}^{U_n(\mathbb{F}_{q^2})}(1) \rangle = \begin{cases} \frac{1}{2} \prod_{\substack{\xi \in \langle \tilde{\sigma} \rangle \backslash L\\ \tilde{d}_\xi = 1\\ 2 \nmid \tilde{m}_\xi}} (\prod_i (m_{2i}(\rho_\xi)+1)) \prod_{\substack{\xi \in \langle \tilde{\sigma} \rangle \backslash L\\ \tilde{d}_\xi = -1\\ 2 \nmid \tilde{m}_\xi}} (\prod_i (m_{2i+1}(\rho_\xi)+1)) \\ \qquad \times \prod_{\substack{\xi \in \langle \tilde{\sigma} \rangle \backslash L\\ \tilde{d}_\xi = 1\\ 2 | \tilde{m}_\xi}} (\prod_i (m_i(\rho_\xi)+1)), \\ \qquad \text{if } \tilde{d}_\xi = 1,\ 2 \nmid \tilde{m}_\xi \Rightarrow 2 | m_{2i+1}(\rho_\xi), \forall i, \\ \qquad \tilde{d}_\xi = -1,\ 2 \nmid \tilde{m}_\xi \Rightarrow 2 | m_{2i}(\rho_\xi), \forall i, \\ \qquad \text{and } \tilde{d}_\xi = -1,\ 2 \,|\, \tilde{m}_\xi \Rightarrow \rho'_\xi \text{ is even} \\ 0, \quad \text{otherwise} \end{cases} + \begin{cases} \frac{1}{2}\epsilon, & \text{if all } \rho'_\xi \text{ are even} \\ 0, & \text{otherwise, and} \end{cases}$$

$$\langle \chi^{\underline{\rho}}, \mathrm{Ind}_{SO^\epsilon_n(\mathbb{F}_q)}^{U_n(\mathbb{F}_{q^2})}(1) \rangle = \langle \chi^{\underline{\rho}}, \mathrm{Ind}_{O^\epsilon_n(\mathbb{F}_q)}^{U_n(\mathbb{F}_{q^2})}(1) \rangle + \langle \chi^{\tilde{\zeta} \cdot \underline{\rho}}, \mathrm{Ind}_{O^\epsilon_n(\mathbb{F}_q)}^{U_n(\mathbb{F}_{q^2})}(1) \rangle.$$

By (1.3.2), it is equivalent to say that for any $\underline{\nu} \in \widehat{\mathcal{P}}_n^{\tilde{\sigma}}$,

(4.4.1)

$$\begin{aligned}\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle = \frac{1}{2}(-1)^{\frac{n}{2}} &\prod_{\substack{\xi\in\langle\tilde{\sigma}\rangle\backslash L\\ \tilde{d}_\xi=1\\ 2\nmid\tilde{m}_\xi}}\left(\sum_{\substack{\rho_\xi\vdash|\nu_\xi|\\ 2|m_{2i+1}(\rho_\xi)}}(-1)^{n(\rho'_\xi)}(\prod_i(m_{2i}(\rho_\xi)+1))\chi_{\nu_\xi}^{\rho_\xi}\right)\\ &\times\prod_{\substack{\xi\in\langle\tilde{\sigma}\rangle\backslash L\\ \tilde{d}_\xi=-1\\ 2\nmid\tilde{m}_\xi}}\left(\sum_{\substack{\rho_\xi\vdash|\nu_\xi|\\ 2|m_{2i}(\rho_\xi)}}(-1)^{n(\rho'_\xi)}(\prod_i(m_{2i+1}(\rho_\xi)+1))\chi_{\nu_\xi}^{\rho_\xi}\right)\\ &\times\prod_{\substack{\xi\in\langle\tilde{\sigma}\rangle\backslash L\\ \tilde{d}_\xi=1\\ 2|\tilde{m}_\xi}}(-1)^{|\nu_\xi|}\left(\sum_{\rho_\xi\vdash|\nu_\xi|}(\prod_i(m_i(\rho_\xi)+1))\chi_{\nu_\xi}^{\rho_\xi}\right)\\ &\times\prod_{\substack{\xi\in\langle\tilde{\sigma}\rangle\backslash L\\ \tilde{d}_\xi=-1\\ 2|\tilde{m}_\xi}}\left(\sum_{\substack{\rho_\xi\vdash|\nu_\xi|\\ \rho'_\xi\text{ even}}}\chi_{\nu_\xi}^{\rho_\xi}\right)\\ &+\frac{1}{2}\epsilon\prod_{\substack{\xi\in\langle\tilde{\sigma}\rangle\backslash L\\ 2\nmid\tilde{m}_\xi}}(-1)^{\frac{|\nu_\xi|}{2}}\left(\sum_{\substack{\rho_\xi\vdash|\nu_\xi|\\ \rho'_\xi\text{ even}}}\chi_{\nu_\xi}^{\rho_\xi}\right)\\ &\times\prod_{\substack{\xi\in\langle\tilde{\sigma}\rangle\backslash L\\ 2|\tilde{m}_\xi}}\left(\sum_{\substack{\rho_\xi\vdash|\nu_\xi|\\ \rho'_\xi\text{ even}}}\chi_{\nu_\xi}^{\rho_\xi}\right),\ \text{and}\end{aligned}$$

$$\langle B_{\underline{\nu}}, \mathrm{Ind}_{(K^\circ)^F}^{G^F}(1)\rangle = \langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle + \langle B_{\tilde{\zeta}.\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle.$$

Here the sign

$$(-1)^{\frac{n}{2}+\sum_{\xi\in\langle\tilde{\sigma}\rangle\backslash L}\tilde{m}_\xi n(\rho'_\xi)+|\rho_\xi|} = (-1)^{\frac{n}{2}+\sum_{\xi\in\langle\tilde{\sigma}\rangle\backslash L}\tilde{m}_\xi(n(\rho'_\xi)+|\rho_\xi|)+|\rho_\xi|}$$

of (1.3.2) has been distributed in an obvious way (in the second term it has been rewritten as $\prod_{\xi\in\langle\tilde{\sigma}\rangle\backslash L}(-1)^{\tilde{m}_\xi\frac{|\nu_\xi|}{2}}$ since $|\rho_\xi|$ and $n(\rho'_\xi)$ are even if ρ'_ξ is even).

The proof of this is similar to that of (4.2.1). Let (T,λ), $\{L_{(\xi,j,i)}\}$ and w_F be as in §2.2. Lemma 4.3.2 again holds, with its usual proof. Corollary 4.2.3 must be modified as follows:

Corollary 4.4.2. *The map $f \mapsto w_f$ induces a map*

$$T^F\backslash\Theta_T^F/K^F \to \{\textit{involutions in } Z_{W(T)}(w_F)\}.$$

If $\mathrm{sign}(w_F) = (-1)^{\frac{n}{2}}\epsilon$, *this map is surjective. If* $\mathrm{sign}(w_F) = -(-1)^{\frac{n}{2}}\epsilon$, *the image consists of all involutions with at least one fixed point.*

Proof. The proof is mostly the same as that of Corollary 4.2.3. Note that if $f \in \Theta_T^F$,

$$\begin{aligned} F_V(f^{-1}L_{(\xi,j,i)}) &= \theta(f)^{-1} \bigcap_{\substack{(\xi',j',i') \\ \neq w_F(\xi,j,i)}} L^{\perp}_{(\xi',j',i')} \\ &= \left(\bigoplus_{\substack{(\xi',j',i') \\ \neq w_F(\xi,j,i)}} f^{-1}L_{(\xi',j',i')} \right)^{\perp} \\ &= f^{-1}L_{w_f^{-1}w_F(\xi,j,i)}. \end{aligned}$$

So in (2) of the proof of Corollary 4.2.3, w_F should be replaced by $w^{-1}w_F$. Since $\mathrm{sign}(w) = (-1)^{\frac{n}{2}}$ if w is a fixed-point free involution, we get the result. □

For w an involution in $Z_{W(T)}(w_F)$, define $\bar{w}$ and $i(w,\xi,j)$ as in the previous subsection.

Lemma 4.4.3. *For $f \in \Theta_T^F$, the number of T^F–K^F double cosets in TfK is*

$$\begin{cases} 1, & \text{if } \bar{w}_f(\xi,j) = (\xi,j),\ i(w_f,\xi,j) = 0 \\ & \text{never occurs} \\ 2^{|\{(\xi,j)\,|\,\bar{w}_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=0\}|-1}, & \text{otherwise,} \end{cases}$$

and the number of T^F–$(K^\circ)^F$ double cosets in TfK° is

$$\begin{cases} 2^{|\{(\xi,j)\,|\,\bar{w}_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=0\}|}, & \text{if } \bar{w}_f(\xi,j) = (\xi,j),\ i(w_f,\xi,j) = 0 \\ & \Rightarrow 2 \,|\, m_\xi(\nu_\xi)_j \\ 2^{|\{(\xi,j)\,|\,\bar{w}_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=0\}|-1}, & \text{otherwise.} \end{cases}$$

Proof. As with Lemma 4.2.4, the proof of Lemma 4.1.4 applies again here. □

Now define Z' and Z'' as in §4.3. We now encounter the first major point of difference from §4.2:

Lemma 4.4.4. *For $f \in \Theta_T^F$, $f \in \Theta^F_{T,K,\lambda} \Leftrightarrow w_f \in Z''$, and $f \in \Theta^F_{T,K^\circ,\lambda} \Leftrightarrow w_f \in Z' \cup Z''$.*

Proof. The descriptions of $(T \cap fKf^{-1})^F$ and $\lambda|_{(T\cap fKf^{-1})^F}$ in the proof of Lemma 4.3.5 remain valid. The new feature when n is even is that

$$\sum_{\substack{(\xi,j) \\ \bar{w}_f(\xi,j)=(\xi,j) \\ i(w_f,\xi,j)=0}} \tilde{m}_\xi(\nu_\xi)_j$$

is even, so that if $t \in (T \cap fKf^{-1})^F$ is such that

$$\sum_{\substack{(\xi,j) \\ \bar{w}_f(\xi,j)=(\xi,j) \\ i(w_f,\xi,j)=0 \\ \alpha_{(\xi,j)}=-1}} \tilde{m}_\xi(\nu_\xi)_j$$

is odd, then

$$\mathbb{F}_q\text{-rank}(Z_G((T \cap fKf^{-1})^\circ)) - \mathbb{F}_q\text{-rank}(Z_G^\circ(t) \cap Z_G((T \cap fKf^{-1})^\circ)) = 1,$$

so $\epsilon_{T,f}(t) = -1$. In fact, $\epsilon_{T,f}$ is the function

$$\big((\alpha_{(\xi,j)}), (\beta_{(\xi,j)}), (\gamma_{(\xi,j)})\big) \mapsto (-1)^{|\{(\xi,j)\,|\,\bar w_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=0, 2\nmid \tilde m_\xi, 2\nmid(\nu_\xi)_j, \alpha_{(\xi,j)}=-1\}|}.$$

Hence $f \in \Theta^F_{T,K,\lambda}$ if and only if for all (ξ,j) such that $\bar w_f(\xi,j) = (\xi,j)$ and $i(w_f,\xi,j) = 0$,

$$\tilde d_\xi = -1, 2 \nmid (\nu_\xi)_j \Leftrightarrow 2 \nmid \tilde m_\xi, 2 \nmid (\nu_\xi)_j,$$

besides the usual conditions on other (ξ,j). Clearly this is equivalent to $w_f \in Z''$. The second statement is proved similarly. □

As in §4.3,

$$\begin{aligned}\mathbb{F}_q\text{-rank}(T)+&\mathbb{F}_q\text{-rank}(Z_G((T\cap fKf^{-1})^\circ))\\ &\equiv \frac{n}{2} + |\{(\xi,j)\,|\,\bar w_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=0, 2|\tilde m_\xi(\nu_\xi)_j\}|\\ &\quad + |\{(\xi,j)\,|\,\bar w_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=\frac{1}{2}\tilde m_\xi(\nu_\xi)_j, 4\nmid \tilde m_\xi(\nu_\xi)_j\}|\\ &\quad + \frac{1}{2}|\{(\xi,j)\,|\,\bar w_f(\xi,j)\neq(\xi,j), 2\nmid \tilde m_\xi(\nu_\xi)_j\}| \bmod 2.\end{aligned}$$

So Lusztig's formula gives (compare §4.2):

$$\begin{aligned}&\langle B_{\underline\nu}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle\\ &= \frac{1}{2}(-1)^{\frac{n}{2}} \sum_{w\in Z''} (-1)^{\substack{|\{(\xi,j)\,|\,\bar w_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=0, 2|\tilde m_\xi(\nu_\xi)_j\}|\\ +|\{(\xi,j)\,|\,\bar w_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=\frac12\tilde m_\xi(\nu_\xi)_j, 4\nmid\tilde m_\xi(\nu_\xi)_j\}|\\ +\frac12|\{(\xi,j)\,|\,\bar w_f(\xi,j)\neq(\xi,j), 2\nmid\tilde m_\xi(\nu_\xi)_j\}|}} \\ &\qquad\qquad \times 2^{|\{(\xi,j)\,|\,\bar w(\xi,j)=(\xi,j), i(w,\xi,j)=0\}|}\\ &\quad + \frac{1}{2}\epsilon \sum_{w\in Z'''} (-1)^{\substack{|\{(\xi,j)\,|\,\bar w_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=\frac12\tilde m_\xi(\nu_\xi)_j, 4|\tilde m_\xi(\nu_\xi)_j\}|\\ +\frac12|\{(\xi,j)\,|\,\bar w_f(\xi,j)\neq(\xi,j), 2\nmid\tilde m_\xi(\nu_\xi)_j\}|}}, \text{ and}\end{aligned}$$

$$\begin{aligned}&\langle B_{\underline\nu}, \mathrm{Ind}_{(K^\circ)^F}^{G^F}(1)\rangle\\ &= \frac{1}{2}(-1)^{\frac{n}{2}} \sum_{w\in Z'} (-1)^{\substack{|\{(\xi,j)\,|\,\bar w_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=0, 2|\tilde m_\xi(\nu_\xi)_j\}|\\ +|\{(\xi,j)\,|\,\bar w_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=\frac12\tilde m_\xi(\nu_\xi)_j, 4\nmid\tilde m_\xi(\nu_\xi)_j\}|\\ +\frac12|\{(\xi,j)\,|\,\bar w_f(\xi,j)\neq(\xi,j), 2\nmid\tilde m_\xi(\nu_\xi)_j\}|}} \\ &\qquad\qquad \times 2^{|\{(\xi,j)\,|\,\bar w(\xi,j)=(\xi,j), i(w,\xi,j)=0\}|}\\ &\quad + \frac{1}{2}(-1)^{\frac{n}{2}} \sum_{w\in Z''} (-1)^{\substack{|\{(\xi,j)\,|\,\bar w_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=0, 2|\tilde m_\xi(\nu_\xi)_j\}|\\ +|\{(\xi,j)\,|\,\bar w_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=\frac12\tilde m_\xi(\nu_\xi)_j, 4\nmid\tilde m_\xi(\nu_\xi)_j\}|\\ +\frac12|\{(\xi,j)\,|\,\bar w_f(\xi,j)\neq(\xi,j), 2\nmid\tilde m_\xi(\nu_\xi)_j\}|}} \\ &\qquad\qquad \times 2^{|\{(\xi,j)\,|\,\bar w(\xi,j)=(\xi,j), i(w,\xi,j)=0\}|}\\ &\quad + \epsilon \sum_{w\in Z'''} (-1)^{\substack{|\{(\xi,j)\,|\,\bar w_f(\xi,j)=(\xi,j), i(w_f,\xi,j)=\frac12\tilde m_\xi(\nu_\xi)_j, 4|\tilde m_\xi(\nu_\xi)_j\}|\\ +\frac12|\{(\xi,j)\,|\,\bar w_f(\xi,j)\neq(\xi,j), 2\nmid\tilde m_\xi(\nu_\xi)_j\}|}}.\end{aligned}$$

As in §4.1, we transform these expressions to get

$$\begin{aligned}
&\langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle \\
&= \frac{1}{2}(-1)^{\frac{n}{2}} \\
&\times \prod_{\substack{\xi\in\langle\tilde\sigma\rangle\backslash L \\ 2\nmid \tilde m_\xi}} \left(\sum_{\substack{\text{involutions } w_\xi \in Z_{S_{|\nu_\xi|}}(\nu_\xi), \\ \text{fixing no odd cycles of } \nu_\xi \text{ if } \tilde d_\xi = 1}} (-1)^{\begin{array}{l}|\{\text{cycles of } \nu_\xi \text{ fixed by } w_\xi \text{ pointwise}\}| \\ +|\{\text{cycles of } \nu_\xi \text{ of length} \equiv 2 \bmod 4 \text{ fixed by } w_\xi, \text{ not pointwise}\}| \\ +\frac{1}{2}|\{\text{odd cycles of } \nu_\xi \text{ moved by } w_\xi\}|\end{array}} \times 2^{|\{\text{cycles of } \nu_\xi \text{ fixed by } w_\xi \text{ pointwise}\}|} \right) \\
&\times \prod_{\substack{\xi\in\langle\tilde\sigma\rangle\backslash L \\ 2|\tilde m_\xi}} \left(\sum_{\substack{\text{involutions } w_\xi \in Z_{S_{|\nu_\xi|}}(\nu_\xi), \\ \text{fixing no odd cycles of } \nu_\xi \text{ if } \tilde d_\xi = -1}} (-2)^{|\{\text{cycles of } \nu_\xi \text{ fixed by } w_\xi \text{ pointwise}\}|} \right) \\
&+ \frac{1}{2}\epsilon \prod_{\substack{\xi\in\langle\tilde\sigma\rangle\backslash L \\ 2\nmid \tilde m_\xi}} \left(\sum_{\substack{\text{fixed-point free} \\ \text{involutions } w_\xi \in Z_{S_{|\nu_\xi|}}(\nu_\xi)}} (-1)^{\begin{array}{l}|\{\text{cycles of } \nu_\xi \text{ divisible by 4, fixed by } w_\xi\}| \\ +\frac{1}{2}|\{\text{odd cycles of } \nu_\xi\}|\end{array}} \right) \\
&\times \prod_{\substack{\xi\in\langle\tilde\sigma\rangle\backslash L \\ 2|\tilde m_\xi}} \left(\sum_{\substack{\text{fixed-point free} \\ \text{involutions } w_\xi \in Z_{S_{|\nu_\xi|}}(\nu_\xi)}} (-1)^{|\{\text{cycles of } \nu_\xi \text{ fixed by } w_\xi\}|} \right),
\end{aligned}$$

and

$$\langle B_{\underline{\nu}}, \mathrm{Ind}_{(K^\circ)^F}^{G^F}(1)\rangle = \langle B_{\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle + \langle B_{\tilde\zeta.\underline{\nu}}, \mathrm{Ind}_{K^F}^{G^F}(1)\rangle.$$

So (4.4.1) follows by applying (4.3.2), (4.3.3), (4.1.2), (4.1.3), (3.3.3) and (3.1.3).

5. Combinatorics of the Symmetric Group

This section is devoted to the proof of the combinatorial facts invoked in §§2-4. We say that a function f on the set of partitions is *multiplicative* if

$$f(\nu) = \prod_i f(i^{m_i(\nu)}),\ \forall \nu.$$

Examples of multiplicative functions of ν are the sign ϵ_ν, $z_\nu = |Z_{S_{|\nu|}}(\nu)|$, and $|\{\text{involutions in } Z_{S_{|\nu|}}(\nu)\}|$.

Our first starting point is [14, VII.(2.4)], which can be rewritten:

$$|\{\text{fixed-point free involutions in } Z_{S_{|\nu|}}(\nu)\}| = \sum_{\substack{\lambda \vdash |\nu| \\ \lambda \text{ even}}} \chi_\nu^\lambda.$$

This is precisely (2.1.2). Now if there exists an involution w in $Z_{S_{|\nu|}}(\nu)$, then $|\{\text{cycles of } \nu \text{ moved by } w\}|$ is even, so

$$(-1)^{|\{\text{cycles of } \nu \text{ fixed by } w\}|} = (-1)^{|\{\text{cycles of } \nu\}|}.$$

As already observed above, if there exists a fixed-point free involution w in $Z_{S_{|\nu|}}(\nu)$, then

$$(-1)^{|\{\text{cycles of } \nu\}|} = (-1)^{|\{\text{even cycles of } \nu\}|} = \epsilon_\nu.$$

Thus

$$\sum_{\substack{\text{fixed-point free} \\ \text{involutions } w \in Z_{S_{|\nu|}}(\nu)}} (-1)^{|\{\text{cycles of } \nu \text{ fixed by } w\}|} = \epsilon_\nu \sum_{\substack{\lambda \vdash |\nu| \\ \lambda \text{ even}}} \chi_\nu^\lambda = \sum_{\substack{\lambda \vdash |\nu| \\ \lambda' \text{ even}}} \chi_\nu^\lambda,$$

which proves (3.1.3). Now (4.1.3) follows from (2.1.2) and the fact that

$$(-1)^{|\{\text{even cycles of } \nu\}|} \sum_{\substack{\text{involutions } w \in Z_{S_{|\nu|}}(\nu) \\ \text{fixing no odd cycles of } \nu}} (-2)^{|\{\text{cycles of } \nu \text{ fixed by } w \text{ pointwise}\}|} = \left| \left\{ \begin{array}{c} \text{fixed-point free} \\ \text{involutions in } Z_{S_{|\nu|}}(\nu) \end{array} \right\} \right|.$$

To see this, note that since both sides are multiplicative, it suffices to consider the case when ν is of the form (a^b), in which case both sides are

$$\begin{cases} 0, & \text{if } a \text{ is odd and } b \text{ is odd,} \\ a^{b/2} \dfrac{b!}{2^{b/2}(\frac{b}{2})!}, & \text{if } a \text{ is odd and } b \text{ is even, and} \\ \displaystyle\sum_{r=0}^{\lfloor \frac{b}{2} \rfloor} \binom{b}{2r} a^r \frac{(2r)!}{2^r r!}, & \text{if } a \text{ is even.} \end{cases}$$

This fact also implies (3.2.2), as clearly

$$\begin{aligned} &\sum_{\substack{\text{involutions } w \in Z_{S_{|\nu|}}(\nu) \\ \text{with signed fixed points,} \\ \text{signs reversed by } \nu}} (-1)^{|\{\text{cycles of } \nu \text{ fixed by } w, \text{ not pointwise}\}|} \\ &\quad = \sum_{\substack{\text{involutions } w \in Z_{S_{|\nu|}}(\nu) \\ \text{fixing no odd cycles of } \nu}} (-1)^{|\{\text{cycles of } \nu \text{ fixed by } w, \text{ not pointwise}\}|} \\ &\qquad\qquad \times 2^{|\{\text{cycles of } \nu \text{ fixed by } w \text{ pointwise}\}|} \\ &\quad = (-1)^{|\{\text{even cycles of } \nu\}|} \sum_{\substack{\text{involutions } w \in Z_{S_{|\nu|}}(\nu) \\ \text{fixing no odd cycles of } \nu}} (-2)^{|\{\text{cycles of } \nu \text{ fixed by } w \text{ pointwise}\}|}, \end{aligned}$$

since if there exists an involution $w \in Z_{S_{|\nu|}}(\nu)$ fixing no odd cycles of ν, then $|\{\text{odd cycles of } \nu\}|$ must be even. Similarly, (3.3.3) follows from (2.1.2) and the

fact that

$$\sum_{\substack{\text{fixed-point free}\\ \text{involutions } w \in Z_{S_{|\nu|}}(\nu)}} (-1)^{|\{\text{cycles of } \nu \text{ divisible by 4, fixed by } w\}|+\frac{1}{2}|\{\text{odd cycles of } \nu\}|}$$
$$= (-1)^{\frac{|\nu|}{2}+|\{\text{even cycles of } \nu\}|} \left| \left\{ \begin{array}{c} \text{fixed-point free} \\ \text{involutions in } Z_{S_{|\nu|}}(\nu) \end{array} \right\} \right|.$$

Again both sides are multiplicative, so it suffices to prove this when $\nu = (a^b)$, in which case it is easy.

Our second starting point is [14, I.8 Example 11], which can be rewritten:

$$\sum_{\text{involutions } w \in Z_{S_{|\nu|}}(\nu)} (-1)^{|\{\text{cycles of } \nu \text{ fixed by } w, \text{ not pointwise}\}|} = \sum_{\lambda \vdash |\nu|} \chi^\lambda_\nu.$$

Now (3.3.4) follows from this and the fact that

$$\left| \left\{ \begin{array}{c} \text{involutions } w \in Z_{S_{|\nu|}}(\nu) \\ \text{fixing no even cycles of } \nu \end{array} \right\} \right|$$
$$= \sum_{\text{involutions } w \in Z_{S_{|\nu|}}(\nu)} (-1)^{|\{\text{cycles of } \nu \text{ fixed by } w, \text{ not pointwise}\}|},$$

which we can again prove simply by observing that when $\nu = (a^b)$ both sides are

$$\begin{cases} 0, & \text{if } a \text{ is even and } b \text{ is odd,} \\ a^{b/2} \dfrac{b!}{2^{b/2}(\frac{b}{2})!}, & \text{if } a \text{ is even and } b \text{ is even, and} \\ \displaystyle\sum_{r=0}^{\lfloor \frac{b}{2} \rfloor} \binom{b}{2r} a^r \frac{(2r)!}{2^r r!}, & \text{if } a \text{ is odd.} \end{cases}$$

Furthermore (3.1.4) follows from (3.3.4), since if there exists an involution $w \in Z_{S_{|\nu|}}(\nu)$ fixing no even cycles of ν, then $|\{\text{even cycles of } \nu\}|$ is even, so

$$(-1)^{|\{\text{cycles of } \nu \text{ fixed by } w\}|} = (-1)^{|\{\text{odd cycles of } \nu\}|}$$
$$= (-1)^{|\nu|}.$$

The remaining identities require a different approach. The prototype is (3.1.2). By definition of induction product ([14, I.7]), and using (3.1.3),

$$\sum_{\substack{\text{involutions } w \in Z_{S_{|\nu|}}(\nu)\\ \text{with signed fixed points,}\\ \text{signs stable under } \nu,\\ \text{signature } (p^+,p^-)+\mathbb{Z}}} (-1)^{|\{\text{cycles of } \nu \text{ fixed by } w, \text{ not pointwise}\}|}$$
$$= \sum_{r=\lceil \frac{p^+ + p^- - |\nu|}{2} \rceil}^{\min\{p^+,p^-\}} \left(\Big(\sum_{\substack{\mu \vdash |\nu|+2r-p^+-p^- \\ \mu' \text{ even}}} \chi^\mu \Big) \cdot \chi^{(p^+ - r)} \cdot \chi^{(p^- - r)} \right)(\nu)$$
$$= \sum_{r=\lceil \frac{p^+ + p^- - |\nu|}{2} \rceil}^{\min\{p^+,p^-\}} \sum_{\lambda \vdash |\nu|} b(p^+ - r, p^- - r, \lambda') \chi^\lambda_\nu,$$

where, by Pieri's formula ([14, (5.16)]), $b(p^+ - r, p^- - r, \lambda')$ is the number of ways of removing first a vertical $(p^- - r)$-strip, then a vertical $(p^+ - r)$-strip, from the Young diagram of λ', to leave a diagram with all rows of even length. Now every signed tableaux of shape λ' and signature $(p^+, p^-) + \mathbb{Z}$ determines uniquely an r as above and such a way of removing strips, as follows:

- order rows of equal length so that rows ending $\boxminus$ are below rows ending $\boxplus$;
- take the vertical $(p^- - r)$-strip to consist of all final boxes signed $\boxminus$;
- take the vertical $(p^+ - r)$-strip to consist of all final boxes signed $\boxplus$ in rows of odd length, including those made odd by removal of the first strip.

This correspondence is clearly bijective, which proves (3.1.2).

The proof of (3.3.2) is similar: using (3.3.3),

$$\begin{aligned}
&\sum_{\substack{\text{involutions } w \in Z_{S_{|\nu|}}(\nu)\\ \text{with signed fixed points,}\\ \text{signs stable under } \nu,\\ \text{signature } (p^+,p^-)+\mathbb{Z}}} (-1)^{\substack{|\{\text{cycles of } \nu \text{ divisible by 4, fixed by } w, \text{ not pointwise}\}|\\ +\frac{1}{2}|\{\text{odd cycles of } \nu \text{ moved by } w\}|}}\\
&= \sum_{r=\lceil \frac{p^+ + p^- - |\nu|}{2} \rceil}^{\min\{p^+,p^-\}} \left(((-1)^{\frac{|\nu|+2r-p^+-p^-}{2}} \sum_{\substack{\mu \vdash |\nu|+2r-p^+-p^-\\ \mu' \text{ even}}} \chi^\mu) . \chi^{(p^+-r)} . \chi^{(p^--r)} \right)(\nu)\\
&= \sum_{\lambda \vdash |\nu|} c(p^+,p^-,\lambda')\chi^\lambda_\nu,
\end{aligned}$$

where by the same bijection as before,

$$\begin{aligned}
&c(p^+,p^-,\lambda')\\
&= \sum_{\substack{\text{signed tableaux } T \text{ of shape } \lambda',\\ \text{signature } (p^+,p^-)+\mathbb{Z}}} (-1)^{\substack{\frac{1}{2}(|\lambda| - |\{\text{rows of } T \text{ ending } \boxminus\}|\\ -|\{\text{odd rows of } T \text{ ending } \boxplus\}| - |\{\text{even rows of } T \text{ ending } \boxminus\}|)}}\\
&= (-1)^{\frac{1}{2}(|\lambda| - |\{\text{odd parts of } \lambda'\}|)} \sum_{\substack{\text{signed tableaux } T \text{ of shape } \lambda',\\ \text{signature } (p^+,p^-)+\mathbb{Z}}} (-1)^{|\{\text{even rows of } T \text{ ending } \boxminus\}|}.
\end{aligned}$$

Now

$$\begin{aligned}
(-1)^{\frac{1}{2}(|\lambda| - |\{\text{odd parts of } \lambda'\}|)} &= (-1)^{|\{\text{parts of } \lambda' \equiv 2 \text{ or } 3 \bmod 4\}|}\\
&= (-1)^{\sum_i \binom{\lambda'_i}{2}} = (-1)^{n(\lambda)},
\end{aligned}$$

and by grouping together signed tableaux which differ only in even rows it is easy to see that the sum equals $\bar{a}(p^+, p^-, \lambda')$. So (3.3.2) is proved.

Our next task is to modify this proof of (3.3.2) to derive (3.4.2). (The reader is warned that the following argument is not very satisfactory, and implored to find a better one.) We may assume that $|\nu|$ is even, for otherwise both sides vanish. Following the above pattern, we need to replace $\chi^{(p^+-r)} . \chi^{(p^--r)}$ with the class function on $S_{|\nu|-2r}$ defined by

$$w \mapsto |\{(A^+, A^-) \,|\, \{1, \cdots, |\nu| - 2r\} = A^+ \coprod A^-, |A^+| = |A^-|, w(A^\pm) = A^\mp\}|.$$

We claim that this class function is

$$\sum_{i=0}^{\frac{|\nu|}{2}-r} (-1)^i \chi^{(|\nu|-2r-i,i)}.$$

To see this, note that the class function

$$w \mapsto |\{(A^+, A^-) \,|\, \{1, \cdots, |\nu|-2r\} = A^+ \coprod A^-, |A^+| = |A^-|, w(A^\pm) = A^\pm\}|$$

is

$$\mathrm{Ind}_{S_{\frac{|\nu|}{2}-r} \times S_{\frac{|\nu|}{2}-r}}^{S_{|\nu|-2r}} (1) = \chi^{(\frac{|\nu|}{2}-r)} . \chi^{(\frac{|\nu|}{2}-r)} = \sum_{i=0}^{\frac{|\nu|}{2}-r} \chi^{(|\nu|-2r-i,i)}$$

by Pieri's formula. So an irreducible representation of $S_{|\nu|-2r}$ has nonzero fixed vectors for $S_{\frac{|\nu|}{2}-r} \times S_{\frac{|\nu|}{2}-r}$ if and only if its character is $\chi^{(|\nu|-2r-i,i)}$ for some i, and if so the space of fixed vectors is one-dimensional. It is easy to check (for instance using explicit models for these representations) that these fixed vectors are also fixed by the larger group $(S_{\frac{|\nu|}{2}-r} \times S_{\frac{|\nu|}{2}-r}) \rtimes S_2$ (where S_2 permutes the factors) if and only if i is even. Hence

$$\mathrm{Ind}_{(S_{\frac{|\nu|}{2}-r} \times S_{\frac{|\nu|}{2}-r}) \rtimes S_2}^{S_{|\nu|-2r}} (1) = \sum_{\substack{i=0 \\ i \text{ even}}}^{\frac{|\nu|}{2}-r} \chi^{(|\nu|-2r-i,i)}.$$

As a class function on $S_{|\nu|-2r}$, this is

$$\begin{aligned} w \mapsto |\{\{A^+, A^-\} \,|\, \{1, \cdots, |\nu|-2r\} &= A^+ \coprod A^-, \\ &|A^+| = |A^-|, w\{A^+, A^-\} = \{A^+, A^-\}\}| \\ = \frac{1}{2}|\{(A^+, A^-) \,|\, \{1, \cdots, |\nu|-2r\} &= A^+ \coprod A^-, |A^+| = |A^-|, w(A^\pm) = A^\pm\}| \\ + \frac{1}{2}|\{(A^+, A^-) \,|\, \{1, \cdots, |\nu|-2r\} &= A^+ \coprod A^-, |A^+| = |A^-|, w(A^\pm) = A^\mp\}|, \end{aligned}$$

so our claim is proved. Thus

$$\sum_{\substack{\text{involutions } w \in Z_{S_{|\nu|}}(\nu) \\ \text{with signed fixed points,} \\ \text{signs reversed under } \nu}} (-1)^{\begin{array}{l}|\{\text{cycles of } \nu \text{ divisible by 4, fixed by } w, \text{ not pointwise}\}| \\ +\frac{1}{2}|\{\text{odd cycles of } \nu \text{ moved by } w\}|\end{array}}$$

$$= \sum_{r=0}^{\frac{|\nu|}{2}} \left(((-1)^r \sum_{\substack{\mu \vdash 2r \\ \mu' \text{ even}}} \chi^\mu) . (\sum_{i=0}^{\frac{|\nu|}{2}-r} (-1)^i \chi^{(|\nu|-2r-i,i)}) \right) (\nu)$$

$$= \sum_{\lambda \vdash |\nu|} d(\lambda') \chi_\nu^\lambda,$$

where, analogously to the above, we can write $d(\lambda')$ as the sum, over signed tableaux of shape λ' and signature $(\frac{|\lambda|}{2}, \frac{|\lambda|}{2})$, of a sign determined by T. As in the proof of (3.3.2), the $(-1)^r$ contribution to the sign is

$$(-1)^{n(\lambda)+|\{\text{even rows of } T \text{ ending } \boxminus\}|}.$$

The $(-1)^i$ contribution is trickier to rephrase in terms of T, but an examination of the bijection

$$\mathrm{Tab}(\lambda' - \mu', (\frac{|\nu|}{2}, \frac{|\nu|}{2})) \xrightarrow{\sim} \coprod_{i=0}^{\frac{|\nu|}{2}-r} \mathrm{Tab}^0(\lambda' - \mu', (|\nu| - 2r - i, i))$$

defined in [14, (9.4)] (in the proof of the Littlewood-Richardson Rule) reveals that the correct reformulation is

$$(-1)^{|\{\text{rows of } T \text{ ending } \boxminus\}| - m(T)},$$

where $m(T)$ is the maximum, over all rows R of T, of the quantity

$$|\{\text{odd rows ending } \boxminus \text{ below or equal to } R\}| - |\{\text{odd rows ending } \boxplus \text{ below or equal to } R\}|$$

(assuming that the rows of T are ordered, as above, so that rows ending $\boxminus$ come below rows ending $\boxplus$ of the same length). Hence

$$d(\lambda') = (-1)^{n(\lambda)} \sum_{\substack{\text{signed tableaux } T \text{ of shape } \lambda', \\ \text{signature } (\frac{|\lambda|}{2}, \frac{|\lambda|}{2})}} (-1)^{|\{\text{odd rows of } T \text{ ending } \boxminus\}| - m(T)},$$

and we are reduced to proving that

$$\sum_{\substack{\text{signed tableaux } T \text{ of shape } \lambda', \\ \text{signature } (\frac{|\lambda|}{2}, \frac{|\lambda|}{2})}} (-1)^{|\{\text{odd rows of } T \text{ ending } \boxminus\}| - m(T)} = \begin{cases} \prod_i (m_{2i}(\lambda') + 1), & \text{if } 2 \mid m_{2i+1}(\lambda'),\ \forall i \\ 0, & \text{otherwise.} \end{cases}$$

Grouping together signed tableaux which differ only in even rows, we see that we may assume that λ' has only odd parts. In lieu of a direct proof, we can deduce this from (3.2.2), proved above. It says that

$$\sum_{\substack{\lambda \vdash |\nu| \\ \lambda \text{ even}}} \chi^\lambda_\nu = \sum_{\substack{\text{involutions } w \in Z_{S_{|\nu|}}(\nu) \\ \text{with signed fixed points,} \\ \text{signs reversed under } \nu}} (-1)^{|\{\text{cycles of } \nu \text{ fixed by } w, \text{ not pointwise}\}|}$$

$$= \sum_{r=0}^{\frac{|\nu|}{2}} \left((\sum_{\substack{\mu \vdash 2r \\ \mu' \text{ even}}} \chi^\mu).(\sum_{i=0}^{\frac{|\nu|}{2}-r} (-1)^i \chi^{(|\nu|-2r-i,i)}) \right)(\nu),$$

whence

$$\sum_{\substack{\text{signed tableaux } T \text{ of shape } \lambda', \\ \text{signature } (\frac{|\lambda|}{2}, \frac{|\lambda|}{2})}} (-1)^{|\{\text{rows of } T \text{ ending } \boxminus\}| - m(T)} = \begin{cases} 1, & \text{if } 2 \mid m_i(\lambda'),\ \forall i \\ 0, & \text{otherwise.} \end{cases}$$

When λ' has only odd parts, this is precisely the statement we want.

It is now easy to prove (4.3.3), since

$$\sum_{\substack{\text{involutions } w \in Z_{S_{|\nu|}}(\nu)\\ \text{fixing no odd cycles of } \nu}} (-1)^{\begin{array}{l}|\{\text{cycles of } \nu \text{ fixed by } w \text{ pointwise}\}|\\ +|\{\text{cycles of } \nu \text{ of length} \equiv 2 \bmod 4 \text{ fixed by } w, \text{ not pointwise}\}|\\ +\frac{1}{2}|\{\text{odd cycles of } \nu \text{ moved by } w\}|\end{array}} \times 2^{|\{\text{cycles of } \nu \text{ fixed by } w \text{ pointwise}\}|}$$

$$= (-1)^{|\{\text{cycles of } \nu\}|} \sum_{\substack{\text{involutions } w \in Z_{S_{|\nu|}}(\nu)\\ \text{with signed fixed points,}\\ \text{signs reversed under } \nu}} (-1)^{\begin{array}{l}|\{\text{cycles of } \nu \text{ divisible by 4, fixed by } w, \text{ not pointwise}\}|\\ +\frac{1}{2}|\{\text{odd cycles of } \nu \text{ moved by } w\}|\end{array}}$$

$$= \epsilon_\nu \sum_{\substack{\lambda \vdash |\nu|\\ 2|m_{2i+1}(\lambda')}} (-1)^{n(\lambda)} (\prod_i (m_{2i}(\lambda')+1)) \chi^\lambda_\nu$$

$$= \sum_{\substack{\lambda \vdash |\nu|\\ 2|m_{2i+1}(\lambda)}} (-1)^{n(\lambda')} (\prod_i (m_{2i}(\lambda)+1)) \chi^\lambda_\nu,$$

as required. Finally, (4.1.2) follows from (3.1.2) by summing over all signatures:

$$\sum_{\text{involutions } w \in Z_{S_{|\nu|}}(\nu)} (-2)^{|\{\text{cycles of } \nu \text{ fixed by } w \text{ pointwise}\}|}$$

$$= (-1)^{|\{\text{cycles of } \nu\}|} \sum_{\substack{\text{involutions } w \in Z_{S_{|\nu|}}(\nu)\\ \text{with signed fixed points,}\\ \text{signs stable under } \nu}} (-1)^{|\{\text{cycles of } \nu \text{ fixed by } w, \text{ not pointwise}\}|}$$

$$= (-1)^{|\nu|} \epsilon_\nu \sum_{\lambda \vdash |\nu|} |\{\text{signed tableaux of shape } \lambda'\}| \chi^\lambda_\nu$$

$$= (-1)^{|\nu|} \sum_{\lambda \vdash |\nu|} |\{\text{signed tableaux of shape } \lambda\}| \chi^\lambda_\nu$$

$$= (-1)^{|\nu|} \sum_{\lambda \vdash |\nu|} (\prod_i (m_i(\lambda)+1)) \chi^\lambda_\nu,$$

and (4.3.2) follows from (3.3.2) in an analogous way:

$$\sum_{\text{involutions } w \in Z_{S_{|\nu|}}(\nu)} (-1)^{\begin{array}{l}|\{\text{even cycles of } \nu \text{ fixed by } w \text{ pointwise}\}|\\ +|\{\text{cycles of } \nu \text{ of length} \equiv 2 \bmod 4 \text{ fixed by } w, \text{ not pointwise}\}|\\ +\frac{1}{2}|\{\text{odd cycles of } \nu \text{ moved by } w\}|\end{array}} \times 2^{|\{\text{cycles of } \nu \text{ fixed by } w \text{ pointwise}\}|}$$

$$= \epsilon_\nu \sum_{\lambda \vdash |\nu|} (-1)^{n(\lambda)} \left| \left\{ \begin{array}{c} \text{signed tableaux of shape } \lambda',\\ \text{fixed under inverting all rows} \end{array} \right\} \right| \chi^\lambda_\nu$$

$$= \sum_{\substack{\lambda \vdash |\nu|\\ 2|m_{2i}(\lambda)}} (-1)^{n(\lambda')} (\prod_i (m_{2i+1}(\lambda)+1)) \chi^\lambda_\nu.$$

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