SUBGROUPS OF THE FREE SEMIGROUP ON A BIORDERED SET IN WHICH PRINCIPAL IDEALS ARE SINGLETONS

Brett McElwee[1]

School of Mathematics and Statistics, University of Sydney,
NSW 2006, Australia
[2]GMO Woolley Ltd, One Angel Court, Throgmorton Street,
London EC2R 7HJ, United Kingdom

Abstract

Easdown has conjectured that the subgroups of the free semigroup on an arbitrary biordered set are free. In this note a weaker conjecture is verified. It is shown that the subgroups of the free semigroup on a biordered set in which principal ideals are singletons are free. In addition, an expression is given for the ranks of the maximal subgroups. This generalizes a result due to Pastijn which involves the free semigroup on a rectangular biset.

1 INTRODUCTION

The biordered set (we use 'biset' for brevity) of an arbitrary semigroup S is the set of idempotents of S equipped with a certain partial product. This partial product is the restriction of the product of S to those pairs (e, f) of

[1]email: `brett_mcelwee@gmow.co.uk`
[2]current postal address, November 2001

idempotents such that $ef = e, ef = f, fe = f$ or $fe = e$. It is readily verified that the resultant structure is indeed a partial algebra.

Nambooripad introduced the notion of biset in [4], where he characterized the bisets of regular semigroups. Easdown subsequently characterized the bisets of arbitrary semigroups in [2]. Specifically, he demonstrated that if E is a partial algebra satisfying certain second order axioms, which had been identified by Nambooripad [4], then the semigroup which is generated by the elements of E, subject to the defining relations given by the product of E, will have E as its biset. Consequently, this is termed the *free semigroup on* E, and the characterizing partial algebras are what we term the *abstract bisets*, or simply *bisets*.

Now Easdown has conjectured that the subgroups of the free semigroup on a biset are free. Pastijn [5] has confirmed this in the special case that the biset is rectangular, and in addition he has obtained, in that case, the ranks of the maximal subgroups. In this note we generalize Pastijn's observations by showing that the subgroups of the free semigroup on a locally trivial biset are free, and by giving an expression for the ranks of those subgroups which are maximal.

2 PRELIMINARIES

Note that when we consider an arbitrary abstract biset there is no loss of generality, due to Easdown's characterization, if the reader considers this to be the biset of an arbitrary semigroup. Thus we can avoid giving the defining axioms of an abstract biset.

Now let us have a precise definition of the free semigroup on a given biset. Let E be an arbitrary biset. Denote by E^+ the free semigroup on E regarded as an alphabet, so that E^+ comprises nonempty words over E under the binary operation of concatenation. Let R be the set of relations of the form $ef = g$, where $e, f, g \in E$ and the biset product ef exists and equals g in E. The *free semigroup on the biset* E, denoted by F_E, is the quotient E^+/γ where γ is the congruence on E^+ generated by R. We denote by $\overline{w}$ the γ-equivalence class of $w \in E^+$.

The free semigroups of concern here are those defined on locally trivial bisets. To say that a biset E is *locally trivial* is simply to say that its principal ideals are singletons, and the reader who is acquainted with bisets can readily verify that E is locally trivial if and only if $ef = e$ lies in R precisely when $fe = f$ does, and $fe = e$ lies in R precisely when $ef = f$ does. Moreover, one can easily verify that any set E equipped with a partial multiplication with domain D_E and equivalence relations $\mathcal{R}$ and $\mathcal{L}$ such that $D_E = \mathcal{R} \cup \mathcal{L}$, $\mathcal{R} \cap \mathcal{L} = \mathrm{id}_E$, and

$$(e, f) \in \mathcal{R}\ [\mathcal{L}] \qquad \text{implies} \qquad ef = f\ [e] \quad \text{and} \quad fe = e\ [f]\ ,$$

satisfies the biset axioms and is locally trivial. It follows that the free semigroup on a locally trivial biset E is simply the semigroup generated by the elements of E subject to, firstly, the idempotent relations $ee = e$, and secondly, any collection of relations satisfying the aforementioned conditions.

Now we can obviously exploit Green's $\mathcal{L}$ and $\mathcal{R}$-class notation when dealing with E by taking $\mathcal{L}$ and $\mathcal{R}$ to be Green's relations restricted to idempotents. We can also exploit his $\mathcal{D}$-class notation by taking $\mathcal{D}$ to be the equivalence on E generated by $\mathcal{L} \cup \mathcal{R}$.

Let E be a locally trivial biset for the duration. We want to demonstrate the uniqueness of the reduced representatives of the γ-classes of E^+, whereby we say a word over E is *reduced* precisely when it contains no consecutive $\mathcal{L}$ or $\mathcal{R}$-related letters.

Our demonstration is an adaptation of an existing proof, due to Artin [1] and van der Waerden [6], of the uniqueness of reduced representative words of free group elements. Firstly, we define a function $[\cdot]$ which operates on the tails of words according to the relations of F_E. Let w be a word, possibly empty, over E. Let e be empty or an element of E, though in the former case only if w is also empty, and let f and g be elements of E. Put

$$[wefg] = \begin{cases} wefg \text{ if } f \not\mathcal{L}\, g \text{ and } f \not\mathcal{R}\, g; \\ wef \quad \text{if } f \,\mathcal{L}\, g \text{ and } f \not\mathcal{R}\, g; \\ weg \quad \text{if } f \,\mathcal{R}\, g \text{ and } e \not\mathcal{L}\, g; \\ we \quad\ \ \text{if } f \,\mathcal{R}\, g \text{ and } e \,\mathcal{L}\, g, \end{cases}$$

where $e \not\mathcal{L}\, g$ is taken to be true if e is the empty word. Now define a function Υ by

$$\Upsilon(\epsilon) = \epsilon$$
$$\Upsilon(e) = e$$
$$\Upsilon(we) = [\Upsilon(w)e],$$

for all e in E, for all nonempty words w over E, and for ϵ the empty word. By a simple induction, for any nonempty word w, $\Upsilon(w)$ is a reduced word the final letter of which is $\mathcal{L}$-related to the final letter of w.

Lemma 2.1 *Let w be a word, possibly empty, over E. Let e be either the empty word or an element of E, though in the former case only if w is also empty, and let f and g be elements of E such that ef is reduced and $f\,\mathcal{L}\,g$. We have $[wefg] = wef$.*

Proof. If $f\,\mathcal{R}\,g$, then

$$\begin{aligned} [wefg] &= weg & [e\,\mathcal{L}\,g \Rightarrow e\,\mathcal{L}\,f, \text{ a contradiction}] \\ &= wef. & [f\,\mathcal{L}\,g\,\mathcal{R}\,f \text{ so that } f = g] \end{aligned}$$

If $f \not\mathcal{R}\, g$, then $[wefg] = wef$ and we are done. $\square$

Lemma 2.2 *Let w be a word, possibly empty, over E. Let e be either the empty word or an element of E, though in the former case only if w is also empty, and let f, g and h be elements of E such that ef is reduced and $g\,\mathcal{R}\,h$. We have $[[wefg]h] = [wefh]$.*

Proof. If $f\,\mathcal{L}\,g$, then the result follows by 2.1. So suppose that $f \not\mathcal{R}\, g$. Consider the case that $f \not\mathcal{R}\, g$ and $f\,\mathcal{L}\,h$. In that case, $[[wefg]h] = [wefgh] = wef$, and $wef = [wefh]$, as if $f\,\mathcal{R}\,h$ were to hold, then $f\,\mathcal{R}\,g$ would hold, yielding a contradiction. Thus that case holds, and the remaining cases can be similarly verified. $\square$

Lemma 2.3 *There exists a unique reduced word in each equivalence class of F_E.*

Proof. Let u and v be words over E, and let f and g be elements of E such that either $f \,\mathcal{L}\, g$ or $f \,\mathcal{R}\, g$. Suppose that $f \,\mathcal{R}\, g$. As $\Upsilon(u)$ is reduced, by 2.2 we have

$$\Upsilon(ufg) = [[\Upsilon(u)f]g] = [\Upsilon(u)g] = \Upsilon(ug).$$

If v is empty then immediately $\Upsilon(ufgv) = \Upsilon(ufv)$. If $v = v'x$ for some word v' and letter x then, by an inductive hypothesis,

$$\Upsilon(ufgv) = [\Upsilon(ufgv')x] = [\Upsilon(ugv')x] = \Upsilon(ugv).$$

On the other hand, suppose that $f \,\mathcal{L}\, g$. By 2.1 we have $[[\Upsilon(u)f]g] = [\Upsilon(u)f]$, as $\Upsilon(u)$ is reduced and its last letter h satisfies $h \,\mathcal{L}\, f \,\mathcal{L}\, g$. Thus $\Upsilon(ufgv) = \Upsilon(ufv)$. It follows that the restriction of Υ to any γ-class is a constant map. As Υ fixes reduced words, the result follows. □

3 THE SUBGROUPS

Let H be an $\mathcal{H}$-class of F_E which contains an idempotent. Denote by l_H the number of $\mathcal{L}_{\mathrm{F}_E}$-classes in the $\mathcal{D}_{\mathrm{F}_E}$-class in which H resides, and denote by r_H the number of $\mathcal{R}_{\mathrm{F}_E}$-classes in the $\mathcal{D}_{\mathrm{F}_E}$-class. Pastijn demonstrates in [5] that if E comprises a single solid $\mathcal{D}$-class, that is, if it is a rectangular biset, then the subgroups of F_E are free, and if E is, in addition, finite, then the maximal subgroup H has rank $(l_H - 1)(r_H - 1)$. Pastijn demonstrates this by employing Rees-matrix semigroups. We demonstrate our result by showing that our subgroups are fundamental groups of graphs.

For a graph Γ and one of its vertices v, denote by $\pi(\Gamma, v)$ the fundamental group based at v. We denote by $\overline{(v, v_1, \ldots v_n, v)}$ the equivalence class of a loop $(v, v_1, \ldots v_n, v)$. The following fact is well known.[1]

Theorem 3.1 *For a finite connected graph* Γ*, each fundamental group* $\pi(\Gamma, v)$ *is free and of rank* $|\mathrm{E}(\Gamma)| - |\mathrm{V}(\Gamma)| + 1$. □

For H as above, denote by d_H the total number of idempotent-free $\mathcal{H}$-classes in the $\mathcal{D}_{\mathrm{F}_E}$-class in which H resides. This number is easily determined because the only idempotents of F_E are the images $\overline{e}$ of elements e of E. The following is our generalization of Pastijn's result.

[1]See [3], for example.

Theorem 3.2 *All subgroups of* F_E *are free, and in the case that* E *is finite, each maximal subgroup* H *has rank* $(l_H - 1)(r_H - 1) - d_H$.

Proof. Since all subgroups of a free group are free, it remains to show that all the $\mathcal{H}_{\mathrm{F}_E}$-classes which contain idempotents are free and that the expression for the rank holds in the case that E is finite. Let H be the $\mathcal{H}_{\mathrm{F}_E}$-class of an idempotent. Let e be the preimage of the idempotent and let D the $\mathcal{D}_E$-class of e. Let the set of $\mathcal{R}$-classes of D be well-ordered by an order $\underset{\mathcal{R}}{\leq}$. Observe that this induces a well ordering on each $\mathcal{L}$-class L by taking

$$f \underset{L}{\leq} g \text{ precisely when } R_f \underset{\mathcal{R}}{\leq} R_g.$$

Now do the same with $\mathcal{L}$ and $\mathcal{R}$ interchanged. For an $\mathcal{L}$-class L of D and $\mathcal{R}$-class R of D, denote by m_L and m_R the least elements of L and R respectively.

Form a graph Γ with vertices the elements of D, and

$$\mathrm{E}(\Gamma) = \big\{\{x, m_{L_x}\} | \, x \neq m_{L_x}\big\} \bigcup \big\{\{x, m_{R_x}\} | \, x \neq m_{R_x}\big\}.$$

We show that $\pi(\Gamma, e)$ is isomorphic to H. Define the map $\zeta \colon \pi(\Gamma, e) \to H$ by putting

$$\zeta\left(\overline{(e, x_1, \ldots, x_n, e)}\right) = \overline{ex_1 \cdots x_n e},$$

for all $\overline{(e, x_1, \ldots, x_n, e)}$ in $\pi(\Gamma, e)$, with the understanding that (e, e) denotes the empty path based at e. The map is evidently well-defined. That it maps into H follows by Green's Lemma and a simple induction. That it is a homomorphism is obvious. So in showing that ζ is an isomorphism, it remains to show that it is injective and surjective.

Consider any $p \neq 1$ in $\pi(\Gamma, e)$. There exists a reduced representative of p, and it can be written $\pi_1 \pi_2 \cdots \pi_{2n}$, where $n \geq 1$, π_i is nonempty, for $i = 1, \ldots, 2n-1$, and one of the following holds.

(i) For all odd i, π_i is contained wholly within an $\mathcal{L}$-class, and for all even j, π_j is contained wholly within an $\mathcal{R}$-class.

(ii) For all even i, π_i is contained wholly within an $\mathcal{L}$-class, and for all odd j, π_j is contained wholly within an $\mathcal{R}$-class.

Evidently none of the subgraphs corresponding to $\mathcal{L}$-classes or $\mathcal{R}$-classes can contain a reduced loop, whence each π_i must not be a loop. It follows that $n \geq 2$. It also follows that if (i) holds, then the length of the reduced word in the γ-class $\zeta(p)$, as given by 2.3, is either n or $n+1$, depending on whether π_{2n} is empty or not. Finally, it follows that if (ii) holds, then the length of the reduced word is n. It follows that $\zeta(p) \neq \overline{e}$, whence ζ is injective.

We show surjectivity. Let $e_1, \ldots, e_n$ be elements of E such that $e_1 \cdots e_n$ is reduced and $\overline{e_1 \cdots e_n} \,\mathcal{H}\, \overline{e}$. If $n = 1$, then $e_1 = e$, as $E \cong E(\mathrm{F}_E)$, whence $\overline{e_1 \cdots e_n}$ is the image of $\overline{(e, e)}$. So suppose not. We have a nonempty reduced word w over E such that $we_1 \cdots e_n \gamma e$. Write $w = f_1 \cdots f_t$, for some f_i in E. By the proof of 2.3, we can transform $f_1 \cdots f_t e_1 \cdots e_n$ into e simply by deleting letters. We want to show that for $i = 1, \ldots, n-1$, there exists f_j such that $e_i \,\mathcal{L}\, f_j \,\mathcal{R}\, e_{i+1}$. Suppose that $f_t \,\mathcal{R}\, e_1$. It cannot be that $f_{t-1} \,\mathcal{R}\, e_1$, for that would contradict the fact that $f_1 \cdots f_t$ is reduced. As $f_1 \cdots f_{t-1} e_1 \cdots e_n \;\gamma\; e$ and $n > 1$, we conclude that $t > 1$ and $f_{t-1} \,\mathcal{L}\, e_1$. So we see that without loss of generality we can suppose that $f_t \,\mathcal{L}\, e_1$. So we have $f_1 \ldots f_t e_2 \cdots e_n \;\gamma\; e$. Now if $f_t \,\mathcal{L}\, e_2$, then $e_2 \,\mathcal{L}\, e_1$, contradicting the fact that $e_1 \cdots e_n$ is reduced. So $f_t \,\mathcal{R}\, e_2$. As $f_1 \cdots f_{t-1} e_2 \cdots e_n \;\gamma\; e$, an inductive hypothesis applies and we are done. Now for $i = 1, \ldots, n-1$, we denote by x_i the letter of $f_1 \cdots f_t$ which is $\mathcal{L}$-related to e_i and $\mathcal{R}$-related to e_{i+1}. Obviously $e_1 x_1 \cdots e_{n-1} x_{n-1} e_n \;\gamma\; e_1 \cdots e_n$, and by Green's Lemma and a simple induction, $e_1 \,\mathcal{R}\, e \,\mathcal{L}\, e_n$. Thus

$$ee_1 x_1 \cdots e_{n-1} x_{n-1} e_n e \;\gamma\; e_1 \cdots e_n,$$

and consecutive letters in the former word are either $\mathcal{L}$ or $\mathcal{R}$-related. So after possibly inserting some letters into that word, we have a word $ey_1 \cdots y_k e$ such that $(e, y_1, \ldots, y_k, e)$ is a loop based at e and

$$ey_1 \cdots y_k e \;\gamma\; e_1 \cdots e_n.$$

Thus ζ is surjective.

Now by 3.1, it remains only to show that the expression holds in the finite case. Let E be finite. We have

$$
\begin{aligned}
(l_H-1)(r_H-1)-d_H &= (l_H r_H - d_H) - l_H - r_H + 1 \\
&= |\mathrm{V}(\Gamma)| - l_H - r_H + 1 \\
&= (|\mathrm{V}(\Gamma)| - l_H + |\mathrm{V}(\Gamma)| - r_H) - |\mathrm{V}(\Gamma)| + 1 \\
&= |\mathrm{E}(\Gamma)| - |\mathrm{V}(\Gamma)| + 1,
\end{aligned}
$$

and so we are done. □

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