Representations of reflection algebras

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Abstract

We study a class of algebras $B(n,l)$ associated with integrable models with boundaries. These algebras can be identified with coideal subalgebras in the Yangian for $gl(n)$. We construct an analog of the quantum determinant and show that its coefficients generate the center of $B(n,l)$. We develop an analog of Drinfeld's highest weight theory for these algebras and give a complete description of their finite-dimensional irreducible representations.

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1 Introduction

In his paper [15] Sklyanin introduced a class of algebras associated with integrable models with boundaries. We call them *reflection algebras* in this paper. His approach was inspired by Cherednik's scattering theory [3] for factorized particles on the half-line. In [15] commutative subalgebras in the reflection algebras (in the case of two dimensions) are constructed and the algebraic Bethe ansatz is described. Moreover, an analog of the *quantum determinant* for these algebras was constructed and some properties of the highest weight representations are discussed. Similar classes of algebras were studied in Kulish–Sklyanin [8], Kuznetsov–Jørgensen–Christiansen [9], Koornwinder–Kuznetsov [6]. Recently, algebras of this kind were discussed in the physics literature in connection with the NLS model; see Liguori–Mintchev–Zhao [10], Mintchev–Ragoucy–Sorba [11].

In this paper we consider a family of reflection algebras $\mathcal{B}(n,l)$. They are defined as associative algebras whose generators satisfy two types of relations: the reflection-type equation and the unitary condition; see (2.9) and (2.10) below. The unitary condition allows us to identify $\mathcal{B}(n,l)$ with a subalgebra in the $\mathfrak{gl}(n)$-Yangian $\mathrm{Y}(n)$; see Theorem 3.1. This condition was not explicitly used in [15]. If we omit it we get a larger algebra $\widetilde{\mathcal{B}}(n,l)$, and it turns out that $\mathcal{B}(n,l)$ is a quotient of $\widetilde{\mathcal{B}}(n,l)$ by an ideal generated by some central elements. The consequence of this fact is that the finite-dimensional irreducible representations of both algebras $\mathcal{B}(n,l)$ and $\widetilde{\mathcal{B}}(n,l)$ are essentially the same.

On the other hand, the advantage of having $\mathcal{B}(n,l)$ as a subalgebra of the Yangian is in the fact that $\mathcal{B}(n,l)$ turns out to be a (left) coideal in the Hopf algebra $\mathrm{Y}(n)$; see Proposition 3.3. This allows us to regard the tensor product $L\otimes V$ of an $\mathrm{Y}(n)$-module L and a $\mathcal{B}(n,l)$-module V as a $\mathcal{B}(n,l)$-module; cf. [15, Proposition 2].

We show that the center of $\mathcal{B}(n,l)$ is generated by the coefficients of an analog of the quantum determinant which we call, following [13], the *Sklyanin determinant*. We derive a formula which expresses the Sklyanin determinant in terms of the quantum determinant for the Yangian $\mathrm{Y}(n)$.

The aforementioned properties of the algebras $\mathcal{B}(n,l)$ exhibit much analogy with the twisted Yangians introduced by Olshanski [14]; see also [13] for a detailed exposition. This also turns out to be true, at some extent, for the representation theory; cf. [12]. We prove here that, as for the twisted Yangians, the Drinfeld highest weight theory [5] is applicable to the algebras $\mathcal{B}(n,l)$. Every finite-dimensional irreducible representation of $\mathcal{B}(n,l)$ is highest weight, and given an irreducible highest weight module we produce necessary and sufficient conditions for it to be finite-dimensional. These conditions are expressed in terms of the Drinfeld polynomials in a way similar to [5] and [12] for the Yangians and twisted Yangians. Note, however, an essential difference: here *all* Drinfeld polynomials must satisfy a symmetry condition; see Theorem 4.6. In particular, all of them have even degree.

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2 Definitions and preliminaries

Recall first the definition of the $\mathfrak{gl}(n)$-*Yangian* $\mathrm{Y}(n)$; see e.g. [7], [5]. We follow the notation from [13] where a detailed account of the properties of $\mathrm{Y}(n)$ is given. The *Yangian* $\mathrm{Y}(n)$ is the complex associative algebra with the generators $t_{ij}^{(1)}, t_{ij}^{(2)}, \dots$ where $1 \le i, j \le n$, and the defining relations

$$[t_{ij}(u), t_{rs}(v)] = \frac{1}{u-v}\Big(t_{rj}(u)t_{is}(v) - t_{rj}(v)t_{is}(u)\Big), \tag{2.1}$$

where

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)}u^{-1} + t_{ij}^{(2)}u^{-2} + \cdots \in \mathrm{Y}(n)[[u^{-1}]]$$

and u is a formal (commutative) variable. Introduce the matrix

$$T(u) = \sum_{i,j=1}^{n} t_{ij}(u) \otimes E_{ij} \in \mathrm{Y}(n)[[u^{-1}]] \otimes \mathrm{End}\, \mathbb{C}^n, \tag{2.3}$$

where the E_{ij} are the standard matrix units. Then the relations (2.1) are equivalent to the single *ternary* (or *RTT*) relation

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v). \tag{2.4}$$

Here $T_1(u)$ and $T_2(u)$ are regarded as elements of $\mathrm{Y}(n)[[u^{-1}]] \otimes \mathrm{End}\, \mathbb{C}^n \otimes \mathrm{End}\, \mathbb{C}^n$, the subindex of $T(u)$ indicates to which copy of $\mathrm{End}\, \mathbb{C}^n$ this matrix corresponds, and

$$R(u) = 1 - Pu^{-1}, \qquad P = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji} \in (\mathrm{End}\, \mathbb{C}^n)^{\otimes 2}.$$

Now we introduce the reflection algebras. Fix a decomposition of the parameter n into the sum of two nonnegative integers, $n = k + l$. Denote by G the diagonal $n \times n$-matrix

$$G = \mathrm{diag}(\varepsilon_1, \dots, \varepsilon_n) \tag{2.6}$$

where $\varepsilon_i = 1$ for $1 \le i \le k$, and $\varepsilon_i = -1$ for $k+1 \le i \le n$. The *reflection algebra* $\mathcal{B}(n, l)$ is a unital associative algebra with the generators $b_{ij}^{(r)}$ where r runs over positive integers and i and j satisfy $1 \le i, j \le n$. To write down the defining relations introduce formal series

$$b_{ij}(u) = \sum_{r=0}^{\infty} b_{ij}^{(r)} u^{-r}, \qquad b_{ij}^{(0)} = \delta_{ij}\, \varepsilon_i \tag{2.7}$$

and combine them into the matrix

$$B(u) = \sum_{i,j=1}^{n} b_{ij}(u) \otimes E_{ij} \in \mathcal{B}(n,l)[[u^{-1}]] \otimes \operatorname{End} \mathbb{C}^n.$$

The defining relations are given by the reflection-type *quaternary relation* [15]

$$R(u-v)B_1(u)R(u+v)B_2(v) = B_2(v)R(u+v)B_1(u)R(u-v) \tag{2.9}$$

together with the *unitary condition*

$$B(u)B(-u) = 1, \tag{2.10}$$

where we have used the notation of (2.4). Rewriting (2.9) and (2.10) in terms of the matrix elements we obtain, respectively,

$$\begin{aligned} [b_{ij}(u), b_{rs}(v)] &= \frac{1}{u-v}\Big(b_{rj}(u)b_{is}(v) - b_{rj}(v)b_{is}(u)\Big) \\ &+ \frac{1}{u+v}\Big(\delta_{rj}\sum_{a=1}^{n} b_{ia}(u)b_{as}(v) - \delta_{is}\sum_{a=1}^{n} b_{ra}(v)b_{aj}(u)\Big) \\ &- \frac{1}{u^2-v^2}\,\delta_{ij}\Big(\sum_{a=1}^{n} b_{ra}(u)b_{as}(v) - \sum_{a=1}^{n} b_{ra}(v)b_{as}(u)\Big) \end{aligned} \tag{2.11}$$

and

$$\sum_{a=1}^{n} b_{ia}(u)b_{aj}(-u) = \delta_{ij}. \tag{2.12}$$

We shall also be using the algebra $\widetilde{\mathcal{B}}(n,l)$ which is defined in the same way as $\mathcal{B}(n,l)$ but with the unitary condition (2.10) dropped. (This corresponds to Sklyanin's original definition [15]). We use the same notation $b_{ij}^{(r)}$ for the generators of $\widetilde{\mathcal{B}}(n,l)$. As we shall see in the following proposition, the algebra $\mathcal{B}(n,l)$ is isomorphic to a quotient of $\widetilde{\mathcal{B}}(n,l)$ by an ideal whose generators are central elements; see also [11].

Proposition 2.1 *In the algebra $\widetilde{\mathcal{B}}(n,l)$ the product $B(u)B(-u)$ is a scalar matrix*

$$B(u)B(-u) = f(u)\,1, \tag{2.13}$$

where $f(u)$ is an even series in u^{-1} whose all coefficients are central in $\widetilde{\mathcal{B}}(n,l)$.

Proof. Multiply both sides of (2.11) by $u^2 - v^2$ and put $v = -u$. We obtain

$$
\begin{aligned}
2u\Big(\delta_{rj}\sum_{a=1}^{n} b_{ia}(u)b_{as}(-u) - \delta_{is}\sum_{a=1}^{n} b_{ra}(-u)b_{aj}(u)\Big)\\
= \delta_{ij}\Big(\sum_{a=1}^{n} b_{ra}(u)b_{as}(-u) - \sum_{a=1}^{n} b_{ra}(-u)b_{as}(u)\Big).
\end{aligned}
\tag{2.14}
$$

Choosing appropriate indices i, j, r, s, it is easy to see that $B(u)B(-u) = B(-u)B(u)$ and that this matrix is scalar. Thus, (2.13) holds for an even series $f(u)$. Now multiply both sides of (2.9) by $B_2(-v)$ from the right:

$$
R(u-v)B_1(u)R(u+v)f(v) = B_2(v)R(u+v)B_1(u)R(u-v)B_2(-v). \tag{2.15}
$$

Applying (2.9) to the right hand side we write it as

$$
\begin{aligned}
B_2(v)R(u+v)B_1(u)R(u-v)B_2(-v) &= B_2(v)B_2(-v)R(u-v)B_1(u)R(u+v)\\
&= f(v)R(u-v)B_1(u)R(u+v).
\end{aligned}
\tag{2.16}
$$

This shows that $f(u)$ is central. □

We would like to comment on the relevance of the choice of the initial matrix G in the expansion

$$
B(u) = G + \sum_{r=1}^{\infty} B^{(r)}u^{-r}; \tag{2.17}
$$

see also [11]. We could take G to be an arbitrary nondegenerate matrix. However, as the proof of Proposition 2.1 shows, the quaternary relation implies that G^2 is a scalar matrix. Since G is nondegenerate, the scalar is nonzero. On the other hand, as can be easily seen, for any constant c and any nondegenerate matrix A the transformations

$$
B(u) \mapsto c\,B(u) \qquad \text{and} \qquad B(u) \mapsto AB(u)A^{-1} \tag{2.18}
$$

preserve the quaternary relation. Accordingly, the matrix G is then transformed as $G \mapsto c\,G$ and $G \mapsto AGA^{-1}$. Therefore, we may assume that $G^2 = 1$ and using an appropriate matrix A we can bring G to the form (2.6).

Note also that both the quaternary relation and unitary condition are preserved by the change of sign $B(u) \mapsto -B(u)$. This implies that the algebras $\mathcal{B}(n, l)$ and $\mathcal{B}(n, n-l)$ are isomorphic. In what follows we assume that the parameter l satisfies $0 \le l \le n/2$.

It is immediate from the definitions of the algebras $\mathcal{B}(n, l)$ and $\widetilde{\mathcal{B}}(n, l)$ that given any formal series $g(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ the mapping

$$
B(u) \mapsto g(u)\,B(u) \tag{2.19}
$$

is an automorphism of the algebra $\widetilde{\mathcal{B}}(n,l)$. If $g(u)$ satisfies $g(u)\,g(-u)=1$ then (2.19) is an automorphism of $\mathcal{B}(n,l)$.

3 Algebraic structure of $\mathcal{B}(n,l)$

Here we show that each $\mathcal{B}(n,l)$ can be identified with a coideal subalgebra in the Yangian $\mathrm{Y}(n)$ and prove an analog of the Poincaré–Birkhoff–Witt theorem for the algebra $\mathcal{B}(n,l)$. Then we describe its center using an analog of the quantum determinant.

3.1 Embedding $\mathcal{B}(n,l)\hookrightarrow \mathrm{Y}(n)$

Denote by $T^{-1}(u)$ the inverse matrix for $T(u)$; see (2.3). It can be easily seen that the matrix $T^{-1}(-u)$ satisfies the ternary relation (2.4). This implies that the mapping $T(u)\to T^{-1}(-u)$ defines an algebra automorphism of the Yangian $\mathrm{Y}(n)$; cf. [13]. The following connection between $\mathcal{B}(n,l)$ and $\mathrm{Y}(n)$ was indicated in [15].

Theorem 3.1 *The mapping*

$$\varphi: B(u)\mapsto T(u)\,G\,T^{-1}(-u) \tag{3.1}$$

defines an embedding of the algebra $\mathcal{B}(n,l)$ into the Yangian $\mathrm{Y}(n)$.

Proof. First we verify that φ is an algebra homomorphism. Denote the matrix $T(u)\,G\,T^{-1}(-u)$ by $\widetilde{B}(u)$. We obviously have $\widetilde{B}(u)\widetilde{B}(-u)=1$ and so (2.10) is satisfied. Further, we have

$$\begin{aligned}R(u-v)&\widetilde{B}_1(u)R(u+v)\widetilde{B}_2(v)\\ &=R(u-v)T_1(u)\,G_1\,T_1^{-1}(-u)R(u+v)T_2(v)\,G_2\,T_2^{-1}(-v).\end{aligned} \tag{3.2}$$

We find from the ternary relation (2.4) that

$$T_1^{-1}(-u)R(u+v)T_2(v)=T_2(v)R(u+v)T_1^{-1}(-u). \tag{3.3}$$

Therefore, since G_i commutes with $T_j(u)$ for $i\neq j$ we bring the expression (3.2) to the form

$$R(u-v)T_1(u)T_2(v)\,G_1R(u+v)\,G_2\,T_1^{-1}(-u)\,T_2^{-1}(-v). \tag{3.4}$$

One easily verifies that

$$R(u-v)G_1R(u+v)G_2=G_2R(u+v)G_1R(u-v). \tag{3.5}$$

Applying (2.4) and (3.5) we write (3.4) as

$$T_2(v)T_1(u)G_2\,R(u+v)\,G_1\,T_2^{-1}(-v)\,T_1^{-1}(-u)R(u-v), \tag{3.6}$$

which coincides with $\widetilde{B}_2(v)R(u+v)\widetilde{B}_1(u)R(u-v)$ due to (3.3). Thus, $\widetilde{B}(u)$ satisfies the quaternary relation (2.9).

We now show that φ has trivial kernel. The Yangian $\mathrm{Y}(n)$ admits two natural filtrations; see [13]. Here we use the one defined by setting $\deg_1 t_{ij}^{(r)} = r$. Similarly, we define the filtration on the algebra $\mathcal{B}(n,l)$ by $\deg_1 b_{ij}^{(r)} = r$. Let us first verify that the homomorphism φ is filtration-preserving. By definition, the matrix elements of $\widetilde{B}(u)$ are expressed as

$$\widetilde{b}_{ij}(u) = \sum_{a=1}^{n} \varepsilon_a t_{ia}(u) t'_{aj}(-u), \tag{3.7}$$

where the $t'_{ij}(u)$ denote the matrix elements of $T^{-1}(u)$. Inverting the matrix $T(u)$ we come to the following expression for $t'_{ij}(u)$:

$$t'_{ij}(u) = \delta_{ij} + \sum_{k=1}^{\infty} (-1)^k \sum_{a_1,\dots,a_{k-1}=1}^{n} t^{\circ}_{ia_1}(u) t^{\circ}_{a_1a_2}(u) \cdots t^{\circ}_{a_{k-1}j}(u), \tag{3.8}$$

where $t^{\circ}_{ij}(u) = t_{ij}(u) - \delta_{ij}$. Taking the coefficient at u^{-r} for $r \geq 1$ we get

$$t'^{(r)}_{ij} = \sum_{k=1}^{r} (-1)^k \sum_{a_1,\dots,a_{k-1}=1}^{n} \sum_{r_1+\dots+r_k=r} t^{(r_1)}_{ia_1} t^{(r_2)}_{a_1a_2} \cdots t^{(r_k)}_{a_{k-1}j}, \tag{3.9}$$

with the last sum taken over positive integers r_i. Therefore, we find from (3.7) that the degree of $\widetilde{b}_{ij}^{(r)}$ does not exceed r. Hence, φ is filtration-preserving and it defines a homomorphism of the corresponding graded algebras

$$\mathrm{gr}_1\, \mathcal{B}(n,l) \to \mathrm{gr}_1\, \mathrm{Y}(n). \tag{3.10}$$

Let $\overline{t}_{ij}^{(r)}$ denote the image of $t_{ij}^{(r)}$ in the r-th component of $\mathrm{gr}_1\, \mathrm{Y}(n)$. The algebra $\mathrm{gr}_1\, \mathrm{Y}(n)$ is obviously commutative, and it was proved in [13, Theorem 1.22] that the elements $\overline{t}_{ij}^{(r)}$ are its algebraically independent generators.

On the other hand, by the defining relations (2.11), the algebra $\mathrm{gr}_1\, \mathcal{B}(n,l)$ is also commutative. Denote by $\overline{b}_{ij}^{(r)}$ the image of $b_{ij}^{(r)}$ in the r-th component of $\mathrm{gr}_1\, \mathcal{B}(n,l)$. We find from the unitary condition (2.12) that the elements

$$\begin{aligned} &\overline{b}_{ij}^{(2p-1)}, \qquad 1 \leq i,j \leq k \qquad \text{or} \qquad k+1 \leq i,j \leq n, \\ &\overline{b}_{ij}^{(2p)}, \qquad 1 \leq i \leq k < j \leq n \qquad \text{or} \qquad 1 \leq j \leq k < i \leq n, \end{aligned} \tag{3.11}$$

with p running over positive integers, generate the algebra $\mathrm{gr}_1\, \mathcal{B}(n,l)$. Now, by (3.7), the image of $\overline{b}_{ij}^{(r)}$ under the homomorphism (3.10) has the form

$$\overline{b}_{ij}^{(r)} \mapsto \big(\, \varepsilon_i\, (-1)^{r-1} + \varepsilon_j\, \big)\, \overline{t}_{ij}^{(r)} + (\ \dots\ ), \tag{3.12}$$

where $(\ \dots\ )$ indicates a linear combination of monomials in the elements $\overline{t}_{ij}^{(s)}$ with $s < r$. This implies that the elements (3.11) are algebraically independent which completes the proof. □

The proposition implies the following analog of the Poincaré–Birkhoff–Witt theorem for the algebra $\mathcal{B}(n,l)$.

Corollary 3.2 *Given any total ordering on the elements*

$$\begin{array}{llll} b_{ij}^{(2p-1)}, & 1 \le i,j \le k & \textit{or} & k+1 \le i,j \le n, \\ b_{ij}^{(2p)}, & 1 \le i \le k < j \le n & \textit{or} & 1 \le j \le k < i \le n, \end{array}$$

with $p = 1, 2, \dots,$ the ordered monomials in these elements constitute a basis of the algebra $\mathcal{B}(n,l)$. □

Due to Proposition 3.1, we may regard $\mathcal{B}(n,l)$ as a subalgebra of $\mathrm{Y}(n)$ so that the generators $b_{ij}(u)$ are identified with the elements $\widetilde{b}_{ij}(u)$ given by (3.7). Recall that the Yangian $\mathrm{Y}(n)$ is a Hopf algebra with the coproduct $\Delta : \mathrm{Y}(n) \to \mathrm{Y}(n) \otimes \mathrm{Y}(n)$ defined by

$$\Delta(t_{ij}(u)) = \sum_{a=1}^{n} t_{ia}(u) \otimes t_{aj}(u). \tag{3.14}$$

Proposition 3.3 *The subalgebra $\mathcal{B}(n,l)$ is a left coideal in $\mathrm{Y}(n)$:*

$$\Delta\big(\mathcal{B}(n,l)\big) \subseteq \mathrm{Y}(n) \otimes \mathcal{B}(n,l). \tag{3.15}$$

Proof. It suffices to show that the images of the generators of $\mathcal{B}(n,l)$ under the coproduct are contained in $\mathrm{Y}(n) \otimes \mathcal{B}(n,l)$. Since Δ is an algebra homomorphism, the images of the matrix elements $t'_{ij}(u)$ of $T^{-1}(u)$ are given by

$$\Delta(t'_{ij}(u)) = \sum_{a=1}^{n} t'_{aj}(u) \otimes t'_{ia}(u). \tag{3.16}$$

Therefore, using (3.7) we calculate that

$$\Delta(b_{ij}(u)) = \sum_{a,c=1}^{n} t_{ia}(u) t'_{cj}(-u) \otimes b_{ac}(u), \tag{3.17}$$

completing the proof. □

3.2 Sklyanin determinant

The *quantum determinant* $\mathrm{qdet}\, T(u)$ of the matrix $T(u)$ is a formal series in u^{-1},

$$\mathrm{qdet}\, T(u) = 1 + d_1 u^{-1} + d_2 u^{-2} + \cdots, \qquad d_i \in \mathrm{Y}(n), \tag{3.18}$$

defined by

$$\mathrm{qdet}\, T(u) = \sum_{p \in \mathfrak{S}_n} \mathrm{sgn}\, p \cdot t_{p(1)1}(u) \cdots t_{p(n)n}(u - n + 1). \tag{3.19}$$

The elements $d_1, d_2, \ldots$ are algebraically independent generators of the center of the algebra $\mathrm{Y}(n)$; see e.g. [13] for the proof and for the details of the following R-matrix definition of $\mathrm{qdet}\, T(u)$. Consider the tensor product

$$\mathrm{Y}(n)[[u^{-1}]] \otimes \mathrm{End}\, \mathbb{C}^n \otimes \cdots \otimes \mathrm{End}\, \mathbb{C}^n \tag{3.20}$$

with n copies of $\mathrm{End}\, \mathbb{C}^n$. For complex parameters $u_1, \ldots, u_n$ set

$$R(u_1, \ldots, u_n) = (R_{n-1,n})(R_{n-2,n} R_{n-2,n-1}) \cdots (R_{1n} \cdots R_{12}), \tag{3.21}$$

where we abbreviate $R_{ij} = R_{ij}(u_i - u_j)$, and the subindices enumerate the copies of $\mathrm{End}\, \mathbb{C}^n$ in (3.20). If we specialize to

$$u_i = u - i + 1, \qquad i = 1, \ldots, n \tag{3.22}$$

then $R(u_1, \ldots, u_n)$ becomes the one-dimensional anti-symmetrization operator A_n in the space $(\mathrm{End}\, \mathbb{C}^n)^{\otimes n}$. The quantum determinant $\mathrm{qdet}\, T(u)$ is defined by the relation

$$A_n\, T_1(u) \cdots T_n(u - n + 1) = A_n\, \mathrm{qdet}\, T(u). \tag{3.23}$$

Now, using (2.9) and the relation

$$R_{ij} R_{ir} R_{jr} = R_{jr} R_{ir} R_{ij}, \tag{3.24}$$

(the Yang–Baxter equation) we derive the following relation for the B-matrices:

$$\begin{aligned} R(u_1, \ldots, u_n) B_1(u_1) \widetilde{R}_{12} \cdots \widetilde{R}_{1n} B_2(u_2) \widetilde{R}_{23} \cdots \widetilde{R}_{2n} B_3(u_3) \cdots \widetilde{R}_{n-1,n} B_n(u_n) = \\ B_n(u_n) \widetilde{R}_{n-1,n} \cdots B_3(u_3) \widetilde{R}_{2n} \cdots \widetilde{R}_{23} B_2(u_2) \widetilde{R}_{1n} \cdots \widetilde{R}_{12} B_1(u_1) R(u_1, \ldots, u_n), \end{aligned} \tag{3.25}$$

where $\widetilde{R}_{ij} = R_{ij}(u_i + u_j)$. When we specialize the parameters u_i as in (3.22), the element (3.25) will be equal to the product of the anti-symmetrizer A_n and a series in u^{-1} with coefficients in $\mathcal{B}(n, l)$. We call this series the *Sklyanin determinant* and denote it by $\mathrm{sdet}\, B(u)$. That is, $\mathrm{sdet}\, B(u)$ is defined by

$$A_n\, B_1(u) \widetilde{R}_{12} \cdots \widetilde{R}_{1n} B_2(u - 1) \cdots \widetilde{R}_{n-1,n} B_n(u - n + 1) = A_n\, \mathrm{sdet}\, B(u). \tag{3.26}$$

As follows from the definition, the constant term of sdet $B(u)$ is $\det G = (-1)^l$, so

$$\operatorname{sdet} B(u) = (-1)^l + c_1\, u^{-1} + c_2\, u^{-2} + \cdots, \qquad c_i \in \mathcal{B}(n,l). \tag{3.27}$$

In the next theorem we regard $\mathcal{B}(n,l)$ as a subalgebra in $\mathrm{Y}(n)$; see Theorem 3.1.

Theorem 3.4 *We have the identity*

$$\operatorname{sdet} B(u) = \theta(u)\operatorname{qdet} T(u)\,\big(\operatorname{qdet} T(-u+n-1)\big)^{-1}, \tag{3.28}$$

where

$$\theta(u) = (-1)^l \prod_{i=1}^{k}(2u-2n+2i)\prod_{i=1}^{l}(2u-2n+2i)\prod_{i=1}^{n}\frac{1}{2u-2n+i+1}. \tag{3.29}$$

In particular, all the coefficients of sdet $B(u)$ *are central in* $\mathcal{B}(n,l)$. *Moreover, the odd coefficients* $c_1, c_3, \dots$ *are algebraically independent and generate the center of the algebra* $\mathcal{B}(n,l)$.

Proof. Substitute $B(u) = T(u)\,G\,T^{-1}(-u)$ into (3.26). Applying relation (3.3) repeatedly, we bring the left hand side of (3.26) to the form

$$\begin{aligned} A_n\, T_1(u)\cdots T_n(u-n+1)\, G_1\, \widetilde{R}_{12}\cdots \widetilde{R}_{1n}\, G_2\, \cdots \widetilde{R}_{n-1,n}\, G_n \\ \times T_1^{-1}(-u)\cdots T_n^{-1}(-u+n-1). \end{aligned} \tag{3.30}$$

Now use (3.23), and then note that

$$A_n\, G_1\, \widetilde{R}_{12}\cdots \widetilde{R}_{1n}\, G_2\, \cdots \widetilde{R}_{n-1,n}\, G_n = A_n\, \theta(u), \tag{3.31}$$

for some scalar function $\theta(u)$. Indeed, this follows e.g. from (3.25) where we specialize u_i as in (3.22) and consider the trivial representation of $\mathcal{B}(n,l)$ such that $B(u) \mapsto G$. Furthermore, we have

$$A_n\, T_1^{-1}(-u)\cdots T_n^{-1}(-u+n-1) = A_n\,\big(\operatorname{qdet} T(-u+n-1)\big)^{-1}. \tag{3.32}$$

This follows from (3.23), if we first multiply both sides by $T_n^{-1}(u-n+1)\cdots T_1^{-1}(u)$ from the right, replace u with $-u+n-1$ and then conjugate the left hand side by the permutation of the indices $1,\dots,n$ which sends i to $n-i+1$.

To complete the proof of (3.28) we need to calculate $\theta(u)$. It suffices to find a diagonal matrix element of the operator on the left hand side of (3.31) corresponding to the vector $e_1 \otimes \cdots \otimes e_n$, where the e_i denote the canonical basis of $\mathbb{C}^n$. We have $(n-1)!\, A_n = A_n\, A'_{n-1}$, where A'_{n-1} is the anti-symmetrizer in the tensor product of the copies of $\operatorname{End}\mathbb{C}^n$ corresponding to the indices $2,\dots,n$. Note that A'_{n-1} commutes with G_1. Furthermore, we have the identity

$$A'_{n-1}\,\widetilde{R}_{12}\cdots\widetilde{R}_{1n}=\widetilde{R}_{1n}\cdots\,\widetilde{R}_{12}A'_{n-1}, \tag{3.33}$$

which easily follows from (3.24). This allows us to use induction on n to find the matrix element and to perform the calculation of $\theta(u)$ which is now straightforward.

Using (3.28) we can conclude that all the coefficients of $\operatorname{sdet}B(u)$ belong to the center of the Yangian $\mathrm{Y}(n)$, and hence to the center of its subalgebra $\mathcal{B}(n,l)$.

Furthermore, if we put

$$c(u)=\operatorname{sdet}B\big(u+\frac{n-1}{2}\big)\,\theta\big(u+\frac{n-1}{2}\big)^{-1}, \tag{3.34}$$

then by (3.28) we have the identity

$$c(u)=d(u)\,d(-u)^{-1},\qquad d(u)=\operatorname{qdet}T\big(u+\frac{n-1}{2}\big). \tag{3.35}$$

The coefficients of $d(u)$ are algebraically independent generators of the center of $\mathrm{Y}(n)$. Since $c(u)\,c(-u)=1$, repeating the argument of the second part of the proof of Theorem 3.1 for the case of $\mathcal{B}(1,0)$ we find that all the even coefficients of $c(u)$ can be expressed in terms of the odd ones, and the latter are algebraically independent. Since

$$\operatorname{sdet}B(u)=\theta(u)\,c\big(u-\frac{n-1}{2}\big), \tag{3.36}$$

the same holds for the coefficients of $\operatorname{sdet}B(u)$.

Finally, let us show that the center of $\mathcal{B}(n,l)$ is generated by the coefficients of $\operatorname{sdet}B(u)$. We use another filtration on $\mathcal{B}(n,l)$ defined by setting $\deg_2 b_{ij}^{(r)}=r-1$. We first verify that the corresponding graded algebra $\operatorname{gr}_2\mathcal{B}(n,l)$ is isomorphic to the universal enveloping algebra for a twisted polynomial current Lie algebra. Consider the involution σ of the Lie algebra $\mathfrak{gl}(n)$ given by $\sigma: E_{ij}\mapsto\varepsilon_i\,\varepsilon_j\,E_{ij}$. Denote by $\mathfrak{a}_0$ and $\mathfrak{a}_1$ the eigenspaces of σ corresponding to the eigenvalues 1 and -1, respectively. In particular, $\mathfrak{a}_0$ is a Lie subalgebra of $\mathfrak{gl}(n)$ isomorphic to $\mathfrak{gl}(k)\oplus\mathfrak{gl}(l)$. Denote by $\mathfrak{gl}(n)[x]^\sigma$ the Lie algebra of polynomials in a variable x of the form

$$a_0+a_1\,x+a_2\,x^2+\dots+a_m\,x^m,\qquad a_{2i}\in\mathfrak{a}_0,\quad a_{2i-1}\in\mathfrak{a}_1. \tag{3.37}$$

We claim that the following is an algebra isomorphism:

$$\operatorname{gr}_2\mathcal{B}(n,l)\simeq\mathrm{U}(\mathfrak{gl}(n)[x]^\sigma). \tag{3.38}$$

Indeed, denote by $\overline{b}_{ij}^{(r)}$ the image of $b_{ij}^{(r)}$ in the $(r-1)$-th component of $\operatorname{gr}_2\mathcal{B}(n,l)$. Then by the unitary condition we have for $r\geq 1$

$$(\varepsilon_i+(-1)^r\,\varepsilon_j)\,\overline{b}_{ij}^{(r)}=0, \tag{3.39}$$

while the quaternary relation gives

$$[\overline{b}_{ij}^{(r)}, \overline{b}_{kl}^{(s)}] = \delta_{kj}\,(\varepsilon_i\,(-1)^{r-1} + \varepsilon_j)\,\overline{b}_{il}^{(r+s-1)} - \delta_{il}\,(\varepsilon_i + \varepsilon_j\,(-1)^{r-1})\,\overline{b}_{kj}^{(r+s-1)}. \tag{3.40}$$

This shows that the mapping

$$\overline{b}_{ij}^{(r)} \mapsto (\varepsilon_i + (-1)^{r-1}\,\varepsilon_j)\,E_{ij}\,x^{r-1} \tag{3.41}$$

defines an algebra homomorphism $\mathrm{gr}_2\,\mathcal{B}(n,l) \to \mathrm{U}(\mathfrak{gl}(n)[x]^\sigma)$. Corollary 3.2 ensures that its kernel is trivial.

Further, it is easily deduced from the definition (3.26) of the Sklyanin determinant that the images of $\overline{c}_{2m+1}$ under the isomorphism (3.38) are given by

$$\overline{c}_{2m+1} \mapsto (-1)^l\,2\,(E_{11} + \cdots + E_{nn})\,x^{2m}, \qquad m \geq 0. \tag{3.42}$$

The theorem will be proved if we show that the center of $\mathrm{U}(\mathfrak{gl}(n)[x]^\sigma)$ is generated by the elements $(E_{11} + \cdots + E_{nn})\,x^{2m}$ with $m \geq 0$. This is equivalent to the claim that the center of $\mathrm{U}(\mathfrak{sl}(n)[x]^\sigma)$ is trivial. Here $\mathfrak{sl}(n)[x]^\sigma$ is the Lie algebra of polynomials of the form (3.37) where now $\mathfrak{a}_0 = \mathfrak{sl}(n) \cap \big(\mathfrak{gl}(k) \oplus \mathfrak{gl}(l)\big)$. However, this follows from a slight modification of a more general result: see [13, Proposition 4.10]. Namely, if $\mathfrak{a}$ is any Lie algebra and σ is its involution, we define $\mathfrak{a}[x]^\sigma$ as the Lie algebra of polynomials of type (3.37). Then an argument similar to [13] proves that if the center of $\mathfrak{a}$ is trivial, and the $\mathfrak{a}_0$-module $\mathfrak{a}_1$ has no nontrivial invariant elements then the center of $\mathrm{U}(\mathfrak{a}[x]^\sigma)$ is trivial. □

Remarks. (1) The algebra $\mathcal{B}(n,l)$ can also be regarded as a deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}(n)[x]^\sigma)$. To see this, introduce the deformation parameter h and rewrite the defining relations for $\mathcal{B}(n,l)$ in terms of the re-scaled generators $b_{ij}^{\prime(r)} = b_{ij}^{(r)}\,h^{r-1}$. These define a family of algebras $\mathcal{B}(n,l)_h$. If $h \neq 0$ the algebra $\mathcal{B}(n,l)_h$ is isomorphic to $\mathcal{B}(n,l)$ while for $h = 0$ one obtains the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}(n)[x]^\sigma)$.

(2) It would be interesting to find an explicit formula for $\mathrm{sdet}\,B(u)$ in terms of the generators $b_{ij}(u)$. □

Recall that the Yangian $\mathrm{Y}(\mathfrak{sl}(n))$ for the special linear Lie algebra $\mathfrak{sl}(n)$ can be defined as the subalgebra of $\mathrm{Y}(n)$ which consists of the elements stable under all automorphisms of the form $T(u) \mapsto h(u)\,T(u)$, where $h(u)$ is a series in u^{-1} with the constant term 1; see [13]. Then one has the tensor product decomposition

$$\mathrm{Y}(n) = \mathrm{Z}(n) \otimes \mathrm{Y}(\mathfrak{sl}(n)), \tag{3.43}$$

where $\mathrm{Z}(n)$ is the center of $\mathrm{Y}(n)$. Define the *special reflection algebra* $\mathcal{SB}(n,l)$ by

$$\mathcal{SB}(n,l) = \mathcal{B}(n,l) \cap \mathrm{Y}(\mathfrak{sl}(n)). \tag{3.44}$$

In other words, $\mathcal{SB}(n,l)$ consists of the elements of $\mathcal{B}(n,l)$ which are stable under all automorphisms (2.19). It is implied by (3.43) (cf. [13, Proposition 4.14]), that the following decomposition holds

$$\mathcal{B}(n,l) = \mathrm{Z}(n,l) \otimes \mathcal{SB}(n,l), \tag{3.45}$$

where $\mathrm{Z}(n,l)$ is the center of $\mathcal{B}(n,l)$.

4 Representations of $\mathcal{B}(n,l)$

Here we show that the Drinfeld highest weight theory [5] applies to the representations of the algebras $\mathcal{B}(n,l)$; see also [12] for the case of twisted Yangians. We then give a complete description of the finite-dimensional irreducible representations of $\mathcal{B}(n,l)$.

4.1 Highest weight representations

Recall first Drinfeld's classification results for representations of the Yangian $\mathrm{Y}(n)$ [5]; see also [12]. A representation L of the Yangian $\mathrm{Y}(n)$ is called *highest weight* if there exists a nonzero vector $\xi \in L$ such that L is generated by ξ,

$$\begin{aligned} &t_{ij}(u)\,\xi = 0 && \text{for} \quad 1 \le i < j \le n, \qquad \text{and} \\ &t_{ii}(u)\,\xi = \lambda_i(u)\,\xi && \text{for} \quad 1 \le i \le n, \end{aligned} \tag{4.1}$$

for some formal series $\lambda_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. The vector ξ is called the *highest vector* of L and the set $\lambda(u) = (\lambda_1(u), \dots, \lambda_n(u))$ is the *highest weight* of L.

Let $\lambda(u) = (\lambda_1(u), \dots, \lambda_n(u))$ be any set of formal series. Then there exists a unique, up to an isomorphism, irreducible highest weight module $L(\lambda(u))$ with the highest weight $\lambda(u)$. Any finite-dimensional irreducible representation of $\mathrm{Y}(n)$ is isomorphic to $L(\lambda(u))$ for some $\lambda(u)$. The representation $L(\lambda(u))$ is finite-dimensional if and only if there exist monic polynomials $Q_1(u), \dots, Q_{n-1}(u)$ in u such that

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \frac{Q_i(u+1)}{Q_i(u)}, \qquad i = 1, \dots, n-1. \tag{4.2}$$

Let us now turn to the reflection algebras. A representation V of the algebra $\mathcal{B}(n,l)$ is called *highest weight* if there exists a nonzero vector $\xi \in V$ such that V is generated by ξ,

$$\begin{aligned} &b_{ij}(u)\,\xi = 0 && \text{for} \quad 1 \le i < j \le n, \qquad \text{and} \\ &b_{ii}(u)\,\xi = \mu_i(u)\,\xi && \text{for} \quad 1 \le i \le n, \end{aligned} \tag{4.3}$$

for some formal series $\mu_i(u) \in \varepsilon_i + u^{-1}\mathbb{C}[[u^{-1}]]$. The vector ξ is called the *highest vector* of V and the set $\mu(u) = (\mu_1(u), \dots, \mu_n(u))$ is the *highest weight* of V.

Theorem 4.1 *Every finite-dimensional irreducible representation V of the algebra $\mathcal{B}(n,l)$ is a highest weight representation. Moreover, V contains a unique (up to a constant factor) highest vector.*

Proof. Our approach is quite standard; cf. [2, Proposition 12.2.3], [12]. Introduce the subspace of V

$$V^0 = \{\eta \in V \mid b_{ij}(u)\,\eta = 0, \qquad 1 \le i < j \le n\}. \tag{4.4}$$

We show first that V^0 is nonzero. The defining relations (2.11) give

$$[b_{ij}^{(1)}, b_{rs}(u)] = (\varepsilon_i + \varepsilon_j)\big(\delta_{rj}\, b_{is}(u) - \delta_{is}\, b_{rj}(u)\big). \tag{4.5}$$

This implies that V^0 can be equivalently defined as

$$V^0 = \{\eta \in V \mid b_{i,i+1}(u)\,\eta = 0, \qquad i = 1, \dots, n-1\}. \tag{4.6}$$

The operators $\varepsilon_i\, b_{ii}^{(1)}$ are pairwise commuting and so they have a common eigenvector $\zeta \ne 0$ in V. If we suppose that $V^0 = 0$ then there exists an infinite sequence of nonzero vectors in V,

$$\zeta, \quad b_{i_1,i_1+1}^{(r_1)}\,\zeta, \quad b_{i_2,i_2+1}^{(r_2)}\, b_{i_1,i_1+1}^{(r_1)}\,\zeta, \quad \cdots. \tag{4.7}$$

By (4.5) they all are eigenvectors for the operators $\varepsilon_i\, b_{ii}^{(1)}$, $i = 1, \dots, n$ of different weights. Thus, they are linearly independent which contradicts to the assumption $\dim V < \infty$. So, V^0 is nontrivial.

Next, we show that all the operators $b_{rr}(u)$ preserve the subspace V^0. We use a reverse induction on r. For $r = n$ we see from (2.11) that if $i < j < n$ then $b_{ij}(u)\, b_{nn}(v) \equiv 0$, where the equivalence is modulo the left ideal in $\mathcal{B}(n,l)$ generated by the coefficients of $b_{ij}(u)$ with $i < j$. Similarly, if $i < n$ then the assertion follows from the equivalence

$$b_{in}(u)\, b_{nn}(v) \equiv \frac{1}{u+v}\, b_{in}(u)\, b_{nn}(v). \tag{4.8}$$

Now let $r < n$. Note the following obvious consequence of (2.11): if $i < m$ and $i \ne j$ then

$$b_{ij}(u)\, b_{jm}(v) \equiv \frac{1}{u+v} \sum_{a=m}^{n} b_{ia}(u)\, b_{am}(v). \tag{4.9}$$

In particular, the right hand side is independent of j and so, if $i < m$ then for any indices $j, j' \neq i$ we have

$$b_{ij}(u)\, b_{jm}(v) \equiv b_{ij'}(u)\, b_{j'm}(v). \tag{4.10}$$

If $i+1 < r$ then it is immediate from (2.11) that $b_{i,i+1}(u)\, b_{rr}(v) \equiv 0$. Further, using (4.9) we derive from (2.11) that

$$b_{r-1,r}(u)\, b_{rr}(v) \equiv \frac{n-r+1}{u+v}\, b_{r-1,r}(u)\, b_{rr}(v), \tag{4.11}$$

which gives $b_{r-1,r}(u)\, b_{rr}(v) \equiv 0$. Next, by (2.11),

$$\begin{aligned} b_{r,r+1}(u)\, b_{rr}(v) \equiv \frac{1}{u-v}\big(b_{r,r+1}(u)\, b_{rr}(v) - b_{r,r+1}(v)\, b_{rr}(u)\big) \\ - \frac{1}{u+v} \sum_{a=r+1}^{n} b_{ra}(v)\, b_{a,r+1}(u). \end{aligned} \tag{4.12}$$

By (4.10), the second sum here is $-\dfrac{n-r}{u+v}\, b_{r,r+1}(v)\, b_{r+1,r+1}(u)$, which is equivalent to zero by the induction hypothesis. So we get

$$\frac{u-v-1}{u-v}\, b_{r,r+1}(u)\, b_{rr}(v) + \frac{1}{u-v}\, b_{r,r+1}(v)\, b_{rr}(u) \equiv 0. \tag{4.13}$$

Swapping u and v we obtain

$$-\frac{1}{u-v}\, b_{r,r+1}(u)\, b_{rr}(v) + \frac{u-v+1}{u-v}\, b_{r,r+1}(v)\, b_{rr}(u) \equiv 0. \tag{4.14}$$

The system of linear equations (4.13) and (4.14) has only zero solution which proves the assertion in this case. Finally, for $i > r$ we get from (2.11)

$$b_{i,i+1}(u)\, b_{rr}(v) \equiv \frac{1}{u-v}\big(b_{r,i+1}(u)\, b_{ir}(v) - b_{r,i+1}(v)\, b_{ir}(u)\big), \tag{4.15}$$

and

$$\begin{aligned} b_{r,i+1}(u)\, b_{ir}(v) \equiv \frac{1}{u-v}\big(b_{i,i+1}(u)\, b_{rr}(v) - b_{i,i+1}(v)\, b_{rr}(u)\big) \\ - \frac{1}{u+v} \sum_{a=i+1}^{n} b_{ia}(v)\, b_{a,i+1}(u). \end{aligned} \tag{4.16}$$

By (4.10), the second sum here is equivalent to $-\dfrac{n-i}{u+v}\, b_{i,i+1}(v)\, b_{i+1,i+1}(u)$, which is equivalent to zero by the induction hypothesis. Therefore, (4.16) implies that $b_{r,i+1}(u)\, b_{ir}(v)$ is symmetric in u and v which shows that the left hand side of (4.15) is equivalent to 0.

Our next step is to show that all the operators $b_{rr}(u)$ on V^0 commute. For any $1 \leq r \leq n$, the defining relations (2.11) give

$$\big(1 - \frac{1}{u+v}\big)\,[b_{rr}(u), b_{rr}(v)] \equiv \frac{1}{u+v}\,\alpha_r(u,v), \tag{4.17}$$

where

$$\alpha_r(u,v) = \sum_{a=r+1}^{n} \big(b_{ra}(u)\,b_{ar}(v) - b_{ra}(v)\,b_{ar}(u)\big). \tag{4.18}$$

Using (2.11) again we obtain for $a \geq r+1$

$$\begin{aligned} b_{ra}(u)\,b_{ar}(v) &- b_{ra}(v)\,b_{ar}(u) \\ &\equiv \frac{1}{u+v}\,\big([b_{rr}(u), b_{rr}(v)] + [b_{aa}(u), b_{aa}(v)] + \alpha_r(u,v) + \alpha_r(u,v)\big). \end{aligned} \tag{4.19}$$

Taking the sum over a gives

$$(u+v-n+r)\alpha_r(u,v) \equiv (n-r)[b_{rr}(u), b_{rr}(v)] + \sum_{a=r+1}^{n} \big([b_{aa}(u), b_{aa}(v)] + \alpha_r(u,v)\big). \tag{4.20}$$

Using (4.17) we easily prove by a reverse induction on r that $[b_{rr}(u), b_{rr}(v)] \equiv 0$ and $\alpha_r(u,v) \equiv 0$. Finally, if $i < r$ then by (2.11)

$$[b_{ii}(u), b_{rr}(v)] \equiv -\frac{1}{u^2 - v^2}\,\big([b_{rr}(u), b_{rr}(v)] + \alpha_r(u,v)\big), \tag{4.21}$$

which is equivalent to 0 as was shown above.

We can now conclude that the subspace V^0 contains a common eigenvector $\xi \neq 0$ for the operators $b_{rr}(u)$, that is, (4.3) holds for some formal series $\mu_i(u)$.

Since V is irreducible, the submodule $\mathcal{B}(n,l)\,\xi$ must coincide with V. The uniqueness of ξ now follows from Corollary 3.2. □

Given any n-tuple $\mu(u) = (\mu_1(u), \dots, \mu_n(u))$, where $\mu_i(u) \in \varepsilon_i + u^{-1}\mathbb{C}[[u^{-1}]]$, we define the *Verma module* $M(\mu(u))$ as the quotient of $\mathcal{B}(n,l)$ by the left ideal generated by all the coefficients of the series $b_{ij}(u)$ with $1 \leq i < j \leq n$, and $b_{ii}(u) - \mu_i(u)$ for $i = 1, \dots, n$. However, contrary to the case of the Yangian $\mathrm{Y}(n)$ the module $M(\mu(u))$ can be trivial for some $\mu(u)$. If $M(\mu(u))$ is nontrivial we denote by $V(\mu(u))$ its unique irreducible quotient. Any irreducible highest weight module with the highest weight $\mu(u)$ is clearly isomorphic to $V(\mu(u))$.

Theorem 4.2 *The Verma module $M(\mu(u))$ is nontrivial (i.e. $V(\mu(u))$ exists) if and only if*

$$\mu_n(u)\,\mu_n(-u) = 1, \tag{4.22}$$

and for each $i = 1, \dots, n-1$ *the following conditions hold*

$$\widetilde{\mu}_i(u)\, \widetilde{\mu}_i(-u+n-i) = \widetilde{\mu}_{i+1}(u)\, \widetilde{\mu}_{i+1}(-u+n-i), \tag{4.23}$$

where

$$\widetilde{\mu}_i(u) = (2u-n+i)\, \mu_i(u) + \mu_{i+1}(u) + \dots + \mu_n(u). \tag{4.24}$$

Proof. Suppose first that $V(\mu(u))$ exists. For each $i = 1, \dots, n$ set

$$\beta_i(u,v) = \sum_{a=i}^{n} b_{ia}(u)\, b_{ai}(v). \tag{4.25}$$

The highest vector of $V(\mu(u))$ is an eigenvector of $\beta_n(u,v)$ with the eigenvalue $\mu_n(u)\,\mu_n(v)$. Due to the unitary condition (2.12) this eigenvalue must be equal to 1 if $v = -u$ which proves (4.22). Using the notation of the proof of Theorem 4.1, we derive from (2.11) that

$$\begin{aligned}\frac{u+v-n+i}{u+v}\, \beta_i(u,v) \equiv b_{ii}(u)\, b_{ii}(v) - \frac{1}{u+v} \sum_{a=i+1}^{n} \beta_a(v,u)& \\ + \frac{1}{u-v} \sum_{a=i+1}^{n} \big(b_{aa}(u)\, b_{ii}(v) - b_{aa}(v)\, b_{ii}(u)\big)&.\end{aligned} \tag{4.26}$$

Therefore,

$$\frac{u+v-n+i}{u+v} \big(\beta_i(u,v) - \beta_i(v,u)\big) \equiv \frac{1}{u+v} \sum_{a=i+1}^{n} \big(\beta_a(u,v) - \beta_a(v,u)\big). \tag{4.27}$$

Since clearly $\beta_n(u,v) \equiv \beta_n(v,u)$, an easy induction implies that $\beta_i(u,v) \equiv \beta_i(v,u)$ for all i. Now consider (4.26) with i replaced by $i+1$ and subtract this from (4.26). This gives

$$\begin{aligned}&\frac{u+v-n+i}{u+v} \big(\beta_i(u,v) - \beta_{i+1}(u,v)\big) \\ &\qquad \equiv b_{ii}(u)\, b_{ii}(v) + \frac{1}{u-v} \sum_{a=i+1}^{n} \big(b_{aa}(u)\, b_{ii}(v) - b_{aa}(v)\, b_{ii}(u)\big) \\ &- b_{i+1,i+1}(u)\, b_{i+1,i+1}(v) - \frac{1}{u-v} \sum_{a=i+2}^{n} \big(b_{aa}(u)\, b_{i+1,i+1}(v) - b_{aa}(v)\, b_{i+1,i+1}(u)\big).\end{aligned} \tag{4.28}$$

Apply both sides to the highest vector of $V(\mu(u))$ and put $u+v=n-i$. We then get the following condition for the components of $\mu(u)$,

$$\begin{aligned}\mu_i(u)\,\mu_i(v)&+\frac{1}{u-v}\sum_{a=i+1}^{n}\big(\mu_a(u)\,\mu_i(v)-\mu_a(v)\,\mu_i(u)\big)\\ =\mu_{i+1}(u)\,\mu_{i+1}(v)&+\frac{1}{u-v}\sum_{a=i+2}^{n}\big(\mu_a(u)\,\mu_{i+1}(v)-\mu_a(v)\,\mu_{i+1}(u)\big),\end{aligned}\tag{4.29}$$

where $v=-u+n-i$. It is now a straightforward calculation to verify that this condition is equivalent to (4.23).

Conversely, suppose that the conditions (4.22) and (4.23) hold. We shall demonstrate that there exist a highest weight module $L(\lambda(u))$ over the Yangian $\mathrm{Y}(n)$ such that $\mathcal{B}(n,l)$-cyclic span of the highest vector is a $\mathcal{B}(n,l)$-module with the highest weight $\mu(u)$. This will prove the existence of $V(\mu(u))$.

Suppose first that $L(\lambda(u))$ is an arbitrary irreducible highest weight module. The *quantum comatrix* $\widehat{T}(u)=(\widehat{t}_{ij}(u))$ is defined by

$$\operatorname{qdet}T(u)=\widehat{T}(u)T(u-n+1).\tag{4.30}$$

It can be deduced from (3.23) that the matrix element $\widehat{t}_{ij}(u)$ equals $(-1)^{i+j}$ times the quantum determinant of the submatrix of $T(u)$ obtained by removing the ith column and jth row. Therefore, the matrix element $t'_{ij}(u)$ of the inverse matrix $T^{-1}(u)$ can be expressed as

$$t'_{ij}(u)=\big(\operatorname{qdet}T(u+n-1)\big)^{-1}\,\widehat{t}_{ij}(u+n-1).\tag{4.31}$$

This implies that the highest vector ξ of the $\mathrm{Y}(n)$-module $L(\lambda(u))$ is annihilated by the elements $t'_{ij}(u)$ with $i<j$, and that ξ is an eigenvector for the $t'_{ii}(u)$. The corresponding eigenvalues are found from the formula (3.19) and given by

$$t'_{ii}(u)\,\xi=\frac{\lambda_{i+1}(u+n-i)\cdots\lambda_n(u+1)}{\lambda_i(u+n-i)\cdots\lambda_n(u)}\,\xi.\tag{4.32}$$

The following relations are easily derived from (2.4); see also [13, Section 7]

$$[t_{ij}(u),t'_{rs}(v)]=\frac{1}{u-v}\Big(\delta_{rj}\sum_{a=1}^{n}t_{ia}(u)\,t'_{as}(v)-\delta_{is}\sum_{a=1}^{n}t'_{ra}(v)\,t_{aj}(u)\Big).\tag{4.33}$$

Hence modulo the left ideal in $\mathrm{Y}(n)$ generated by the elements $t_{ij}(u)$ with $i<j$ we can write for $a>i$

$$t_{ia}(u)\,t'_{ai}(-u)\equiv\frac{1}{2u}\sum_{c=i}^{n}t_{ic}(u)\,t'_{ci}(-u)-\frac{1}{2u}\sum_{c=a}^{n}t'_{ac}(-u)\,t_{ca}(u).\tag{4.34}$$

Due to (3.7) this implies that

$$b_{ii}(u) \equiv \varepsilon_i\, t_{ii}(u)\, t'_{ii}(-u) + \frac{1}{2u} \sum_{a=i+1}^{n} \varepsilon_a \Big(\sum_{c=i}^{n} t_{ic}(u)\, t'_{ci}(-u) - \sum_{c=a}^{n} t'_{ac}(-u) t_{ca}(u) \Big). \tag{4.35}$$

Now suppose that $i \geq k+1$ and set

$$f_{ii}(u) = - \sum_{c=i}^{n} t'_{ic}(-u) t_{ci}(u). \tag{4.36}$$

Then (4.35) can be written as

$$\frac{2u-n+i}{2u}\, b_{ii}(u) \equiv -t_{ii}(u)\, t'_{ii}(-u) - \frac{1}{2u} \sum_{a=i+1}^{n} f_{aa}(u). \tag{4.37}$$

A similar transformation for $f_{ii}(u)$ gives

$$\frac{2u-n+i}{2u}\, f_{ii}(u) \equiv -t'_{ii}(-u)\, t_{ii}(u) - \frac{1}{2u} \sum_{a=i+1}^{n} b_{aa}(u). \tag{4.38}$$

Since $b_{nn}(u) \equiv f_{nn}(u)$, an easy induction proves that $b_{ii}(u) \equiv f_{ii}(u)$ for all $i = k+1, \dots, n$. By (4.37), we have for those i,

$$\frac{2u-n+i}{2u}\, b_{ii}(u) + \frac{1}{2u} \sum_{a=i+1}^{n} b_{aa}(u) \equiv -t_{ii}(u)\, t'_{ii}(-u). \tag{4.39}$$

In the same way for $i = 1, \dots, k$ we get

$$\frac{2u-n+i}{2u-2l}\, b_{ii}(u) + \frac{1}{2u-2l} \sum_{a=i+1}^{n} b_{aa}(u) \equiv t_{ii}(u)\, t'_{ii}(-u). \tag{4.40}$$

Now, applying both sides of (4.39) and (4.40) to ξ and using (4.32) we obtain

$$\mu_n(u) = \begin{cases} \lambda_n(u)\lambda_n(-u)^{-1} & \text{if} \quad l = 0, \\ -\lambda_n(u)\lambda_n(-u)^{-1} & \text{if} \quad l > 0. \end{cases} \tag{4.41}$$

Moreover,

$$\frac{\widetilde{\mu}_i(u)}{\widetilde{\mu}_{i+1}(u)} = \frac{\lambda_i(u)\, \lambda_{i+1}(-u+n-i)}{\lambda_{i+1}(u)\, \lambda_i(-u+n-i)}, \tag{4.42}$$

if $i \neq k$; while

$$\frac{\widetilde{\mu}_k(u)}{\widetilde{\mu}_{k+1}(u)} = \frac{l-u}{u} \cdot \frac{\lambda_k(u)\, \lambda_{k+1}(-u+l)}{\lambda_{k+1}(u)\, \lambda_k(-u+l)}. \tag{4.43}$$

The proof of the theorem is now completed as follows. By (4.22) there exists a series $\lambda_n(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ such that (4.41) holds. Similarly, (4.23) ensures that there exist series $\lambda_1(u), \dots, \lambda_{n-1}(u)$ satisfying (4.42) and (4.43). Thus, $V(\mu(u))$ is isomorphic to the irreducible quotient of $\mathcal{B}(n,l)\, \xi$. □

4.2 Representations of $\mathcal{B}(2,0)$ and $\mathcal{B}(2,1)$

The algebras $\mathcal{B}(2,0)$ and $\mathcal{B}(2,1)$ turn out to be isomorphic to the symplectic and orthogonal *twisted Yangians* $\mathrm{Y}^-(2)$ and $\mathrm{Y}^+(2)$, respectively (see the Remark at the end of this section). Thus the representations of $\mathcal{B}(2,0)$ and $\mathcal{B}(2,1)$ can be described by using the results of [12] for the twisted Yangians. In our argument we use the corresponding isomorphisms of the extended algebras (Proposition 4.3). Note also that this similarity between the reflection algebras and the twisted Yangians does not extend to higher dimensions.

Recall (see e.g. [13]) that the twisted Yangian $\mathrm{Y}^{\pm}(2)$ is an associative algebra with generators $s_{ij}^{(1)}, s_{ij}^{(2)}, \dots$, where $i,j \in \{1,2\}$. Introduce the generating series

$$s_{ij}(u) = \delta_{ij} + s_{ij}^{(1)} u^{-1} + s_{ij}^{(2)} u^{-2} + \cdots \tag{4.44}$$

and combine them into the matrix $S(u) = (s_{ij}(u))$. Then the defining relations have the form of a quaternary relation analogous to (2.9),

$$R(u-v) S_1(u) R^t(-u-v) S_2(v) = S_2(v) R^t(-u-v) S_1(u) R(u-v), \tag{4.45}$$

as well as the symmetry relation

$$S^t(-u) = S(u) \pm \frac{S(u) - S(-u)}{2u}. \tag{4.46}$$

Here we have used the notation of Section 2, and

$$R^t(u) = 1 - Q u^{-1}, \qquad Q = \sum_{i,j=1}^{2} E_{ij}^t \otimes E_{ji} \in (\mathrm{End}\,\mathbb{C}^2)^{\otimes 2},$$

where the matrix transposition is defined by

$$A^t = \begin{pmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{pmatrix} \qquad \text{and} \qquad A^t = \begin{pmatrix} a_{22} & -\, a_{12} \\ -a_{21} & a_{11} \end{pmatrix}, \tag{4.48}$$

for $\mathrm{Y}^+(2)$ and $\mathrm{Y}^-(2)$, respectively.

Another pair of associative algebras $\widetilde{\mathrm{Y}}^+(2)$ and $\widetilde{\mathrm{Y}}^-(2)$ is defined in the same way as $\mathrm{Y}^{\pm}(2)$, but with the symmetry relation (4.46) dropped. Then the algebra $\mathrm{Y}^{\pm}(2)$ is isomorphic to the quotient of $\widetilde{\mathrm{Y}}^{\pm}(2)$ by the ideal generated by all coefficients of the series $\delta(u)$ defined by the relation

$$\Big(1 \mp \frac{1}{2u}\Big)\, \delta(u)\, Q = Q\, S_1(u)\, R(2u)\, S_2^{-1}(-u). \tag{4.49}$$

Moreover, these coefficients belong to the center of $\widetilde{\mathrm{Y}}^{\pm}(2)$; see [13, Theorems 6.3 and 6.4] for the proofs of these assertions. Note also that $\delta(u)$ satisfies $\delta(u)\,\delta(-u) = 1$ which is easy to deduce from (4.49).

Recall that $\widetilde{\mathcal{B}}(n,l)$ is the algebra with the defining relations (2.9); see Section 2. Here we let $G = \mathrm{diag}\,(1,-1)$.

Proposition 4.3 *The mappings*

$$S(u) \mapsto B\big(u+\frac{1}{2}\big) \qquad \textit{and} \qquad S(u) \mapsto B\big(u+\frac{1}{2}\big)\, G \tag{4.50}$$

define algebra isomorphisms $\widetilde{\mathrm{Y}}^{-}(2) \to \widetilde{\mathcal{B}}(2,0)$ *and* $\widetilde{\mathrm{Y}}^{+}(2) \to \widetilde{\mathcal{B}}(2,1)$, *respectively.*

Proof. We have to show that the matrix $B\big(u+\frac{1}{2}\big)$ or $B\big(u+\frac{1}{2}\big)\, G$, respectively, satisfies the relation (4.45). Note that in the case of $\widetilde{\mathrm{Y}}^{-}(2)$ the operator $P+Q$ is the identity on $(\mathrm{End}\,\mathbb{C}^2)^{\otimes 2}$. This implies that

$$R^t(-u-v) = \frac{u+v+1}{u+v}\, R(u+v+1) \tag{4.51}$$

proving the assertion. In the case of $\widetilde{\mathrm{Y}}^{+}(2)$ both $G_1\, Q\, G_1 + P$ and $G_2\, Q\, G_2 + P$ are the identity operators which implies that

$$G_1\, R^t(-u-v)\, G_1 = G_2\, R^t(-u-v)\, G_2 = \frac{u+v+1}{u+v}\, R(u+v+1). \tag{4.52}$$

It remains to note that $G_1\, G_2$ commutes with $R(u-v)$. $\square$

Suppose now that V is an irreducible finite-dimensional representation of the twisted Yangian $\mathrm{Y}^{\pm}(2)$. Then V is naturally extended to $\widetilde{\mathrm{Y}}^{\pm}(2)$ and, by Proposition 4.3, to the algebra $\widetilde{\mathcal{B}}(2,0)$ or $\widetilde{\mathcal{B}}(2,1)$, respectively. The central element $f(u)$ defined in (2.13) acts in V as a scalar series. Then there exists a series $g(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ such that $g(u)\, g(-u)\, f(u) = 1$ as an operator in V. This means that the composition of V with the corresponding automorphism (2.19) of $\widetilde{\mathcal{B}}(2,0)$ or $\widetilde{\mathcal{B}}(2,1)$ can be regarded as an irreducible representation of the algebra $\mathcal{B}(2,0)$ or $\mathcal{B}(2,1)$, respectively.

Conversely, any irreducible finite-dimensional representation V of $\mathcal{B}(2,0)$ or $\mathcal{B}(2,1)$ is naturally extended to the corresponding algebra $\widetilde{\mathcal{B}}(2,0)$ or $\widetilde{\mathcal{B}}(2,1)$ and hence to $\widetilde{\mathrm{Y}}^{\pm}(2)$. The composition of V with an appropriate automorphism $S(u) \mapsto h(u)\, S(u)$ can be regarded as an irreducible representation of the twisted Yangian $\mathrm{Y}^{\pm}(2)$. Indeed, the central element $\delta(u)$ defined by (4.49) acts as a scalar series on V, and since $\delta(u)\,\delta(-u) = 1$, the series $h(u)$ is found from the relation $h(u)\, h(-u)^{-1}\, \delta(u) = 1$.

This argument allows us to carry over the description of representations of $\mathrm{Y}^{\pm}(2)$ to the case of the algebras $\mathcal{B}(2,0)$ or $\mathcal{B}(2,1)$. The following proposition is easily derived from [12, Theorems 5.4 and 6.4]. Suppose that two formal series $\mu_1(u)$ and $\mu_2(u)$ satisfy the conditions of Theorem 4.2 so that the irreducible highest weight module $V(\mu(u))$ exists.

Proposition 4.4 *(i) The* $\mathcal{B}(2,0)$*-module* $V(\mu(u))$ *is finite-dimensional if and only if there exists a monic polynomial* $P(u)$ *in* u *such that* $P(-u+2) = P(u)$ *and*

$$\frac{(2u-1)\,\mu_1(u) + \mu_2(u)}{2u\,\mu_2(u)} = \frac{P(u+1)}{P(u)}. \tag{4.53}$$

In this case $P(u)$ *is unique.*

(ii) The $\mathcal{B}(2,1)$*-module* $V(\mu(u))$ *is finite-dimensional if and only if there exist* $\gamma \in \mathbb{C}$ *and a monic polynomial* $P(u)$ *in* u *such that* $P(-u+2) = P(u)$, $P(\gamma) \neq 0$ *and*

$$\frac{(2u-1)\,\mu_1(u) + \mu_2(u)}{2u\,\mu_2(u)} = \frac{P(u+1)}{P(u)} \cdot \frac{\gamma - u}{\gamma + u - 1}. \tag{4.54}$$

In this case the pair $(P(u), \gamma)$ *is unique.* □

Given $\alpha, \beta \in \mathbb{C}$ denote by $L(\alpha, \beta)$ the irreducible $\mathfrak{gl}(2)$-module with the highest weight (α, β). It is well-known that any finite-dimensional irreducible representation of $\mathrm{Y}(2)$ is isomorphic to a tensor product of the form

$$L = L(\alpha_1, \beta_1) \otimes \cdots \otimes L(\alpha_k, \beta_k), \tag{4.55}$$

up to the twisting by an automorphism $T(u) \mapsto h(u)\,T(u)$ of $\mathrm{Y}(2)$, where $h(u)$ is a formal series [16], [1]; see also [12]. The corresponding analogs of this result for the algebras $\mathcal{B}(2,0)$ and $\mathcal{B}(2,1)$ are also immediate from [12] due to the above argument. For any $\gamma \in \mathbb{C}$ denote by $V(\gamma)$ the one-dimensional representation of $\mathcal{B}(2,1)$ such that the generators act by

$$b_{11}(u) \mapsto \frac{u+\gamma}{u-\gamma}, \qquad b_{22}(u) \mapsto -1, \qquad b_{12}(u) \mapsto 0, \qquad b_{21}(u) \mapsto 0. \tag{4.56}$$

Given any $\mathrm{Y}(n)$-module L and any $\mathcal{B}(n,l)$-module V we shall regard $L \otimes V$ as a $\mathcal{B}(n,l)$-module by using Proposition 3.3.

Proposition 4.5 *Any finite-dimensional irreducible* $\mathcal{B}(2,0)$*-module is isomorphic to the restriction of a* $\mathrm{Y}(2)$*-module of the form* (4.55)*. Any finite-dimensional irreducible* $\mathcal{B}(2,1)$*-module is isomorphic to the tensor product* $L \otimes V(\gamma)$*, where* L *is a* $\mathrm{Y}(2)$*-module of the form* (4.55)*.* □

Remarks. (1) It is proved in [13, Proposition 4.14] that the twisted Yangian admits the decomposition

$$\mathrm{Y}^{\pm}(2) = \mathrm{Z}^{\pm} \otimes \mathrm{SY}^{\pm}(2), \tag{4.57}$$

where $\mathrm{Z}^{\pm}$ is the center of $\mathrm{Y}^{\pm}(2)$, and $\mathrm{SY}^{\pm}(2)$ is a subalgebra called the *special twisted Yangian*. Using the decomposition (3.45) we conclude from Proposition 4.3 that the algebras $\mathrm{SY}^{+}(2)$ and $\mathrm{SY}^{-}(2)$ are respectively isomorphic to $\mathcal{SB}(2,1)$ and $\mathcal{SB}(2,0)$. Indeed, each special subalgebra is the quotient of the corresponding algebra $\widetilde{\mathrm{Y}}^{\pm}(2)$ or $\widetilde{\mathcal{B}}(2,l)$ by the ideal generated by the center. On the other hand, both centers $\mathrm{Z}^{\pm}$ and $\mathrm{Z}(2,l)$ are polynomial algebras in countably many variables; see Theorem 3.4 above and [13, Theorem 4.11]. This implies the isomorphisms $\mathrm{Y}^{+}(2) \cong \mathcal{B}(2,1)$ and $\mathrm{Y}^{-}(2) \cong \mathcal{B}(2,0)$.

(2) Criteria of irreducibility of the modules L and $L \otimes V(\gamma)$ (with L given by (4.55)) over the algebras $\mathcal{B}(2,0)$ and $\mathcal{B}(2,1)$, respectively, can also be obtained from the corresponding results in [12].

4.3 Classification theorem for $\mathcal{B}(n,l)$-modules

Theorem 4.1 together with the following result will complete the description of finite-dimensional irreducible representations of $\mathcal{B}(n,l)$. Let $V(\mu(u))$ be an irreducible highest weight module over the algebra $\mathcal{B}(n,l)$, so that the conditions of Theorem 4.2 for the components $\mu_i(u)$ hold. We shall keep using the notation (4.24).

Theorem 4.6 (*i*) *The $\mathcal{B}(n,0)$-module $V(\mu(u))$ is finite-dimensional if and only if there exist monic polynomials $P_1(u),\dots,P_{n-1}(u)$ in u such that $P_i(-u+n-i+1)=P_i(u)$ and*

$$\frac{\widetilde{\mu}_i(u)}{\widetilde{\mu}_{i+1}(u)}=\frac{P_i(u+1)}{P_i(u)},\qquad i=1,\dots,n-1. \tag{4.58}$$

(*ii*) *The $\mathcal{B}(n,l)$-module $V(\mu(u))$ (with $l>0$) is finite-dimensional if and only if there exist an element $\gamma\in\mathbb{C}$ and monic polynomials $P_1(u),\dots,P_{n-1}(u)$ in u such that $P_i(-u+n-i+1)=P_i(u)$, $P_k(\gamma)\neq 0$ and*

$$\frac{\widetilde{\mu}_i(u)}{\widetilde{\mu}_{i+1}(u)}=\frac{P_i(u+1)}{P_i(u)},\qquad i=1,\dots,n-1,\quad i\neq k, \tag{4.59}$$

while

$$\frac{\widetilde{\mu}_k(u)}{\widetilde{\mu}_{k+1}(u)}=\frac{P_k(u+1)}{P_k(u)}\cdot\frac{\gamma-u}{\gamma+u-l}. \tag{4.60}$$

Proof. Set $V=V(\mu(u))$ and suppose first that $\dim V<\infty$. We shall use induction on n. In the case $n=2$ the result holds by Proposition 4.4. Suppose now that $n\geq 3$ and consider the subspace

$$V_+=\{\eta\in V\mid b_{1i}(u)\,\eta=0\qquad\text{for}\quad i=2,\dots,n\}. \tag{4.61}$$

The calculations similar to those used in the proof of Theorem 4.1 show that V_+ is stable under the operators $b_{ij}(u)$ with $2 \le i, j \le n$. Moreover, the operators

$$b^{\circ}_{ij}(u) = b_{i+1,j+1}(u), \qquad i, j = 1, \dots, n-1 \tag{4.62}$$

form a representation of the algebra $\mathcal{B}(n-1, l)$ in V_+ (recall that by our assumptions, $l \le n/2$). The cyclic span $\mathcal{B}(n-1, l)\, \xi$ of the highest vector $\xi \in V$ is a module with the highest weight $\mu_+(u) = (\mu_2(u), \dots, \mu_n(u))$. Since this module is finite-dimensional the conditions of the theorem must be satisfied for the components of $\mu_+(u)$.

Similarly, consider the subspace

$$V_- = \{\eta \in V \mid b_{in}(u)\, \eta = 0 \qquad \text{for} \quad i = 1, \dots, n-1\}. \tag{4.63}$$

Now the operators $b_{nn}(u)$ and $b_{ij}(u)$ with $1 \le i, j \le n-1$ preserve V_-. Moreover, the operators

$$b^{\circ}_{ij}(u) = b_{ij}\big(u + \frac{1}{2}\big) + \frac{\delta_{ij}}{2u} \cdot b_{nn}\big(u + \frac{1}{2}\big), \qquad i, j = 1, \dots, n-1 \tag{4.64}$$

form a representation of the algebra $\mathcal{B}(n-1, 0)$ or $\mathcal{B}(n-1, l-1)$ in V_-, respectively for $l = 0$ and $l > 0$. Again, the cyclic span of ξ is a finite-dimensional module over $\mathcal{B}(n-1, 0)$ or $\mathcal{B}(n-1, l-1)$, respectively, with the highest weight $\mu_-(u) = (\mu^{\circ}_1(u), \dots, \mu^{\circ}_{n-1}(u))$, where

$$\mu^{\circ}_i(u) = \mu_i\big(u + \frac{1}{2}\big) + \frac{1}{2u} \cdot \mu_n\big(u + \frac{1}{2}\big). \tag{4.65}$$

By the induction hypothesis, the components of $\mu_-(u)$ must satisfy the conditions of the theorem. Rewriting them in terms of the components of $\mu(u)$ we complete the proof of the 'only if' part.

Conversely, suppose that the conditions of the theorem hold for the components of the highest weight $\mu(u)$. Since $P_i(u) = P_i(-u + n - i + 1)$ for each i, there exist monic polynomials $Q_i(u)$ in u such that

$$P_i(u) = (-1)^{\deg Q_i}\, Q_i(u)\, Q_i(-u + n - i + 1). \tag{4.66}$$

Then there exists a representation $L(\lambda(u))$ of the Yangian $\mathrm{Y}(n)$ such that the components $\lambda_i(u)$ of $\lambda(u)$ satisfy the conditions (4.2) for the polynomials $Q_i(u)$. The argument of the proof of Theorem 4.2 shows that the $\mathcal{B}(n, 0)$-span of the highest vector ξ of $L(\lambda(u))$ is a module with the highest weight $\mu(u)$ such that the relations (4.42) hold. Since $V(\mu(u))$ is isomorphic to the irreducible quotient of this span we conclude that $V(\mu(u))$ is finite-dimensional.

Suppose now that $l > 0$. It is easy to verify that for any $\gamma \in \mathbb{C}$ the assignment

$$b_{ij}(u) \mapsto \delta_{ij} \, \frac{u+\gamma}{\varepsilon_i \, u - \gamma} \tag{4.67}$$

defines a one-dimensional representation of $\mathcal{B}(n,l)$ which we denote by $V(\gamma)$. Now consider the $\mathcal{B}(n,l)$-module $L(\lambda(u)) \otimes V(l-\gamma)$; see Proposition 3.3. Let η be a basis vector of $V(l-\gamma)$. Repeating again the calculation of the proof of Theorem 4.2 we find that the $\mathcal{B}(n,l)$-span of the vector $\xi \otimes \eta$ is a module with the highest weight $\mu(u)$ such that the relations (4.42) hold for $i \neq k$, while

$$\frac{\widetilde{\mu}_k(u)}{\widetilde{\mu}_{k+1}(u)} = \frac{\gamma - u}{\gamma + u - l} \cdot \frac{\lambda_k(u)\, \lambda_{k+1}(-u+l)}{\lambda_{k+1}(u)\, \lambda_k(-u+l)}. \tag{4.68}$$

This implies that $V(\mu(u))$ is finite-dimensional. $\square$

Using the decomposition (3.45) we can deduce the following parametrization of the representations of the special reflection algebra $\mathcal{SB}(n,l)$.

Corollary 4.7 (i) *The finite-dimensional irreducible representations of the special reflection algebra $\mathcal{SB}(n,0)$ are in a one-to-one correspondence with the families of monic polynomials $(P_1(u),\dots,P_{n-1}(u))$ such that $P_i(-u+n-i+1) = P_i(u)$.*

(ii) *The finite-dimensional irreducible representations of the special reflection algebra $\mathcal{SB}(n,l)$ with $l > 0$ are in a one-to-one correspondence with the families $(P_1(u),\dots,P_{n-1}(u),\gamma)$, where $\gamma \in \mathbb{C}$ and the $P_i(u)$ are monic polynomials such that $P_i(-u+n-i+1) = P_i(u)$ and $P_k(\gamma) \neq 0$.* $\square$

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