Degenerate affine Hecke algebras and centralizer construction for the symmetric groups

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Abstract

In our recent papers the centralizer construction was applied to the series of classical Lie algebras to produce the quantum algebras called (twisted) Yangians. Here we extend this construction to the series of the symmetric groups $S(n)$. We study the 'stable' properties of the centralizers of $S(n-m)$ in the group algebra $\mathbb{C}[S(n)]$ as $n \to \infty$ with m fixed. We construct a limit centralizer algebra A and describe its algebraic structure. The algebra A turns out to be closely related with the degenerate affine Hecke algebras. We also show that the so-called tame representations of $S(\infty)$ yield a class of natural A-modules.

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1 Introduction

The *centralizer construction* proposed in [29] shows that certain "quantum" algebras can be obtained as projective limits of centralizers in classical enveloping algebras. This approach has been applied to the series of matrix Lie algebras to construct the quantum algebras called the *Yangians* and *twisted Yangians* which are originally defined as certain deformations of enveloping algebras; cf. [5]. The type A case is treated in [29], [30] and extended to the B, C, D types in [32] and [18]. A modified version of the A type construction is given in [17].

In the A type case, one considers the centralizer $\mathcal{A}_m(n)$ of the subalgebra $\mathfrak{gl}(n-m)$ in the enveloping algebra $\mathrm{U}(\mathfrak{gl}(n))$. It turns out that for each pair $n > m$ there is a natural algebra homomorphism $\mathcal{A}_m(n) \to \mathcal{A}_m(n-1)$ so that one can define a projective limit algebra $\mathcal{A}_m$ by using the chain of homomorphisms

$$\cdots \longrightarrow \mathcal{A}_m(n) \longrightarrow \mathcal{A}_m(n-1) \longrightarrow \cdots \longrightarrow \mathcal{A}_m(m+1) \longrightarrow \mathcal{A}_m(m). \tag{1.1}$$

The algebra $\mathcal{A}_m$ has a large center $\mathcal{A}_0$ which is isomorphic to the algebra of shifted symmetric functions Λ^* (see [24]) and one has an isomorphism

$$\mathcal{A}_m \simeq \mathcal{A}_0 \otimes \mathrm{Y}_m, \tag{1.2}$$

where Y_m is the Yangian for the Lie algebra $\mathfrak{gl}(m)$; [30, Theorem 2.1.15]. In particular, for each $n \geq m$ there is a natural homomorphism

$$\mathrm{Y}_m \to \mathcal{A}_m(n). \tag{1.3}$$

This result was used in [2, 22, 17] to study the class of representations of Y_m which naturally arises from this construction. A similar result for the B, C, D types [32, 18] was used in [14, 15, 16] to construct weight bases of Gelfand–Tsetlin type for representations of the classical Lie algebras.

Our aim in the present paper is to extend these constructions to the series of the symmetric groups $S(n)$. Denote by $B_m(n)$ the centralizer of the subgroup $S(n-m)$ in the group algebra $\mathbb{C}[S(n)]$, where $S(n-m)$ consists of the permutations which fix each of the indices $1, 2, \ldots, m$. It turns out that no natural analog of the chain (1.1) exists for the algebras $B_m(n)$. Indeed, note that $B_n(n) = B_{n-1}(n) = \mathbb{C}[S(n)]$ and so, by analogy with (1.1) we would have a homomorphism $\mathbb{C}[S(n)] \to \mathbb{C}[S(n-1)]$ identical on $\mathbb{C}[S(n-1)]$. But such a homomorphism does not exist for $n > 4$.

On the other hand, it was observed in [29] that an analog of (1.3) still exists: for any $n \geq m$ there is a homomorphism $\mathcal{H}_m \to B_m(n)$, where $\mathcal{H}_m$ is the *degenerate affine Hecke algebra* introduced by Drinfeld [6] and Lusztig [12]. This fact was used by Okounkov and Vershik [25] to develop a new approach to the representation theory of the symmetric groups; see also earlier results by Cherednik [1].

This observation together with the semigroup method [28] allows one to expect that an analog of the centralizer construction for the symmetric group should exist, with the algebras $\mathbb{C}[S(n)]$ replaced with the *semigroup algebras* $A(n) = \mathbb{C}[\Gamma(n)]$, where $\Gamma(n)$ is the *semigroup of partial bijections* of the set $\{1, \ldots, n\}$. Alternatively, the elements of $\Gamma(n)$ can be identified with $(0,1)$-matrices which have at most one 1 in each row and column. The semigroups $\Gamma(n)$ are studied in [28] and used to prove Lieberman's classification theorem [11] for unitary representations of the complete infinite symmetric group.

Taking the centralizer $A_m(n)$ of $\Gamma(n-m)$ in $A(n)$ instead of the algebras $B_m(n)$ we do obtain an analog of the chain of homomorphisms (1.1) for the algebras $A_m(n)$. The corresponding projective limit algebra A_m has a decomposition analogous to (1.2)

$$A_m \simeq A_0 \otimes \widetilde{\mathcal{H}}_m, \tag{1.4}$$

where $\widetilde{\mathcal{H}}_m$ is a 'semigroup analog' of the degenerate affine Hecke algebra $\mathcal{H}_m$. The algebra $\widetilde{\mathcal{H}}_m$ can be presented by generators and defining relations. Moreover, the algebra $\mathcal{H}_m$ is a homomorphic image of $\widetilde{\mathcal{H}}_m$. The subalgebra $A_0 \subseteq A_m$ is commutative and isomorphic to the algebra of shifted symmetric functions Λ^*; see [24] for a detailed study of the algebra Λ^*.

The mentioned above homomorphisms $\mathcal{H}_m \to B_m(n)$ can be regarded as 'retractions' of the homomorphisms $\widetilde{\mathcal{H}}_m \to A_m(n)$ whose existence is provided with the centralizer construction for the symmetric groups.

Finally, the algebra A is defined as the inductive limit of A_m as $m \to \infty$. We show that A naturally acts in the so-called *tame* representations of $S(\infty)$ and it can be regarded as the 'true' analog of the group algebra $\mathbb{C}[S(\infty)]$. Indeed, contrary to the finite case of $\mathbb{C}[S(n)]$, the algebra $\mathbb{C}[S(\infty)]$ has a trivial center while A contains a large center whose elements act by scalar operators in the tame representations of $S(\infty)$. Moreover, the central elements separate the irreducible tame representations.

Some other generalizations of the degenerate affine Hecke algebra $\mathcal{H}_m$ have been constructed by Nazarov [20, 21].

The paper is organized as follows. Section 2 is preliminary. Here we define tame representations of $S(\infty)$ and describe the properties of the semigroups $\Gamma(n)$; most of these results are contained in [28]. In Section 3 we construct the algebras A_m as projective limits of the centralizers $A_m(n)$. Section 4 describes the algebra A_0 and establishes its isomorphism with the algebra of shifted symmetric functions Λ^*. The main results are given in Section 5 where we investigate the structure of A_m and describe its relationship with the degenerate affine Hecke algebras.

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2 Tame representations and semigroups

Here we give constructions of the *tame* representations of the infinite symmetric group and describe the semigroups of partial bijections. The material of this section is based on [28] and [27, §2]. About applications of the semigroup method to representations of “big” groups, see Olshanski [31], Neretin [23].

2.1 Constructing tame representations

Let $\mathbb{N} = \{1, 2, \dots\}$ and $n \in \mathbb{N}$. We denote by $S(n)$ the group of permutations of the set $\mathbb{N}_n := \{1, \dots, n\}$. We also regard $S(n)$ as the group of permutations of $\mathbb{N}$ fixing $n+1, n+2, \dots$, and set

$$S(\infty) = \bigcup_{n \geq 1} S(n).$$

This is the group of all *finite* permutations of the set $\mathbb{N}$.

For any $m \leq n$, we denote by $S_m(n)$ the subgroup of permutations in $S(n)$ fixing $1, \dots, m$, and set

$$S_m(\infty) = \bigcup_{n \geq m} S_m(n).$$

Note that the subgroups $S(n)$ and $S_n(\infty)$ of $S(\infty)$ commute. This simple observation will play an important role later.

For a (unitary) representation T, we denote by $H(T)$ its (Hilbert) space.

Let T be a unitary representation of the group $S(\infty)$. For $n = 0, 1, \dots$, denote by $H_n(T)$ the subspace of $S_n(\infty)$-invariant vectors in $H(T)$. Since $\{S_n(\infty)\}$ is an descending chain of subgroups, $\{H_n(T)\}$ is an ascending chain of subspaces. Let

$$H_\infty(T) = \bigcup_{n \geq 0} H_n(T).$$

Since $S(n)$ and $S_n(\infty)$ commute, the subspace $H_n(T)$ is invariant with respect to $S(n)$, and so, $H_\infty(T)$ is invariant with respect to the whole group $S(\infty)$.

Definition 2.1 A unitary representation T of the group $S(\infty)$ is said to be *tame* if $H_\infty(T)$ is dense in $H(T)$. $\square$

Note that for an irreducible representation T, this is equivalent to saying that $H_\infty(T)$ is nonzero.

Clearly, the trivial representation of $S(\infty)$ is tame, and another one–dimensional representation, $s \mapsto \operatorname{sgn} s$, is not tame. Less trivial examples follow.

Example 2.2 (i) Let H be the space l_2 with its canonical basis $e_1, e_2, \dots$ and let $S(\infty)$ operate in H by permuting the basis vectors. The representation H is tame. It is irreducible and for any n the subspace H_n is spanned by $e_1, \dots, e_n$.

(ii) The right (or left) regular representation of the group $S(\infty)$ in the Hilbert space $l_2\big(S(\infty)\big)$ is not tame. □

For any $n = 0, 1, 2, \dots$ and any partition $\lambda \vdash n$, we will construct a tame representation T_λ. First, if $n = 0$ then $\lambda = \emptyset$ and $T_\emptyset$ is the one-dimensional trivial representation. Let now $n \geq 1$ and let π_λ denote the irreducible representation of $S(n)$ corresponding to λ. Then T_λ is defined as the induced representation

$$T_\lambda = \operatorname{Ind}_{S(n)\times S_n(\infty)}^{S(\infty)}(\pi_\lambda \otimes 1) \tag{2.1}$$

where 1 stands for the trivial representation of $S_n(\infty)$. The representation T_λ can be realized as follows. Let $\Omega(n)$ denote the set of injective maps $\omega : \mathbb{N}_n \to \mathbb{N}$. Define a right action of $S(\infty)$ on $\Omega(n)$ by

$$\omega \cdot s = s^{-1} \circ \omega, \qquad s \in S(\infty), \tag{2.2}$$

and a left action of $S(n)$ by

$$t \cdot \omega = \omega \circ t^{-1}, \qquad t \in S(n). \tag{2.3}$$

Note that these two actions commute. Consider the Hilbert space $l_2\big(\Omega(n), H(\pi_\lambda)\big)$ of $H(\pi_\lambda)$-valued square-integrable functions on $\Omega(n)$, and let $H(n, \lambda)$ be its subspace formed by the functions $f(\omega)$ such that

$$f(t \cdot \omega) = \pi_\lambda(t) f(\omega), \qquad t \in S(n), \quad \omega \in \Omega(n). \tag{2.4}$$

The action of T_λ in $H(n, \lambda)$ is given by

$$(T_\lambda(s) f)(\omega) = f(\omega \cdot s). \tag{2.5}$$

The space $H(T_\lambda)$ may now be identified with $H(n, \lambda)$. For any $l \geq n$ set

$$\Omega(n, l) = \{\omega \in \Omega(n) \mid \omega(\mathbb{N}_n) \subseteq \mathbb{N}_l\} \tag{2.6}$$

and note that

$$\Omega(n) = \bigcup_{l \geq n} \Omega(n, l) \tag{2.7}$$

Also, set

$$H'_l(T_\lambda) = \{f \in H(n, \lambda) \mid \operatorname{supp} f \subseteq \Omega(n, l)\}, \tag{2.8}$$

where $\operatorname{supp} f = \{\omega \in \Omega(n) \mid f(\omega) \neq 0\}$.

Proposition 2.3 *For any $n \in \mathbb{N}$ we have*

$$H_l(T_\lambda) = \begin{cases} \{0\} & \text{if } l < n, \\ H'_l(T_\lambda) & \text{if } l \geq n. \end{cases} \tag{2.9}$$

Proof. Since $S_l(\infty)$ acts trivially on $\Omega(n, l)$, we have $H'_l(T_\lambda) \subseteq H_l(T_\lambda)$. Conversely, let $f \in H_l(T_\lambda)$. Then the function $\|f(\omega)\|^2$ is constant on any orbit of the subgroup $S_l(\infty)$ in $\Omega(n)$. Since the sum of the $\|f(\omega)\|^2$ taken over $\omega \in \Omega(n)$ must be finite, we have $\|f(\omega)\| = 0$ unless the orbit containing ω is finite. But if $\omega \notin \Omega(n, l)$, then its orbit is clearly infinite, so that $f(\omega) = 0$. This proves the opposite inclusion $H_l(T_\lambda) \subseteq H'_l(T_\lambda)$. □

Proposition 2.4 *For any $n \in \mathbb{N}$ and any $\lambda \vdash n$, the representation T_λ of $S(\infty)$ is tame and irreducible.*

Proof. By Proposition 2.3,

$$H_\infty(T_\lambda) = \bigcup_l H_l(T_\lambda) = \bigcup_l H'_l(T_\lambda). \tag{2.10}$$

This is the subspace of finitely supported functions in $H(T_\lambda) = H(n, \lambda)$ which is clearly dense. So, T_λ is tame.

The subspace $H_n(T_\lambda) = H'_n(T_\lambda)$ is both cyclic in $H(T_\lambda)$ and irreducible under the action of the subgroup $S(n)$. This implies that T_λ is irreducible. □

We shall identify any partition λ with its Young diagram; see e.g. [13]. We write $|\lambda| = n$ if λ has n boxes. Given two diagrams λ and μ the notation $\mu \nearrow \lambda$ will mean that μ can be obtained from λ by removing one box, i.e. $\mu \subset \lambda$ and $|\mu| = |\lambda| - 1$. An *infinite tableau* is defined as an infinite chain of diagrams

$$\tau = (\lambda^{(1)} \nearrow \lambda^{(2)} \nearrow \dots), \qquad |\lambda^{(n)}| = n. \tag{2.11}$$

Two infinite tableaux will be called *equivalent* if the corresponding chains of diagrams differ in a finite number of places only.

Given an infinite tableau τ, we may construct an inductive limit unitary representation $\Pi(\tau)$ of the group $S(\infty)$ as follows. By the branching rule for the symmetric groups (see e.g. [8], [25]), for any n the representation $\pi_{\lambda^{(n)}}$ occurs in the decomposition of $\pi_{\lambda^{(n+1)}} \downarrow S(n)$ with multiplicity one. Hence there is an infinite chain of embeddings

$$\pi_{\lambda^{(1)}} \hookrightarrow \pi_{\lambda^{(2)}} \hookrightarrow \dots \tag{2.12}$$

which are defined uniquely up to scalar multiples. So we may set

$$\Pi(\tau) = \lim \operatorname{ind} \pi_{\lambda^{(n)}}, \qquad n \to \infty. \tag{2.13}$$

One can show that any $\Pi(\tau)$ is irreducible (cf. [26, Theorem 2.1] and [27, §2.7]), and that $\Pi(\tau)$ and $\Pi(\tau')$ are isomorphic if and only if τ and τ' are equivalent.

This construction provides a large class of pairwise non-equivalent irreducible representations of the group $S(\infty)$. We will be only interested in some special representations of this class. Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be an arbitrary diagram. Consider an infinite tableau $\tau = (\lambda^{(i)})$ such that

$$\lambda^{(i)} = (i - |\lambda|, \lambda_1, \dots , \lambda_l) \quad \text{for} \quad i \geq |\lambda| + \lambda_1, \tag{2.14}$$

and set $\Pi_\lambda = \Pi(\tau)$. The representation Π_λ is well defined since the equivalence class of Π_λ does not depend on the choice of $\lambda^{(i)}$ for small values of i.

Proposition 2.5 *The representations T_λ and Π_λ are isomorphic for any λ.*

Proof. For any $l \geq n + \lambda_1$, the natural representation of $S(l)$ in the space $H_l(T_\lambda)$ is isomorphic to the induced representation

$$\operatorname{Ind}_{S(n)\times S(l-n)}^{S(l)} (\pi_\lambda \otimes 1) \tag{2.15}$$

where $S(l-n)$ is identified with $S_n(l)$, and 1 stands for the trivial representation of $S(l-n)$. This follows immediately from (2.9). It is well known [8] that the representation (2.15) is multiplicity free and that its spectrum consists of the representations π_μ such that $\mu \vdash l$ and

$$\mu_{i+1} \leq \lambda_i \leq \mu_i, \qquad i \geq 1. \tag{2.16}$$

It follows from (2.16) that $\pi_{\lambda^{(l)}}$ occurs in the decomposition of (2.15). Let $H_l^0(T_\lambda)$ denote the corresponding subspace of $H_l(T_\lambda)$. It remains to prove that $H_m^0(T_\lambda)$ is contained in $H^0_l(T_\lambda)$ for any $m < l$ provided that m is large enough. This follows from the fact that

$$\pi_{\lambda^{(m)}} \subset \pi_{\lambda^{(l)}}\big|_{S(m)}, \tag{2.17}$$

and $\pi_{\lambda^{(m)}} \not\subset \pi_\mu\big|_{S(m)}$ if μ satisfies (2.16) and $\mu \neq \lambda^{(l)}$. Indeed, property (2.17) follows from the definition of the diagrams $\lambda^{(l)}$ for large l and the branching rule. Finally, note that there exists i such that $\lambda_i > \mu_{i+1}$ (otherwise $\mu = \lambda^{(l)}$). Applying the branching rule again we complete the proof. □

2.2 The semigroup method

Definition 2.6 Let X be a set.

(i) A *partial bijection* of X is a bijection $\gamma : D \to R$ between two (possibly empty) subsets of X. The subset $D \subseteq X$ is called the *domain* of γ and denoted by $\operatorname{dom}\gamma$. The subset $R \subseteq X$ is called the *range* of γ and denoted by $\operatorname{range}\gamma$. If $x \in X$ belongs to $\operatorname{dom}\gamma$, then we will say that γ *is defined on* x. The set of partial bijections of X is denoted by $\Gamma(X)$.

(ii) Given $\gamma \in \Gamma(X)$, we define $\gamma^* \in \Gamma(X)$ as the inverse bijection $\gamma^{-1} : \operatorname{range}\gamma \to \operatorname{dom}\gamma$, so that $\operatorname{dom}\gamma^* = \operatorname{range}\gamma$ and $\operatorname{range}\gamma^* = \operatorname{dom}\gamma$. The mapping $\gamma \to \gamma^*$ is involutive: $(\gamma^*)^* = \gamma$.

(iii) Given $\gamma_1, \gamma_2 \in \Gamma(X)$ with $\operatorname{dom}\gamma_i = D_i$ and $\operatorname{range}\gamma_i = R_i$, $i = 1, 2$, their *product* $\gamma_1\gamma_2$ is a partial bijection on X with $D = \operatorname{dom}\gamma_1\gamma_2 = \gamma_2^{-1}(D_1 \cap R_2)$ and $R = \operatorname{range}\gamma_1\gamma_2 = \gamma_1(D_1 \cap R_2)$:

$$\gamma_1\gamma_2 = (\gamma_1\big|_{\gamma_2(D)}) \circ (\gamma_2\big|_D). \tag{2.18}$$

That is, $\gamma_1\gamma_2$ is defined on $x \in X$ if and only if γ_2 is defined on x and γ_1 is defined on $\gamma_2(x)$; then $(\gamma_1\gamma_2)(x) = \gamma_1(\gamma_2(x))$. □

Any $\gamma \in \Gamma(X)$ may be regarded as a *relation* on X. The product defined above is the *product of relations*; see [3]. With this product $\Gamma(X)$ becomes a semigroup, called the *semigroup of partial bijections* of X.

The involution $\gamma \to \gamma^*$ is an anti-homomorphism of $\Gamma(X)$, so that $\Gamma(X)$ is an *involutive semigroup*.

The semigroup $\Gamma(X)$ possesses the *unity* 1, which is the identity bijection $X \to X$, and the *zero* 0, which is the (unique) bijection of the empty subset of X onto itself. Note that

$$1\,\gamma = \gamma\,1 = \gamma, \quad 0\,\gamma = \gamma\,0 = 0, \qquad \text{for any } \gamma \in \Gamma(X). \tag{2.19}$$

For any subset $Y \subseteq X$, let $1_Y \in \Gamma(X)$ denote the identity bijection $Y \to Y$. In particular, $1_X = 1$ and $1_\emptyset = 0$. Then 1_Y is a self-adjoint idempotent, i.e.,

$$(1_Y)^* = 1_Y, \quad (1_Y)^2 = 1_Y. \tag{2.20}$$

Moreover, all idempotents of this type are pairwise commuting. For any $\gamma \in \Gamma(X)$ we obviously have

$$\gamma^* \gamma = 1_{\operatorname{dom}\gamma}, \quad \gamma \gamma^* = 1_{\operatorname{range}\gamma}. \tag{2.21}$$

The subset of those $\gamma \in \Gamma(X)$ for which $\operatorname{dom}\gamma = \operatorname{range}\gamma = X$ is clearly identified with the group $S(X)$ of all permutations of the set X.

Remark. The semigroup $\Gamma(X)$ is a model example of an *inverse semigroup* (each of its elements has an inverse); see [3]. The class of inverse semigroups is very closed to that of groups, and the role of the semigroups of partial bijections $\Gamma(X)$ is quite similar to that of the symmetric groups $S(X)$. In particular, any inverse semigroup is isomorphic to an involutive subsemigroup of some $\Gamma(X)$: this is an analog of Cayley theorem; see [3]. □

There is a convenient realization of partial bijections by (0,1)-matrices, i.e., by the matrices whose entries are 0 or 1. A (0,1)-matrix is said to be *monomial* if any of its rows or columns contains at most one 1. Given a set X, we will consider (0,1)-matrices whose rows and columns are labeled by the points of X. Then to any $\gamma \in \Gamma(X)$, we assign a monomial (0,1)-matrix $[\gamma_{xy}]$ as follows: for $x, y \in X$

$$\gamma_{xy} = \begin{cases} 1 & \text{if } y \in \operatorname{dom}\gamma \text{ and } \gamma(y) = x, \\ 0 & \text{otherwise.} \end{cases} \tag{2.22}$$

In particular, the unity $1 \in \Gamma(X)$ and the zero $0 \in \Gamma(X)$ are represented by the unit and the zero matrices, respectively. We thus obtain an isomorphism between the semigroup $\Gamma(X)$ and the semigroup of monomial matrices over X with the usual matrix multiplication. Note that in this matrix realization, the involution $\gamma \mapsto \gamma^*$ is represented by the matrix transposition.

Definition 2.7 For $Y \subseteq X$ we define three mappings as follows:

(i) The *canonical projection* $\theta : \Gamma(X) \to \Gamma(Y)$. Let $\gamma \in \Gamma(X)$. Then $\theta(\gamma)$ is defined at $y \in Y$ if γ is defined at y and $\gamma(y) \in Y$. The image of y with respect to $\theta(\gamma)$ is $\gamma(y)$. In matrix terms: the matrix of $\theta(\gamma)$ is obtained from that of γ by striking the rows and columns corresponding to points of $X \setminus Y$.

(ii) The *canonical embedding* $\phi : \Gamma(Y) \hookrightarrow \Gamma(X)$. Let $\gamma \in \Gamma(Y)$. Then $\phi(\gamma)$ is defined at $x \in X$ if and only if $x \in \operatorname{dom}\gamma \subseteq Y$ or $x \in X \setminus Y$. In the former case $\phi(\gamma)$ sends x to $\gamma(x)$, and in the latter case it fixes x. In matrix terms, for $x, y \in X$,

$$\phi(\gamma)_{xy} = \begin{cases} \gamma_{xy} & \text{if } x, y \in Y, \\ \delta_{xy} & \text{otherwise.} \end{cases} \tag{2.23}$$

(iii) The 0-*embedding* $\psi : \Gamma(Y) \hookrightarrow \Gamma(X)$. Let $\gamma \in \Gamma(Y)$. Then $\psi(\gamma)$ is obtained by regarding $\operatorname{dom}\gamma$ and $\operatorname{range}\gamma$ as subsets of X. In matrix terms, for $x, y \in X$,

$$\psi(\gamma)_{xy} = \begin{cases} \gamma_{xy} & \text{if } x, y \in Y, \\ 0 & \text{otherwise.} \end{cases} \tag{2.24}$$

□

For a positive integer n we shall denote by $\Gamma(n)$ the semigroup $\Gamma(\mathbb{N}_n)$. Given a tame representation T of $S(\infty)$ we now construct a representation $\mathcal{T}_n$ of the semigroup $\Gamma(n)$. Regarding $S(\infty)$ as the group of infinite monomial matrices which have only a finite number of 1's off the diagonal, define the map $\theta^{(n)} : S(\infty) \to \Gamma(n)$ which assigns to each infinite matrix its upper left $n \times n$ submatrix. It is easy to check that the map $\theta^{(n)}$ is surjective and it induces a bijection between the set of double cosets $S_n(\infty)\backslash S(\infty)/S_n(\infty)$ and $\Gamma(n)$; see [28].

Let T be a tame representation of $S(\infty)$. Denote by P_n the orthogonal projection $P_n : H(T) \to H_n(T)$. Suppose that n is so large that $H_n(T) \neq \{0\}$. For any $\sigma_1, \sigma_2 \in S(\infty)$ we have

$$\theta^{(n)}(\sigma_1) = \theta^{(n)}(\sigma_2) \quad \Rightarrow \quad P_n T(\sigma_1) P_n = P_n T(\sigma_2) P_n. \tag{2.25}$$

Therefore there exists a unique map $\mathcal{T}_n : \Gamma(n) \to \operatorname{End} H_n(T)$ such that

$$\mathcal{T}_n\big(\theta^{(n)}(\sigma)\big) = P_n T(\sigma)|_{H_n(T)}, \qquad \sigma \in S(\infty). \tag{2.26}$$

Proposition 2.8 *The map $\mathcal{T}_n = \mathcal{T}_n(T)$ defined by (2.26) is a representation of the semigroup $\Gamma(n)$ in $H_n(T)$.*

Proof. We give a sketch of the proof. The details and one more proof can be found in [28]. We show first that the tame representation T of $S(\infty)$ can be extended to a representation $\mathcal{T}$ of the semigroup $\Gamma(\infty)$ of partial bijections of the set $\mathbb{N}$.

Further, we prove that P_n coincides with the operator $\mathcal{T}(1_n)$ where 1_n is the identity bijection of the subset $\mathbb{N}_n$.

Finally, consider the 0-embedding $\psi : \Gamma(n) \to \Gamma(\infty)$; see (2.24). We have for any $\gamma \in \Gamma(n)$

$$\mathcal{T}_n(\gamma) = P_n \mathcal{T}\big(\psi(\gamma)\big) P_n = \mathcal{T}(1_n)\mathcal{T}\big(\psi(\gamma)\big)\mathcal{T}(1_n) = \mathcal{T}(1_n \psi(\gamma) 1_n) = \mathcal{T}(\psi(\gamma)). \quad (2.27)$$

This proves the multiplicativity of $\mathcal{T}_n$. □

Consider the tame representations T_λ of $S(\infty)$ constructed in the previous section. Proposition 2.8 yields a representation $\mathcal{T}_n(\lambda) := \mathcal{T}_n(T_\lambda)$ of $\Gamma(n)$ provided that $H_n(T_\lambda) \neq 0$, i.e., $n \geq |\lambda|$.

We now outline the proof of the classification theorem for representations of $\Gamma(n)$; see [28].

Theorem 2.9 *The representations $\mathcal{T}_n(\lambda)$ where λ is a partition with $|\lambda| \leq n$ exhaust all irreducible representations of $\Gamma(n)$.*

Proof. Let $\mathcal{T}$ be a representation of $\Gamma(n)$. Then for any $m \leq n$ the operator $\mathcal{T}(1_m)$ is a projection. Denote its image by $H_m(\mathcal{T})$. We let $m(\mathcal{T})$ denote the minimum value of m such that $H_m(\mathcal{T}) \neq \{0\}$.

Further, if $\mathcal{T}$ is irreducible then one shows that $\dim \mathcal{T} < \infty$. If $m = m(\mathcal{T})$ then the subspace $H_m(\mathcal{T})$ is invariant under $S(m)$ and irreducible. So, $H_m(\mathcal{T})$ corresponds to a partition λ with $|\lambda| = m$. A standard argument proves that $\mathcal{T}$ is uniquely determined by λ.

Conversely, given a partition λ with $|\lambda| = m \leq n$ one uses the following argument to construct an irreducible representation $\mathcal{T}$ of $\Gamma(n)$ such that $m = m(\mathcal{T})$ and the representation of $S(m)$ in $H_m(\mathcal{T})$ corresponds to λ.

Denote by $\Omega(m,n)$ the set of injective maps $\omega : \mathbb{N}_m \to \mathbb{N}_n$. We define $H(\mathcal{T})$ to be the space of functions $f : \Omega(m,n) \to H(\pi_\lambda)$ such that

$$f(t \cdot \omega) = \pi_\lambda(t) f(\omega), \qquad t \in S(m), \qquad \omega \in \Omega(m,n); \quad (2.28)$$

see (2.3). The action of $\gamma \in \Gamma(n)$ is given by

$$(\mathcal{T}(\gamma)f)(\omega) = \begin{cases} f(\gamma^* \omega) & \text{if } \omega = (\omega_1, \dots, \omega_m) \subseteq \operatorname{dom} \gamma^*, \\ 0 & \text{otherwise.} \end{cases} \quad (2.29)$$

One easily checks that $\mathcal{T}$ is a representation of $\Gamma(n)$. Moreover, it is isomorphic to the representation $\mathcal{T}_n(\lambda)$. □

Note that the representation of $\Gamma(n)$ corresponding to $m = 0$ and the empty diagram is the trivial representation sending all elements of $\Gamma(n)$ to 1.

Theorem 2.9 leads to the following result.

Theorem 2.10 *Let λ range over the set of all Young diagrams including the empty diagram. The representations T_λ constructed in Section 2.1 exaust, within equivalence, all the irreducible tame representations of the group $S(\infty)$. Moreover, any tame representation of $S(\infty)$ can be decomposed into a direct sum of irreducible tame representations.*

Proof. See Theorem 6.7 and §7.2 in [28]. $\square$

This is equivalent to Lieberman's theorem [11] concerning continuous unitary representations of the complete infinite symmetric group (this group consists of all permutations of the set $\mathbb{N}$); see [28, §7].

3 Centralizer construction

We shall denote by θ_n the canonical projection $\Gamma(n) \to \Gamma(n-1)$; see Definition 2.7. So, if $\gamma \in \Gamma(n)$ then $\theta_n(\gamma)$ is the upper left corner of γ of order $n-1$. Here the elements of $\Gamma(n)$ and $\Gamma(n-1)$ are regarded as $(0,1)$-matrices of order n and $n-1$, respectively.

We set $A(n) = \mathbb{C}[\Gamma(n)]$, the semigroup algebra of $\Gamma(n)$. The canonical embedding $\Gamma(n-1) \hookrightarrow \Gamma(n)$ is extended to an embedding $A(n-1) \hookrightarrow A(n)$ by linearity. Further, set $A(\infty) = \mathbb{C}[\Gamma(\infty)] = \bigcup_{n\geq 1} A(n)$, the semigroup algebra of $\Gamma(\infty)$.

For each $i = 1, \ldots, n$ denote by ε_i the diagonal $n \times n$-matrix whose ii-th entry is 0 and all other diagonal entries are equal to 1. The corresponding element of $\Gamma(n)$ is the identity bijection of the set $\mathbb{N}_n \setminus \{i\}$. The algebra $A(n)$ is obviously generated by $S(n)$ and the elements ε_i, $i = 1, \ldots, n$. We have for any i and any $\sigma \in S(n)$:

$$\sigma\, \varepsilon_i\, \sigma^{-1} = \varepsilon_{\sigma(i)}. \tag{3.1}$$

Proposition 3.1 *The algebra $A(n)$ is isomorphic to the abstract algebra with generators $s_1, \ldots, s_{n-1}$, $\varepsilon_1, \ldots, \varepsilon_n$ and the defining relations*

$$s_k^2 = 1, \qquad s_k\, s_{k+1}\, s_k = s_{k+1}\, s_k\, s_{k+1}, \qquad s_k\, s_l = s_l\, s_k, \quad |k-l| > 1, \tag{3.2}$$

$$\varepsilon_k^2 = \varepsilon_k, \qquad \varepsilon_k\, \varepsilon_l = \varepsilon_l\, \varepsilon_k, \tag{3.3}$$

$$s_k\, \varepsilon_k = \varepsilon_{k+1}\, s_k, \qquad s_k\, \varepsilon_k\, \varepsilon_{k+1} = \varepsilon_k\, \varepsilon_{k+1}. \tag{3.4}$$

Proof. Denote the abstract algebra by $\mathcal{A}(n)$. The assignments $s_k \mapsto (k, k+1)$ and $\varepsilon_k \mapsto \varepsilon_k$ obviously define an algebra epimorphism $\mathcal{A}(n) \to A(n)$. Note also that (3.2) are defining relations for the symmetric group $S(n)$ and (3.1) holds. To complete the proof we verify that $\dim \mathcal{A}(n) \le \dim A(n)$. We have

$$\dim A(n) = |\,\Gamma(n)| = \sum_{r=0}^{n} \binom{n}{r}^2 r!\ . \tag{3.5}$$

On the other hand, we see from the relations (3.3) and (3.4) that

$$\mathcal{A}(n) = \mathbb{C}[S(n)]\,\mathbb{C}[\varepsilon_1, \dots, \varepsilon_n]. \tag{3.6}$$

Here $\mathbb{C}[\varepsilon_1, \dots, \varepsilon_n]$ is the subalgebra of $\mathcal{A}(n)$ generated by the ε_k. It is spanned by the monomials $\varepsilon_{k_1} \cdots \varepsilon_{k_r}$ with $k_1 < \cdots < k_r$. Given such a monomial, consider the subspace $\mathbb{C}[S(n)]\,\varepsilon_{k_1} \cdots \varepsilon_{k_r}$ in $\mathcal{A}(n)$. Using (3.1) if necessary, we may assume without loss of generality that $k_i = i$ for each i. Observe that by (3.4) we have in $\mathcal{A}(n)$

$$\sigma\, S(r)\, \varepsilon_1 \cdots \varepsilon_r = \sigma\, \varepsilon_1 \cdots \varepsilon_r, \qquad \sigma \in S(n). \tag{3.7}$$

Hence the dimension of the subspace $\mathbb{C}[S(n)]\,\varepsilon_1 \cdots \varepsilon_r$ does not exceed the number of left cosets of $S(n)$ over the subgroup $S(r)$. Therefore, $\dim \mathcal{A}(n)$ does not exceed

$$\sum_{r=0}^{n} \binom{n}{r} \frac{n!}{r!} \tag{3.8}$$

which coincides with (3.5). □

Using Proposition 3.1 we shall sometimes identify $A(n)$ with the algebra $\mathcal{A}(n)$.

Corollary 3.2 *The mapping*

$$(k, k+1) \mapsto (k, k+1), \qquad \varepsilon_k \mapsto 0 \tag{3.9}$$

defines an algebra homomorphism $A(n) \to \mathbb{C}[S(n)]$ which is identical on the subalgebra $\mathbb{C}[S(n)]$. □

We shall call (3.9) the *retraction homomorphism*. It can be equivalently defined as follows. For any $\gamma \in \Gamma(n)$, define its *rank*, denoted by $\operatorname{rank}\gamma$, as the number of 1's in the $(0,1)$-matrix representing γ. That is,

$$\operatorname{rank}\gamma = |\operatorname{dom}\gamma| = |\operatorname{range}\gamma|. \tag{3.10}$$

The rank of an element $a = \sum a_\gamma \gamma \in A(n)$ is defined as the maximum of the ranks $\operatorname{rank} \gamma$ with $a_\gamma \neq 0$. Now (3.9) can also be defined by setting for $\gamma \in \Gamma(n)$

$$\gamma \to \begin{cases} \gamma & \text{if } \operatorname{rank} \gamma = n, \\ 0 & \text{if } \operatorname{rank} \gamma < n, \end{cases} \tag{3.11}$$

and extending this to $A(n)$ by linearity.

For any $0 \leq m \leq n$ denote by $\Gamma_m(n)$ the subsemigroup of $\Gamma(n)$ which consists of the matrices with first m diagonal entries equal to 1. Set $A_m(n) = A(n)^{\Gamma_m(n)}$, the centralizer of $\Gamma_m(n)$ in the algebra $A(n)$. In particular, $A_0(n)$ is the center of $A(n)$.

We extend θ_n to a linear map $A(n) \to A(n-1)$.

Proposition 3.3 *The restriction of θ_n to $A_{n-1}(n) \subseteq A(n)$ defines a unital algebra homomorphism*

$$\theta_n : A_{n-1}(n) \to A(n-1). \tag{3.12}$$

Moreover,

$$\theta_n(A_m(n)) \subseteq A_m(n-1) \qquad \textit{for} \ \ m = 0, 1, \ldots, n-1. \tag{3.13}$$

Proof. Note that any $a \in A_{n-1}(n)$ commutes with ε_n because ε_n is contained in $\Gamma_{n-1}(n)$, and $A_{n-1}(n)$ is the centralizer of $\Gamma_{n-1}(n)$ in $A(n)$. Since ε_n is an idempotent we have

$$\theta_n(a) = a\,\varepsilon_n = \varepsilon_n\,a = \varepsilon_n\,a\,\varepsilon_n, \qquad a \in A_{n-1}(n). \tag{3.14}$$

Now, if $a, b \in A_{n-1}(n)$ then

$$\theta_n(ab) = \varepsilon_n\,ab\,\varepsilon_n = \varepsilon_n\,a\,\varepsilon_n\,\varepsilon_n\,b\,\varepsilon_n = \theta_n(a)\theta_n(b). \tag{3.15}$$

It is clear that (3.12) preserves the unity.

Finally, we need to show that if $a \in A_m(n)$ and $b \in \Gamma_m(n-1)$ then $\theta_n(a)$ and b commute. Regard b as an element of $\Gamma(n)$; then it lies in $\Gamma_m(n)$ and commutes with ε_n. This implies that $a\,\varepsilon_n$ and b commute, and so do $\theta_n(a)$ and b. □

For $\gamma \in \Gamma(n)$, set

$$\begin{aligned} J(\gamma) &= \{i \mid 1 \leq i \leq n,\ \gamma_{ii} = 0\}, \\ \deg \gamma &= |\,J(\gamma)|. \end{aligned} \tag{3.16}$$

Proposition 3.4 *Let $\gamma, \gamma' \in \Gamma(n)$. Then*

$$\deg \gamma\gamma' \leq \deg \gamma + \deg \gamma'. \tag{3.17}$$

Moreover, the equality in (3.17) *implies $\gamma\gamma' = \gamma'\gamma$.*

Proof. Regard γ and γ' as partial bijections of the set $\mathbb{N}_n = \{1, \ldots, n\}$. If $\delta \in \Gamma(n)$ then $\mathbb{N}_n \setminus J(\delta)$ is the set of δ-invariant elements in the domain of δ. This implies

$$\big(\mathbb{N}_n \setminus J(\gamma)\big) \cap \big(\mathbb{N}_n \setminus J(\gamma')\big) \subseteq \big(\mathbb{N}_n \setminus J(\gamma\gamma')\big). \tag{3.18}$$

Therefore,

$$J(\gamma) \cup J(\gamma') \supseteq J(\gamma\gamma'), \tag{3.19}$$

and (3.17) follows. Finally, the equality in (3.17) implies that $J(\gamma)$ and $J(\gamma')$ are disjoint. But then γ and γ' must commute. □

Using (3.16), we define a filtration of the space $A(n)$:

$$\mathbb{C} = A^0(n) \subseteq A^1(n) \subseteq \ldots \subseteq A^n(n) = A(n) \tag{3.20}$$

where $A^M(n)$ is spanned by the subset $\{\gamma \mid \deg \gamma \leq M\} \subseteq \Gamma(n)$. By Proposition 3.4 this filtration is compatible with the algebra structure of $A(n)$, and the corresponding graded algebra

$$\operatorname{gr} A(n) = \bigoplus_M \big(A^M(n)/A^{M-1}(n)\big) \tag{3.21}$$

is commutative. Note that for $\gamma \in \Gamma(n)$ the degree $\deg \theta_n(\gamma)$ can be equal either to $\deg \gamma$ or $\deg \gamma - 1$. Therefore the homomorphisms (3.13) are compatible with the filtration on $A(n)$.

Definition 3.5 For $m = 0, 1, 2, \ldots$ let A_m be the projective limit of the infinite sequence

$$\cdots \longrightarrow A_m(n) \xrightarrow{\theta_n} A_m(n-1) \longrightarrow \cdots \longrightarrow A_m(m+1) \xrightarrow{\theta_{m+1}} A_m(m) \tag{3.22}$$

taken in the category of filtered algebras. □

By the definition, an element $a \in A_m$ is a sequence $(a_n \mid n \geq m)$ such that

$$a_n \in A_m(n), \quad \theta_n(a_n) = a_{n-1}, \qquad \deg a := \sup_{n \geq m} \deg a_n < \infty \tag{3.23}$$

with the componentwise operations. For $n \geq m$ we shall denote by $\theta^{(n)}$ the projection $A_m \to A_m(n)$ such that

$$\theta^{(n)}(a) = a_n. \tag{3.24}$$

The M-th term of the filtered algebra A_m will be denoted by A_m^M.

There are natural algebra homomorphisms $A_m \to A_{m+1}$ defined by

$$(a_n \mid n \geq m) \mapsto (a_n \mid n \geq m+1) \tag{3.25}$$

where we use the inclusions $A_m(n) \subset A_{m+1}(n)$ for $n > m$. These homomorphisms are injective because a_m is uniquely determined by a_{m+1}.

Definition 3.6 The algebra A is defined as the inductive limit (the union) of the algebras A_m taken with respect to the embeddings $A_m \hookrightarrow A_{m+1}$, $m \geq 0$, defined in (3.25). □

Since these embeddings preserve the filtration, A is a filtered algebra. We will denote by A^M the M-th term of the filtration, so that

$$A^M = \bigcup_{m \geq 0} A_m^M. \tag{3.26}$$

Proposition 3.7 *There exists a natural embedding $A(\infty) \hookrightarrow A$ whose image consists of stable sequences $a = (a_n) \in A$.*

Proof. Let $b \in A(\infty)$. There exists m such that $b \in A(m)$. Note that $b \in A_m(n)$ for any $n \geq m$ since $A(m)$ and $\Gamma_m(n)$ commute. Set

$$a = (a_n \mid n \geq m) \in A_m \subset A \quad \text{with } a_n \equiv b. \tag{3.27}$$

The sequence $a \in A$ only depends on b and not on the choice of m. The mapping $b \mapsto a$ is clearly an algebra embedding. □

Corollary 3.8 *There is a natural algebra embedding $\mathbb{C}[S(\infty)] \hookrightarrow A$.* □

Proposition 3.9 *The center of the algebra A coincides with A_0.*

Proof. Recall that $A_0(n)$ is the center of $A(n)$. The subalgebra A_0 is contained in the center of A since the sequences $a = (a_n) \in A$ are multiplied componentwise.

Conversely, if a belongs to the center of A then a commutes with the subalgebra $A(\infty) \subset A$. This implies that for any n the element a_n is contained in $A_0(n)$, and so $a \in A_0$. □

Remark. The same argument shows that the subalgebra $A_m \subset A$ coincides with the centralizer in A of the subalgebra

$$\bigcup_{n \geq m} \mathbb{C}[\Gamma_m(n)] \subset A(\infty) \subset A. \tag{3.28}$$

□

Note that the centers of both algebras $\mathbb{C}[S(\infty)]$ and $A(\infty)$ are trivial. However, as it will be shown in the next section, the center A_0 of the algebra A has a rich structure.

Proposition 3.10 *For any tame representation T of the group $S(\infty)$, the subspace $H_\infty(T) \subseteq H(T)$ admits a natural structure of an A-module such that for any m the subspace $H_m(T)$ is invariant with respect to the subalgebra A_m (and hence $H_n(T)$ is invariant with respect to A_m for $n \geq m$).*

Proof. Let $a \in A$ and $h \in H_\infty(T)$. Choose m such that $a \in A_m$. Then we may write $a = (a_n \mid n \geq m)$. Let us prove that

$$a_n h = a_{n+1} h, \quad n \geq m. \tag{3.29}$$

Consider the family of representations $\{\mathcal{T}_n\}$ associated with T, which has been introduced in Section 2.2. Each $\mathcal{T}_n$ is a representation of the semigroup $\Gamma(n)$ in the space $H_n(T)$ and so it can be extended to a representation of the semigroup algebra $A(n)$ in the same space. Recall that $\mathcal{T}_{n+1}(\varepsilon_{n+1})$ projects $H_{n+1}(T)$ onto its subspace $H_n(T)$. Since h is already contained in $H_n(T)$ (as we assume $n \geq m$), we have $\mathcal{T}_{n+1}(\varepsilon_{n+1})h = h$, so that

$$T_{n+1}(1 - \varepsilon_{n+1})h = 0. \tag{3.30}$$

This implies that h is annihilated by the left ideal $I(n+1) \subset A(n+1)$. Since $a_n - a_{n+1} \in I(n+1)$, this implies (3.29).

Define a mapping

$$A \times H_\infty(T) \to H_\infty(T), \qquad (a,h) \mapsto a_m h \tag{3.31}$$

where m is so large that $a \in A_m$ and $h \in H_m(T)$. Note that under this assumption $ah \in H_m(T)$.

The mapping (3.31) is clearly bilinear and $1\,h = h$. The multiplicativity property $(ab)h = a(bh)$ follows from the definition of the multiplication in A_m. □

Proposition 3.11 *If T is an irreducible tame representation of $S(\infty)$ then $H_\infty(T)$ is irreducible as an A-module. In particular, the center A_0 of A acts by scalar operators.*

Proof. The first claim is obvious because $H_\infty(T)$ is already irreducible as a $A(\infty)$-module (see Proposition 3.7). To prove the second claim consider an element $a \in A_0$ as an operator in $H_\infty(T)$. It suffices to show that a has an eigenvalue. The result will then follow by a standard argument using Schur's lemma.

Assume that a has no eigenvalues. Let us show first that a is algebraically independent over $\mathbb{C}$. Indeed, let $P(x) \in \mathbb{C}[x]$ be a nonzero polynomial of a minimum degree such that $P(a) = 0$. Then $P(x) = (x - \alpha)\,Q(x)$ for a certain $\alpha \in \mathbb{C}$ and a polynomial $Q(x) \in \mathbb{C}[x]$. Since $Q(a)$ is a nonzero operator, there is a vector $v \in H_\infty(T)$ such that $w := Q(a)v \neq 0$. Then w is an eigenvector for a with the eigenvalue α. Contradiction.

We note now that the space $H_\infty(T)$ has countable dimension and then use a version of Dixmier's argument [4] as follows.

Since a is algebraically independent over $\mathbb{C}$, the field $\mathbb{C}(a)$ is embedded in the endomorphism algebra of $H_\infty(T)$. This implies that the dimension of $H_\infty(T)$ is at least as large as the dimension of $\mathbb{C}(a)$ over $\mathbb{C}$, but the latter is continuum. This contradiction completes the proof. □

4 The structure of the algebra A_0

In the last two sections we aim to describe the structure of the algebras A_m. Here we consider the commutative algebra A_0; see Proposition 3.9. We construct generators of A_0 and show that it is isomorphic to the algebra of shifted symmetric functions.

4.1 Generators of A_0

Let $Z(S(n))$ denote the center of the algebra $\mathbb{C}[S(n)]$. For $0 \le M \le n$ denote by $Z^M(S(n))$ the M-th term of the filtration on $Z(S(n))$ inherited from the algebra $A(n)$; see (3.20). Note that

$$Z^0(S(n)) = Z^1(S(n)) = \mathbb{C}1 \tag{4.1}$$

because, for a permutation $s \in S(n)$ the inequality $\deg s \le 1$ implies $s = 1$.

For any partition $\mathcal{M} = (M_1, \dots, M_r)$ with

$$|\mathcal{M}| = M_1 + \dots + M_r \le n \tag{4.2}$$

introduce the element $c_n^{\mathcal{M}}$ of the group algebra $\mathbb{C}[S(n)]$ as follows

$$c_n^{\mathcal{M}} = \sum (i_1, \dots, i_{M_1})(j_1, \dots, j_{M_2}) \cdots (k_1, \dots, k_{M_r}) \tag{4.3}$$

where the sum is taken over the sequences $i_1, \dots, i_{M_1}; j_1, \dots, j_{M_2}; \dots; k_1, \dots, k_{M_r}$ of $|\mathcal{M}|$ pairwise distinct indices taken from $\mathbb{N}_n$. By $(i_1, \dots, i_{M_1})$ etc. in (4.3) we denote cycles in the symmetric group. For the empty partition $\emptyset$ we set $c_n^{\emptyset} = 1$. Note that $c_n^{(1)} = \sum_{i=1}^n (i) = n\,1$. Given two partitions $\mathcal{M}$ and $\mathcal{L}$ we denote by $\mathcal{M} \cup \mathcal{L}$ the partition whose parts are those of $\mathcal{M}$ and $\mathcal{L}$ rewritten in the decreasing order. We have

$$c_n^{\mathcal{M} \cup 1 \cup \dots \cup 1} = (n - |\mathcal{M}|) \cdots (n - |\mathcal{M}| - p + 1)\, c_n^{\mathcal{M}} \tag{4.4}$$

where p stands for the number of 1's in the left hand side of the relation.

By definition (3.16) of the degree of an element of $\Gamma(n)$ we have

$$\deg c_n^{(1)} = 0, \qquad \text{and} \qquad \deg c_n^{(M)} = M \quad \text{for } M \ge 2. \tag{4.5}$$

More generally,

$$\deg c_n^{(M_1, \dots, M_r)} = \sum_{i,\, M_i \ge 2} M_i. \tag{4.6}$$

Proposition 4.1 *Each of the families*

$$c_n^{\mathcal{M}}, \qquad |\mathcal{M}| = n, \tag{4.7}$$

and

$$c_n^{\mathcal{M}}, \qquad |\mathcal{M}| \le n \ \ \textit{and } \mathcal{M} \textit{ has no part equal to } 1, \tag{4.8}$$

forms a basis of $Z(S(n))$. Moreover, the elements of degree $\le M$ of each family form a basis of $Z^M(S(n))$.

Proof. The elements (4.7) are proportional to the characteristic functions of the conjugacy classes of the group $S(n)$ and so, they form a basis of $Z(S(n))$. By (4.4) the elements of type (4.8) are proportional to those of type (4.7). $\square$

Proposition 4.2 *For any two partitions $\mathcal{M} = (M_1, \dots, M_r)$ and $\mathcal{L} = (L_1, \dots, L_t)$ with $|\mathcal{M}| + |\mathcal{L}| \le n$ we have*

$$c_n^{\mathcal{M}} c_n^{\mathcal{L}} = c_n^{\mathcal{M} \cup \mathcal{L}} + (\dots), \tag{4.9}$$

where $(\dots)$ stands for a linear combination of the elements $c_n^{\mathcal{K}}$ with $|\mathcal{K}| < |\mathcal{M}| + |\mathcal{L}|$.

Proof. For a permutation $s \in S(n)$ or $s \in S(\infty)$ define its *support* as

$$\operatorname{supp} s = \{i \in \mathbb{N}_n \mid s(i) \ne i\} \quad \text{or} \quad \operatorname{supp} s = \{i \in \mathbb{N} \mid s(i) \ne i\}, \tag{4.10}$$

respectively. (The degree of a permutation is then given by $\deg s = |\operatorname{supp} s|$; cf. (3.16)).

Let $s \in S(n)$ be a permutation which occurs in the expansion of $c_n^{\mathcal{M}}$, that is, s is of cycle type $\mathcal{M} \cup 1 \cup \dots \cup 1$ (with $n - |\mathcal{M}|$ units). Similarly, let s' be a permutation occurring in $c_n^{\mathcal{L}}$. If the supports $\operatorname{supp} s$ and $\operatorname{supp} s'$ are disjoint then s and s' commute, and the product ss' occurs in the expansion of $c_n^{\mathcal{M} \cup \mathcal{L}}$. In particular, $\deg ss' = |\mathcal{M}| + |\mathcal{L}|$. If $\operatorname{supp} s$ and $\operatorname{supp} s'$ have a non-empty intersection then the degree of ss' is strictly less than $|\mathcal{M}| + |\mathcal{L}|$. $\square$

Remark. A detailed investigation of the structure constants for the products of type (4.9) have been recently given by Ivanov and Kerov [7]. $\square$

Corollary 4.3 *Let $k = (k_1, \dots, k_n)$ run over the n-tuples of non-negative integers such that $2k_2 + \dots + nk_n \le n$. Then the monomials*

$$(c_n^{(2)})^{k_2} \cdots (c_n^{(n)})^{k_n} \tag{4.11}$$

form a basis of $Z(S(n))$. Moreover, for any $M \ge 0$, the elements (4.11) with $2k_2 + \dots + nk_n \le M$ form a basis of $Z^M(S(n))$.

Proof. It suffices to prove that

$$(c_n^{(2)})^{k_2} \cdots (c_n^{(n)})^{k_n} = c_n^{\mathcal{M}} + (\dots) \tag{4.12}$$

where $\mathcal{M} = 2^{k_2} \cdots n^{k_n}$ and $(\dots)$ stands for a certain linear combination of the elements $c_n^{\mathcal{M}'}$ with $|\mathcal{M}'| < |\mathcal{M}| = 2k_2 + \cdots + nk_n$. But this follows from Proposition 4.2. $\square$

Now we will define analogs of the elements $c_n^{\mathcal{M}}$ for the algebra $A_0(n)$. Namely, for any partition $\mathcal{M} = (M_1, \dots, M_r)$ with $|\mathcal{M}| \le n$ set

$$\Delta_n^{\mathcal{M}} = \sum (i_1, \dots, i_{M_1})(j_1, \dots, j_{M_2}) \cdots (k_1, \dots, k_{M_r})(1 - \varepsilon_{i_1}) \cdots (1 - \varepsilon_{k_{M_r}}) \quad (4.13)$$

where, as in (4.3), the sum is taken over all sequences of $|\mathcal{M}|$ pairwise distinct indices taken from $\mathbb{N}_n$. In particular,

$$\Delta_n^{(1)} = \sum_{i=1}^{n} (1 - \varepsilon_i). \quad (4.14)$$

For the empty partition $\emptyset$ we set $\Delta_n^{\emptyset} = 1$. By (3.16), we have

$$\deg \Delta_n^{\mathcal{M}} = |\mathcal{M}| \qquad \text{for any partition } \mathcal{M}, \quad (4.15)$$

cf. (4.6). Note that $\Delta_n^{\mathcal{M}}$ can also be written as

$$\Delta_n^{\mathcal{M}} = \sum (1 - \varepsilon_{i_1}) \cdots (1 - \varepsilon_{k_{M_r}})(i_1, \dots, i_{M_1})(j_1, \dots, j_{M_2}) \cdots (k_1, \dots, k_{M_r}), \quad (4.16)$$

and as

$$\begin{aligned} \Delta_n^{\mathcal{M}} = \sum (1 - \varepsilon_{i_1}) \cdots (1 - \varepsilon_{k_{M_r}})(i_1, \dots, i_{M_1})(j_1, \dots, j_{M_2}) \\ \cdots (k_1, \dots, k_{M_r})(1 - \varepsilon_{i_1}) \cdots (1 - \varepsilon_{k_{M_r}}). \end{aligned} \quad (4.17)$$

Indeed, $(1 - \varepsilon_{i_1}) \cdots (1 - \varepsilon_{k_{M_r}})$ is invariant under the conjugation by the permutation $(i_1, \dots, i_{M_1})(j_1, \dots, j_{M_2}) \cdots (k_1, \dots, k_{M_r})$ which implies (4.16). To derive (4.17) it suffices to note that $(1 - \varepsilon_{i_1}) \cdots (1 - \varepsilon_{k_r})$ is an idempotent.

Proposition 4.4 *The element $\Delta_n^{\mathcal{M}}$ belongs to $A_0(n)$ for any $\mathcal{M}$.*

Proof. Since $\Gamma(n)$ is generated by the group $S(n)$ and the pairwise commuting idempotents $\varepsilon_1, \dots, \varepsilon_n$, it suffices to show that $\Delta_n^{\mathcal{M}}$ commutes both with $S(n)$ and with the ε_i. The first claim is clear since $\Delta_n^{\mathcal{M}}$ is invariant under the conjugation by the elements of $S(n)$. To prove the second claim, we observe that any ε_l, $1 \le l \le n$, commutes with any term

$$\sigma = (i_1, \dots, i_{M_1})(j_1, \dots, j_{M_2}) \cdots (k_1, \dots, k_{M_r})(1 - \varepsilon_{i_1}) \cdots (1 - \varepsilon_{k_{M_r}}) \quad (4.18)$$

in (4.13). Indeed, this is clear if l does not occur in the set of indices in (4.18) because ε_l commutes with the corresponding cycle. But if l coincides with one of the indices $i_1, \dots, k_{M_r}$, then $\varepsilon_l\,\sigma = \sigma\,\varepsilon_l = 0$. This follows from (4.17) and the relation $(1-\varepsilon_l)\varepsilon_l = 0$. □

Proposition 4.5 *We have*

$$\theta_n(\Delta_n^{\mathcal{M}}) = \Delta_{n-1}^{\mathcal{M}} \tag{4.19}$$

where we adopt the convention that

$$\Delta_k^{\mathcal{M}} = 0 \quad \textit{if} \quad |\mathcal{M}| > k. \tag{4.20}$$

Proof. By the definition of the projection θ_n (see Section 3) we need to calculate $\Delta_n^{\mathcal{M}}\,\varepsilon_n$. However, as it follows from the proof of Proposition 4.4, the effect of multiplying $\Delta_n^{\mathcal{M}}$ by ε_n reduces to striking from (4.13) all terms (4.18) such that n occurs among the corresponding indices. If $|\mathcal{M}| = n$, then all the terms are vanished, so that the result of the multiplication is 0. If $|\mathcal{M}| < n$, then the terms that survive are just the terms of the sum defining $\Delta_{n-1}^{\mathcal{M}}$. □

Corollary 4.6 *For any partition $\mathcal{M}$, there exists an element $\Delta^{\mathcal{M}} \in A_0$ such that*

$$\theta^{(n)}(\Delta^{\mathcal{M}}) = \Delta_n^{\mathcal{M}} \quad \textit{for any } n \geq 1 \tag{4.21}$$

with the convention (4.20)*.*

Proof. By Proposition 4.4, $\Delta_n^{\mathcal{M}} \in A_0(n)$. Now we apply Proposition 4.5 and note that the degrees of the elements $\Delta_n^{\mathcal{M}}$ are uniformly bounded by (4.15). □

We now aim to prove an analog of Proposition 4.1 for the algebra $A_0(n)$; see Proposition 4.10 below. For this we need the following three lemmas.

Let $I(n) = A(n)(1-\varepsilon_n)$ denote the left ideal of the algebra $A(n)$ generated by the element $1-\varepsilon_n$.

Lemma 4.7 *For any n,*

$$I(n) \cap A_0(n) = Z(S(n))(1-\varepsilon_1)\cdots(1-\varepsilon_n). \tag{4.22}$$

Proof. First suppose that $x \in A(n)$ can be written as $y\,(1-\varepsilon_1)\cdots(1-\varepsilon_n)$ where $y \in Z(S(n))$. The argument of the proof of Proposition 4.4 shows that $x \in A_0(n)$. Moreover, we obviously have $x \in I(n)$.

Conversely, suppose $x \in I(n) \cap A_0(n)$. Then $x\,\varepsilon_n = 0$. Using the invariance of x under the conjugation by elements of $S(n)$ we also obtain $x\,\varepsilon_i = 0$ for $i = 1, \dots, n$. Therefore x is invariant under the right multiplication by $(1-\varepsilon_1)\cdots(1-\varepsilon_n)$. Further, we may write

$$x = y + \sum_{r=1}^{n} \sum_{1 \le i_1 < \cdots < i_r \le n} y_{i_1 \dots i_r}\, \varepsilon_{i_1} \cdots \varepsilon_{i_r}, \tag{4.23}$$

where y and all the $y_{i_1 \dots i_r}$ are elements of $\mathbb{C}[S(n)]$; see Proposition 3.1. Multiplying this relation by $(1-\varepsilon_1)\cdots(1-\varepsilon_n)$ on the right we obtain

$$x = y\,(1-\varepsilon_1)\cdots(1-\varepsilon_n). \tag{4.24}$$

Finally, for any $s \in S(n)$ we may write

$$x = sxs^{-1} = sys^{-1}(1-\varepsilon_1)\cdots(1-\varepsilon_n). \tag{4.25}$$

Averaging over $s \in S(n)$ turns y into an element of $Z(S(n))$. □

For a subset $I = \{i_1, \dots, i_k\}$ in $\mathbb{N}_n$ we put $\varepsilon_I = \varepsilon_{i_1} \cdots \varepsilon_{i_k}$, and for $s \in S(n)$ set

$$Q(s) = \{i \in \mathbb{N}_n \mid s_{ii} = 1\} = \mathbb{N}_n \setminus \operatorname{supp} s. \tag{4.26}$$

Lemma 4.8 *The mapping*

$$s \mapsto \gamma, \qquad \gamma = s\,\varepsilon_{Q(s)} = \varepsilon_{Q(s)} s, \tag{4.27}$$

defines a bijection of $S(n)$ onto the set of all $\gamma \in \Gamma(n)$ satisfying the conditions

$$\operatorname{dom}\gamma = \operatorname{range}\gamma, \tag{4.28}$$

$$\deg\gamma = n. \tag{4.29}$$

Proof. The effect of the multiplication of s by $\varepsilon_{Q(s)}$ from the left or from the right consists of replacing all the 1's on the diagonal by zeros. This implies (4.28), and (4.29) is obvious.

Conversely, let $\gamma \in \Gamma(n)$ satisfy (4.28) and (4.29). Relation (4.28) means that for any $i \in \mathbb{N}_n$ the i-th row and the i-th column are zero or non-zero at the same time, whereas (4.29) means that all the diagonal entries of γ are zero. Now, let the matrix σ be defined as follows. Set $\sigma_{ii} = 1$ if the i-th row (and the i-th column) of γ is zero, and set $\sigma_{ij} = \gamma_{ij}$ for $i \ne j$. It is easy to see that $\sigma \in S(n)$ and that γ is the image of σ under the mapping (4.27). □

Lemma 4.9 *The restriction of the projection $\theta_n : A_0(n) \to A_0(n-1)$ to the subspace $A_0^{n-1}(n)$ is injective.*

Proof. Let $x \in A_0(n)$ and $\theta_n(x) = 0$. We will show that $\deg x = n$ unless $x = 0$. By Lemma 4.7, x can be written as a linear combination of the elements

$$s\,(1-\varepsilon_1)\cdots(1-\varepsilon_n) = \sum_{I\subseteq \mathbb{N}_n} (-1)^{|I|} s\,\varepsilon_I, \qquad s \in S(n). \tag{4.30}$$

Rewrite this as

$$s\,(1-\varepsilon_1)\cdots(1-\varepsilon_n) = \sum_{I\supseteq Q(s)} (-1)^{|I|} s\,\varepsilon_I + \sum_{I\not\supseteq Q(s)} (-1)^{|I|} s\,\varepsilon_I. \tag{4.31}$$

Then the terms of the first sum are of degree n whereas those of the second sum are of degree strictly less than n. So it suffices to prove that the elements

$$\sum_{I\supseteq Q(s)} (-1)^{|I|} s\,\varepsilon_I, \qquad s \in S(n), \tag{4.32}$$

are linearly independent. Note that

$$\operatorname{rank} s\,\varepsilon_I = n - |Q(s)| \qquad \text{if } I = Q(s), \tag{4.33}$$

$$\operatorname{rank} s\,\varepsilon_I < n - |Q(s)| \qquad \text{if } I \supset Q(s); \tag{4.34}$$

see (3.10). Write $S(n)$ as the disjoint union of $n+1$ subsets:

$$S(n) = \bigcup_{k=0}^{n} \{s \in S(n) \mid n - |Q(s)| = k\}. \tag{4.35}$$

If s belongs to the k-th subset then

$$\operatorname{rank}\left(\sum_{I\supseteq Q(s)} (-1)^{|I|} s\,\varepsilon_I\right) = k. \tag{4.36}$$

Moreover, only one term of the sum in (4.36) has rank k, namely that with $I = Q(s)$. Finally, it remains to note that by Lemma 4.8 all the elements $s\varepsilon_{Q(s)}$ with $s \in S(n)$ are pairwise distinct elements of $\Gamma(n)$. $\square$

Proposition 4.10 *For any n the elements $\Delta_n^{\mathcal{M}}$, where $\mathcal{M}$ is any partition with $|\mathcal{M}| \le n$, form a basis of $A_0(n)$. Furthermore, for any M such that $0 \le M \le n$ these elements with $|\mathcal{M}| \le M$ form a basis of $A_0^M(n)$.*

Proof. The first claim of the proposition will follow from the second one. We will prove the second claim using induction on n. The claim is obviously true for $n = 1$. Assume that $n \geq 2$ and $M \leq n-1$. By the induction hypothesis the elements $\Delta_{n-1}^{\mathcal{M}}$ with $|\mathcal{M}| \leq M$ form a basis of $A_0^M(n-1)$. By Proposition 4.5 the image of $\Delta_n^{\mathcal{M}}$ under θ_n is $\Delta_{n-1}^{\mathcal{M}}$. By Lemma 4.9 the restriction $\theta_n \downarrow A_0^M(n)$ is injective. Therefore, the elements $\Delta_n^{\mathcal{M}}$ with $|\mathcal{M}| \leq M$ form a basis in $A_0^M(n)$.

Further, let us show that the elements $\Delta_n^{\mathcal{M}}$ with $|\mathcal{M}| = n$ form a basis of $I(n) \cap A_0(n)$. Note that

$$\Delta_n^{\mathcal{M}} = c_n^{\mathcal{M}} (1-\varepsilon_1)\cdots(1-\varepsilon_n). \tag{4.37}$$

By Proposition 4.1 the elements $c_n^{\mathcal{M}}$, where $\mathcal{M}$ runs over the set of partitions of n, form a basis of $Z(S(n))$. Due to Lemma 4.7 it now remains to check that the elements $c_n^{\mathcal{M}}$, being multiplied by $(1-\varepsilon_1)\cdots(1-\varepsilon_n)$, remain linearly independent. However, this follows from the fact that the composite map

$$\mathbb{C}[S(n)] \to A(n) \to \mathbb{C}[S(n)] \tag{4.38}$$

is the identity map; here the first arrow is the multiplication by $(1-\varepsilon_1)\cdots(1-\varepsilon_n)$, and the second arrow is the retraction homomorphism (3.9).

Finally, let us show that

$$A_0(n) = A_0^{n-1}(n) \oplus \big(I(n) \cap A_0(n)\big). \tag{4.39}$$

Indeed, as it was shown above, θ_n maps $A_0^{n-1}(n)$ onto $A_0^{n-1}(n-1) = A_0(n-1)$. Since $I(n) \cap A_0(n)$ is the kernel of the restriction $\theta_n \downarrow A_0(n)$ and since $\theta_n(A_0(n))$ is contained in $A_0(n-1)$, we obtain the decomposition

$$A_0(n) = A_0^{n-1}(n) + \big(I(n) \cap A(n)\big). \tag{4.40}$$

Lemma 4.9 implies that

$$A_0^{n-1}(n) \cap I(n) = \{0\} \tag{4.41}$$

and (4.39) follows.

To complete the proof we need to show that the elements $\Delta_n^{\mathcal{M}}$ with $0 \leq |\mathcal{M}| \leq n$ form a basis of $A_0(n)$. However, the elements with $|\mathcal{M}| < n$ form a basis of the first component of the decomposition (4.39), whereas the elements with $|\mathcal{M}| = n$ form a basis in the second component of this decomposition. $\square$

The following is an analog of Corollary 4.3.

Corollary 4.11 *Let $k = (k_1, \ldots, k_n)$ run over the n-tuples of non-negative integers such that $k_1 + 2k_2 + \cdots + nk_n \le n$. Then the monomials*

$$(\Delta_1^{(1)})^{k_1} \cdots (\Delta_n^{(n)})^{k_n} \tag{4.42}$$

form a basis of $A_0(n)$. Moreover, for any $M \ge 0$, the monomials (4.42) with $k_1 + 2k_2 + \cdots + nk_n \le M$ form a basis of $A_0^M(n)$.

Proof. It suffices to prove that

$$(\Delta_1^{(1)})^{k_1} \cdots (\Delta_n^{(n)})^{k_n} = \Delta_n^{\mathcal{M}} + (\ldots) \tag{4.43}$$

where $\mathcal{M} = 1^{k_1}2^{k_2}\cdots n^{k_n}$ and $(\ldots)$ stands for a linear combination of the elements $\Delta_n^{\mathcal{M}'}$ with $|\mathcal{M}'| < |\mathcal{M}|$. Then our claim will follow from Proposition 4.10. To prove (4.43) we verify that for any partitions $\mathcal{M} = (M_1, \ldots, M_r)$ and $\mathcal{L} = (L_1, \ldots, L_t)$ with $|\mathcal{M}| + |\mathcal{L}| \le n$

$$\Delta_n^{\mathcal{M}}\Delta_n^{\mathcal{L}} = \Delta_n^{\mathcal{M}\cup\mathcal{L}} + (\ldots), \tag{4.44}$$

where the rest term $(\ldots)$ has degree strictly less than $|\mathcal{M}| + |\mathcal{L}|$ and so, it is a linear combination of elements $\Delta_n^{\mathcal{K}}$ with $|\mathcal{K}| < |\mathcal{M}| + |\mathcal{L}|$. Write

$$\Delta_n^{\mathcal{M}} = \sum \delta_I, \qquad \Delta_n^{\mathcal{L}} = \sum \delta'_J. \tag{4.45}$$

Here I is a sequence $i_1, \ldots, i_{|\mathcal{M}|}$ of pairwise distinct indices taken from $\mathbb{N}_n$ and

$$\delta_I = (i_1, \ldots, i_{M_1}) \ldots (i_{M_1+\ldots+M_{r-1}+1}, \ldots, i_{|\mathcal{M}|}) \prod_{p=1}^{|\mathcal{M}|}(1 - \varepsilon_{i_p}); \tag{4.46}$$

the δ'_J are the corresponding elements for the partition $\mathcal{L}$. Then

$$\Delta_n^{\mathcal{M}}\Delta_n^{\mathcal{L}} = \sum_{I,\,J} \delta_I\delta'_J = \sum_{I\cap J=\emptyset} \delta_I\delta'_J + \sum_{I\cap J\neq\emptyset} \delta_I\delta'_J. \tag{4.47}$$

The first sum on the right hand side of (4.47) is $\Delta_n^{\mathcal{M}\cup\mathcal{L}}$ whereas the second sum is of degree strictly less than $|\mathcal{M}| + |\mathcal{L}|$. □

Consider the elements $\Delta^{\mathcal{M}} \in A_0$ introduced in Corollary 4.6. We shall denote by $\mathbb{P}$ the set of all partitions.

Theorem 4.12 *The elements $\Delta^{\mathcal{M}}$, $\mathcal{M} \in \mathbb{P}$ form a basis of the algebra A_0. Moreover, for any $M \ge 0$, the elements $\Delta^{\mathcal{M}}$ with $|\mathcal{M}| \le M$ form a basis of the M-th subspace A_0^M in A_0.*

Proof. The first claim follows from the second one. The second claim follows from Proposition 4.10 and the definition of A_0^M as the projective limit of the spaces $A_0^M(n)$. $\square$

Corollary 4.13 *For $n > M$, the mapping*

$$\theta_n : A_0^M(n) \to A_0^M(n-1) \tag{4.48}$$

is an isomorphism of vector spaces and so is the mapping

$$\theta^{(n)} : A_0^M \to A_0^M(n), \qquad n \geq M. \tag{4.49}$$

In particular, $\dim A_0^M < \infty$. $\square$

Theorem 4.14 *The monomials*

$$(\Delta^{(1)})^{k_1}(\Delta^{(2)})^{k_2}\cdots \tag{4.50}$$

with $k_1, k_2, \ldots \in \mathbb{Z}_+$ and $k_1 + 2k_2 + \ldots < \infty$ form a basis of the algebra A_0. Moreover, for any $M \geq 0$ the monomials (4.50) *with $k_1 + 2k_2 + \ldots \leq M$ form a basis of the subspace A_0^M.*

Proof. It suffices to check that

$$(\Delta^{(1)})^{k_1}(\Delta^{(2)})^{k_2}\ldots \equiv \Delta^{\mathcal{M}} \qquad \mathrm{mod}\ A_0^{M-1} \tag{4.51}$$

where $\mathcal{M} = 1^{k_1}2^{k_2}\cdots$ and $M = |\mathcal{M}|$. However, this follows from the relation (4.43). $\square$

Corollary 4.15 *The elements $\Delta^{(1)}, \Delta^{(2)}, \ldots$ are algebraically independent and generate the algebra A_0.* $\square$

4.2 The algebra Λ^* of shifted symmetric functions, and the isomorphism $A_0 \simeq \Lambda^*$

Let $\Lambda^*(n) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ denote the subalgebra of polynomials in n variables $x_1, \ldots, x_n$ which are symmetric in the new variables

$$y_1 = x_1 - 1, \quad y_2 = x_2 - 2, \ \ldots, \ y_n = x_n - n. \tag{4.52}$$

Following [24], we refer to $\Lambda^*(n)$ as the *algebra of shifted symmetric polynomials* in n variables. We equip $\Lambda^*(n)$ with the filtration with respect to the usual degree of polynomials. Set $\Lambda^*(0) = \mathbb{C}$ and for $n \geq 1$ define the projection $\Lambda^*(n) \to \Lambda^*(n-1)$ by specializing $x_n = 0$. Note that this projection preserves the filtration.

Definition 4.16 The *algebra* Λ^* *of shifted symmetric functions* is the projective limit of the *filtered* algebras $\Lambda^*(n)$ as $n \to \infty$. □

In other words, an element $f \in \Lambda^*$ is a sequence $(f_n \mid n \geq 0)$ such that
(i) $f_n \in \Lambda^*(n)$ for any n;
(ii) for any $n \geq 1$, $f_n \mapsto f_{n-1}$ under the projection $\Lambda^*(n) \to \Lambda^*(n-1)$;
(iii) $\deg f_n$ remains bounded as $n \to \infty$.
For an element $f = (f_n) \in \Lambda^*$, we define its *degree* by

$$\deg f = \sup_n \deg f_n, \tag{4.53}$$

and for $M = 0, 1, \dots$ we denote by $(\Lambda^*)^M$ the subspace in Λ^* consisting of the elements of degree $\leq M$. The algebra Λ^* was first introduced in [26]. A detailed study of Λ^* is contained in [24].

Note an evident similarity between the shifted symmetric functions and the symmetric functions. Recall (see [13]) that the algebra Λ of symmetric functions is defined as the projective limit as $n \to \infty$ of the *graded* algebras $\Lambda(n) \subseteq \mathbb{C}[x_1, \dots, x_n]$ of symmetric polynomials in n variables. A difference between Λ^* and Λ consists in a shift of variables and the replacement of the gradation by a filtration. The algebra Λ^* may be viewed as a deformation of the algebra Λ. Indeed, let h be a numerical parameter, and let Λ^*_h be defined similarly to Λ^* but with $y_i = x_i - ih$ instead of (4.52). Then the algebras Λ^*_h with $h \neq 0$, are naturally isomorphic to each other. Moreover, Λ^*_1 coincides with Λ^* while Λ^*_0 coincides with Λ. Another relation between Λ^* and Λ is given by

Proposition 4.17 *The graded algebra*

$$\operatorname{gr} \Lambda^* = \mathbb{C} \oplus \bigoplus_{M=1}^{\infty} \Big((\Lambda^*)^M / (\Lambda^*)^{M-1} \Big) \tag{4.54}$$

is isomorphic to the algebra Λ.

Proof. For any $M \geq 1$ and any n, $\Lambda^*(n)^M / \Lambda^*(n)^{M-1}$ is naturally isomorphic to the M-th homogeneous component of the algebra $\Lambda(n) \subseteq \mathbb{C}[x_1, \dots, x_n]$. Moreover, this isomorphism is compatible with the projections $\Lambda^*(n) \to \Lambda^*(n-1)$ and $\Lambda(n) \to \Lambda(n-1)$. This yields an isomorphism $\operatorname{gr}\Lambda^* \to \Lambda$. □

In the following example we give some families of generators of the algebra Λ^*. Note that there also exist other important families analogous to the basic symmetric functions; see [24].

Example 4.18 For $M = 1, 2, \dots$, elements e_M, h_M, and p_M defined by the formulas below, are shifted symmetric functions:

$$E(t) = 1 + \sum_{M=1}^{\infty} e_M t^M = \prod_{k=1}^{\infty} \frac{1 + (x_k - k)t}{1 - kt},$$

$$H(t) = 1 + \sum_{M=1}^{\infty} h_M t^M = \prod_{k=1}^{\infty} \frac{1 + kt}{1 - (x_k - k)t},$$

$$p_M = \sum_{k=1}^{\infty} \left((x_k - k)^M - (-k)^M \right).$$

□

The generating functions satisfy the following relations; cf. [13]:

$$E(t)H(-t) = 1, \qquad \sum_{k=1}^{\infty} p_M t^M = t \frac{d}{dt} \log H(t). \tag{4.55}$$

Proposition 4.19 *The algebra Λ^* is isomorphic to the algebra of polynomials in countably many generators. Furthermore, we have*

$$\Lambda^* = \mathbb{C}[e_1, e_2, \dots] = \mathbb{C}[h_1, h_2, \dots] = \mathbb{C}[p_1, p_2, \dots]. \tag{4.56}$$

Proof. The corresponding statement for the algebra Λ of symmetric functions is well known, see [13, Ch. 1, Section 2]. Now, we apply Proposition 4.17 and note that the image of the shifted symmetric function e_M, h_M or p_M in the space $(\Lambda^*)^M/(\Lambda^*)^{M-1} \simeq \Lambda^M$ is the corresponding M-th symmetric function (elementary, complete or power sum). This implies that each of the three families is algebraically independent and generates the algebra Λ^*. □

Let $\operatorname{Fun}\mathbb{P}$ denote the algebra of complex functions on the set of partitions $\mathbb{P}$. By Propositions 3.10 and 3.11, there is an algebra homomorphism

$$A_0 \to \operatorname{Fun}\mathbb{P}, \qquad a \mapsto \widehat{a}, \tag{4.57}$$

such that for $a \in A_0$ and $\lambda \in \mathbb{P}$, the element a acts in $H_\infty(T_\lambda)$ as the scalar operator $\widehat{a}(\lambda) \cdot 1$. On the other hand, any $\lambda \in \mathbb{P}$ can be viewed as a sequence $(\lambda_1, \lambda_2, \dots, 0, 0, \dots)$ with finitely many non-zero coordinates, and so, any element of Λ^* may be viewed as a function on $\mathbb{P}$. Thus we obtain an algebra homomorphism $\Lambda^* \to \operatorname{Fun}\mathbb{P}$ which is clearly an embedding.

Let λ be a partition with $m = |\lambda| \le n$. Consider the corresponding irreducible representation π_λ of $S(m)$, and the representation $\mathcal{T}_n(\lambda)$ of the semigroup $\Gamma(n)$; see Section 2.2.

Proposition 4.20 *The eigenvalue of the central element $\Delta_n^{(r)}$ in $\mathcal{T}_n(\lambda)$ is 0 if $r > m$. If $r \le m$ then the eigenvalue coincides with that of the element $c_m^{(r)}$ in the representation π_λ of $S(m)$.*

Proof. Recall the construction of $\mathcal{T}_n(\lambda)$ given in Section 2.2. Let ω be an injective map from $\{1, \dots, m\}$ to $\{1, \dots, n\}$. Regarding ω as an m-tuple $\omega = (\omega_1, \dots, \omega_m)$ we have

$$\varepsilon_a f(\omega) = \begin{cases} 0 & \text{if } a \in \omega, \\ f(\omega) & \text{if } a \notin \omega. \end{cases} \tag{4.58}$$

Therefore, the product $(1-\varepsilon_{i_1}) \cdots (1-\varepsilon_{i_r})$ is a projection to the subspace of functions f such that the indices $i_1, \dots, i_r$ belong to any $\omega \in \operatorname{supp} f$. This implies the first statement. The second follows from the obvious embedding $H(\pi_\lambda) \subseteq H(\mathcal{T}_n(\lambda))$ whose image consists of the functions supported by the maps ω such that $\{\omega_1, \dots, \omega_m\} = \{1, \dots, m\}$. $\square$

It was proved in [10] (see also [24]) that the eigenvalue of $c_n^{(r)}$ in the irreducible representation π_λ of $S(n)$ is a shifted symmetric function whose highest homogeneous component is the power sum symmetric function p_r.

Theorem 4.21 *Let Λ^* be identified with its image in* $\operatorname{Fun}\mathbb{P}$. *Then the mapping* (4.57) *is an isomorphism $A_0 \to \Lambda^*$ of filtered algebras.*

Proof. By Proposition 4.20 the images of the generators $\Delta^{(r)} \in A_0$ with respect to the homomorphism (4.57) are shifted symmetric functions which are algebraically independent generators of the algebra Λ^*. The map obviously respects the filtrations. $\square$

Recall that by Propositions 3.10 and 3.11, elements of the center A_0 act in irreducible tame representations of $S(\infty)$ by scalar operators. Hence, any such representation determines a homomorphism $A_0 \to \mathbb{C}$.

Corollary 4.22 *The center A_0 separates irreducible tame representations of $S(\infty)$. That is, non-equivalent irreducible tame representations give rise to distinct homomorphisms $A_0 \to \mathbb{C}$.*

Proof. By Theorem 2.10, the irreducible tame representations are precisely the representations T_λ. Hence, our claim is equivalent to the fact that the map $\Lambda^* \to \operatorname{Fun}\mathbb{P}$ defined above is an embedding. □

5 The structure of the algebra A_m, $m > 0$

Here we generalize the results of Section 4 to the algebra A_m, where $m = 1, 2, \dots$. Throughout the section we assume $0 \le m \le n$ and use the notation

$$\mathbb{N}_{mn} = \{m+1, \dots, n\}. \tag{5.1}$$

For $\gamma \in \Gamma(n)$, set

$$\begin{aligned} J_m(\gamma) &= \{i \mid i \in \mathbb{N}_{mn},\ \gamma_{ii} = 0\}, \\ \deg_m \gamma &= |\, J_m(\gamma)|. \end{aligned} \tag{5.2}$$

We shall call $\deg_m \gamma$ the *m-degree* of γ.

Proposition 5.1 *For $\gamma, \delta \in \Gamma(n)$,*

$$\deg_m \gamma\delta \le \deg_m \gamma + \deg_m \delta. \tag{5.3}$$

Proof. For any $i \in \mathbb{N}_{mn}$ we have

$$(\gamma\delta)_{ii} = 0 \quad \Rightarrow \quad \gamma_{ij}\delta_{ji} = 0 \qquad \text{for all } j = 1, \dots, n. \tag{5.4}$$

In particular, $(\gamma\delta)_{ii} = 0$ implies $\gamma_{ii}\delta_{ii} = 0$, i.e.,

$$J_m(\gamma\delta) \subseteq J_m(\gamma) \cup J_m(\delta), \tag{5.5}$$

and (5.3) follows. □

Definition 5.2 Using the m-degree we define a new filtration in $A(n)$, called the *m-filtration*, by

$$A(m) = F_m^0(A(n)) \subseteq F_m^1(A(n)) \subseteq \ldots \subseteq F_m^{n-m}(A(n)) = A(n). \tag{5.6}$$

Here $F_m^M(A(n))$, the M-th term of the filtration, is formed by the elements $a \in A(n)$ which are linear combinations of the elements of $\Gamma(n)$ of m-degree $\leq M$. For any subspace S of $A(n)$ we will use the symbol $F_m^M(S)$ to indicate the M-th term of the induced filtration. □

By Proposition 5.1, the m-filtration is compatible with the algebra structure of $A(n)$, so the corresponding graded algebra exists. But contrary to the case $m = 0$, this graded algebra is *not* commutative for $m \geq 1$ since it contains, as the 0-component, the non-commutative algebra $A(m)$.

Let D be a multiplicative semigroup with unity 1. Consider the union $D \cup \{0\}$, where 0 is an extra symbol, and adopt the convention that

$$d\,0 = 0\,d = 0, \quad d + 0 = 0 + d = d \quad \text{for any } d \in D. \tag{5.7}$$

Definition 5.3 (i) The semigroup $S(m, D)$ consists of the $m \times m$ matrices $\alpha = [\alpha_{ij}]$ with entries in $D \cup \{0\}$ such that any row and column contains exactly one non-zero entry. The product is the matrix multiplication with the conventions (5.7).

(ii) The semigroup $\Gamma(n, D)$ is defined as in (i) by allowing any row and column contain *at most* one non-zero entry. □

Note that if $D = \{1\}$, then $S(n, D)$ and $\Gamma(n, D)$ coincide with $S(n)$ and $\Gamma(n)$, respectively. If D is a group, then $S(n, D)$ is the wreath product of $S(n)$ and D.

We shall be assuming now that D is the free abelian semigroup $\{1, z, z^2, \dots\}$ with unity 1 and one generator z. This semigroup is isomorphic to the additive semigroup $\mathbb{Z}_+$. We denote the corresponding semigroups introduced in Definition 5.3 by $S(m, \mathbb{Z}_+)$ and $\Gamma(m, \mathbb{Z}_+)$.

Set $\operatorname{ord} z^k = k$ for $k = 0, 1, \dots,$ and for $\alpha \in \Gamma(m, \mathbb{Z}_+)$, set

$$\operatorname{ord} \alpha = \sum_{i,j;\, \alpha_{ij} \neq 0} \operatorname{ord} \alpha_{ij}. \tag{5.8}$$

Definition 5.4 (i) Set

$$\Gamma(m, n) = \{\sigma \in \Gamma(n) \mid \operatorname{dom} \sigma \text{ and } \operatorname{range} \sigma \text{ contain } \mathbb{N}_{mn}\}. \tag{5.9}$$

(ii) Consider the linear span of $\Gamma(m, n)$ and let $Z_m(n) \subset A(n)$ denote the subspace in this span formed by the elements invariant under the conjugation by the elements of the group $S_m(n)$. □

In particular, $\Gamma(0,n) = S(n)$ and $Z_0(n) = Z\big(S(n)\big)$ is the center of $\mathbb{C}[S(n)]$. The role of $\Gamma(m,n)$ and $Z_m(n)$ will be similar to that of $S(n)$ and $Z\big(S(n)\big)$ in Section 4. Note also that $Z_m(n)$ contains $\mathbb{C}[S(n)]^{S_m(n)}$, the centralizer of $S_m(n)$ in the group algebra $\mathbb{C}[S(n)]$.

Now our purpose is to construct a convenient basis in $Z_m(n)$. To do this, we need to classify the $S_m(n)$-orbits in $\Gamma(m,n)$ where the elements of $S_m(n)$ act by conjugations.

Proposition 5.5 *There is a natural parameterization of the $S_m(n)$-orbits in $\Gamma(m,n)$ by the couples $(\alpha, \mathcal{M})$, where $\alpha \in \Gamma(m, \mathbb{Z}_+)$ and $\mathcal{M}$ is a partition such that*

$$\operatorname{ord}\alpha + |\mathcal{M}| = n - m. \tag{5.10}$$

Proof. Fix an arbitrary element $\sigma \in \Gamma(m,n)$ and assign to it an $m \times m$-matrix $\alpha = \alpha(\sigma)$ as follows. For $i, j \notin \mathbb{N}_{mn}$ set

$$\alpha_{ij} = 0 \qquad \text{if } j \notin \operatorname{dom}\sigma, \tag{5.11}$$

$$\alpha_{ij} = 1 \qquad \text{if } j \in \operatorname{dom}\sigma \text{ and } \sigma(j) = i, \tag{5.12}$$

$$\alpha_{ij} = z^k \qquad \text{if } j \in \operatorname{dom}\sigma, \tag{5.13}$$

and there exist k points $p_1, \dots, p_k \in \mathbb{N}_{mn}$ such that $\sigma(j) = p_1$, $\sigma(p_1) = p_2$, $\dots$, $\sigma(p_{k-1}) = p_k$, $\sigma(p_k) = i$. Thus, to any $j \in \operatorname{dom}\sigma$ with $\sigma(j) \in \mathbb{N}_{mn}$ we have assigned a subset $\{p_1, \dots, p_k\} \subseteq \mathbb{N}_{mn}$. It is clear that these subsets are pairwise disjoint. Let $P = P(\sigma)$ denote their union. Then $\operatorname{ord}\alpha = |\,P| \le n - m$. It is also clear that $\alpha \in \Gamma(m, \mathbb{Z}_+)$.

Further, let $P^* = P^*(\sigma)$ be the complement of P in $\mathbb{N}_{mn}$. Then P^* is contained in the domain of σ, and P^* is σ-invariant. Therefore, the restriction of σ to P^* defines a permutation of P^*. Let $\mathcal{M} = \mathcal{M}(\sigma)$ be the partition of the number $|\,P^*|$ which is defined by the lengths of the cycles of this permutation. Then the couple

$$(\alpha, \mathcal{M}) = (\alpha(\sigma), \mathcal{M}(\sigma)) \tag{5.14}$$

satisfies (5.10). It is clear that the couple (5.14) remains unchanged if σ is replaced by $s\sigma s^{-1}$ with $s \in S_m(n)$. Moreover, it is also clear that if the couples (5.14) corresponding to two elements of $\Gamma(m,n)$ are the same, then these elements belong to the same orbit. Finally, any couple satisfying (5.10) can be obtained from an element of $\Gamma(m,n)$. $\square$

Remark. A couple (5.14) corresponds to an element of $S(n) \subseteq \Gamma(m,n)$ if and only if $\alpha \in S(m, \mathbb{Z}_+)$. $\square$

We shall now define analogs of the elements $c_n^{\mathcal{M}}$. First, for any subset $Q \subseteq \mathbb{N}_{mn}$ and any partition $\mathcal{M} = (M_1, \ldots, M_r)$ such that $|\mathcal{M}| = |\,Q|$ we set

$$c_Q^{\mathcal{M}} = \sum (i_1, \ldots, i_{M_1})(j_1, \ldots, j_{M_2}) \ldots (k_1, \ldots, k_{M_r}), \tag{5.15}$$

where $(i_1, \ldots, i_{M_1})$ etc. are cyclic permutations of the corresponding indices and the summation is taken over all the orderings $(i_1, \ldots, i_{M_1}; j_1, \ldots, j_{M_2}; \ldots; k_1, \ldots, k_{M_r})$ of the elements of Q. We shall suppose that $c_\emptyset^\emptyset = 1$.

Second, for any $\alpha \in \Gamma(m, \mathbb{Z}_+)$ and any subset $P \subseteq \mathbb{N}_{mn}$ such that $\operatorname{ord} \alpha = |\,P|$, we set

$$\Gamma(\alpha, P) = \{\sigma \in \Gamma(m,n) \mid \alpha(\sigma) = \alpha,\ P(\sigma) = P,\ \mathcal{M}(\sigma) = (1^{n-m-|\,P|})\}, \tag{5.16}$$

i.e., σ has to fix all the points in $\mathbb{N}_{mn} \setminus P$.

Definition 5.6 For any couple $(\alpha, \mathcal{M})$, where $\alpha \in \Gamma(m, \mathbb{Z}_+)$ and $\mathcal{M}$ is a partition such that $\operatorname{ord} \alpha + |\mathcal{M}| \leq n - m$ we set

$$c_n^{\alpha,\mathcal{M}} = \sum_{P,\,Q} \sum_{\sigma \in \Gamma(\alpha,P)} \sigma\, c_Q^{\mathcal{M}}, \tag{5.17}$$

where P, Q are disjoint subsets in $\mathbb{N}_{mn}$ such that

$$|\,P| = \operatorname{ord} \alpha, \qquad |\,Q| = |\mathcal{M}|. \tag{5.18}$$

$\square$

Proposition 5.7 *Each of the families*

$$c_n^{\alpha,\mathcal{M}} \qquad \textit{with} \quad \operatorname{ord} \alpha + |\mathcal{M}| = n - m, \tag{5.19}$$

and

$$c_n^{\alpha,\mathcal{M}} \qquad \textit{with} \quad \operatorname{ord} \alpha + |\mathcal{M}| \leq n - m, \quad \textit{and } \mathcal{M} \textit{ has no part equal to } 1, \tag{5.20}$$

forms a basis of $Z_m(n)$.

Proof. Note that $c_n^{\alpha,\mathcal{M}\cup 1 \cup \ldots \cup 1}$ is proportional to $c_n^{\alpha,\mathcal{M}}$. Therefore, it suffices to consider the family (5.19). By Proposition 5.5 the elements $c_n^{\alpha,\mathcal{M}}$ with $\operatorname{ord} \alpha + |\mathcal{M}| = n - m$ are proportional to characteristic functions of the $S_m(n)$-orbits in $\Gamma(m,n)$. $\square$

Now we introduce analogs of the elements $\Delta_n^{\mathcal{M}}$.

Definition 5.8 For any couple $(\alpha, \mathcal{M})$, where $\alpha \in \Gamma(m, \mathbb{Z}_+)$ and $\mathcal{M}$ is a partition such that $\operatorname{ord} \alpha + |\mathcal{M}| \leq n - m$ we set

$$\Delta_n^{\alpha,\mathcal{M}} = \sum_{P,\, Q} \sum_{\sigma \in \Gamma(\alpha, P)} \varepsilon(P)\, \sigma\, c_Q^{\mathcal{M}}\, \varepsilon(Q)\, \varepsilon(P), \tag{5.21}$$

where $\varepsilon(I) := (1 - \varepsilon_{i_1}) \cdots (1 - \varepsilon_{i_k})$ for $I = \{i_1, \dots, i_k\}$. Here P, Q are disjoint subsets in $\mathbb{N}_{mn}$ satisfying (5.18). We set $\Delta_n^{1,\emptyset} = 1$, where $\emptyset$ stands for the empty partition.

□

Note that (5.21) can be written in an equivalent form where the term $\varepsilon(Q)$ takes the leftmost position; cf. (4.13) and (4.16).

Proposition 5.9 *The elements $\Delta_n^{\alpha,\mathcal{M}}$ belong to the algebra $A_m(n)$.*

Proof. The semigroup $\Gamma_m(n)$ is generated by the subgroup $S_m(n)$ and the idempotents $\varepsilon_{m+1}, \dots, \varepsilon_n$. Therefore, it suffices to check that $\Delta_n^{\alpha,\mathcal{M}}$ is stable under the conjugation by the elements of $S_m(n)$ and commutes with the idempotents. The first claim is immediate from (5.21). The second claim is verified exactly as its counterpart for the elements $\Delta_n^{\mathcal{M}}$; see the proof of Proposition 4.4. □

The following is an analog of Proposition 4.5 and it is proved by the same argument.

Proposition 5.10 *We have*

$$\theta_n(\Delta_n^{\alpha,\mathcal{M}}) = \Delta_{n-1}^{\alpha,\mathcal{M}}, \tag{5.22}$$

where we adopt the convention that

$$\Delta_k^{\alpha,\mathcal{M}} = 0 \quad \textit{if} \quad \operatorname{ord} \alpha + |\mathcal{M}| > k - m. \tag{5.23}$$

□

Our aim now is to prove an analog of Propositions 4.1 and 4.10; see Proposition 5.14 below. We need the following three lemmas.

Lemma 5.11 *For $m < n$*

$$I(n) \cap A_m(n) = (1 - \varepsilon_{m+1}) \cdots (1 - \varepsilon_n)\, Z_m(n)\, (1 - \varepsilon_{m+1}) \cdots (1 - \varepsilon_n). \tag{5.24}$$

Proof. Suppose that $x \in A(n)$ can be written as

$$x = (1 - \varepsilon_{m+1}) \cdots (1 - \varepsilon_n)\, y\, (1 - \varepsilon_{m+1}) \cdots (1 - \varepsilon_n), \tag{5.25}$$

where $y \in Z_m(n)$. Then $x \in A_m(n)$ since x is invariant under the conjugation by the elements of $S_m(n)$ and is annihilated when multiplied (from the left or from the right) by any idempotent $\varepsilon_{m+1}, \dots, \varepsilon_n$. Moreover, this also implies that $x \in I(n)$.

Conversely, suppose $x \in I(n) \cap A_m(n)$. Then $x\, \varepsilon_n = \varepsilon_n\, x = 0$. Using the invariance of x under the conjugation by the elements of $S_m(n)$ we obtain $x\, \varepsilon_i = \varepsilon_i\, x = 0$ for $i = m+1, \dots, n$. Thus x is invariant under the multiplication by $(1-\varepsilon_{m+1}) \cdots (1-\varepsilon_n)$ both from the left and from the right.

Further, we can write $x = y + y'$ where y and y' are spanned by elements of $\Gamma(m, n)$ and $\Gamma(n) \setminus \Gamma(m, n)$, respectively. However,

$$(1 - \varepsilon_{m+1}) \cdots (1 - \varepsilon_n)\, y'\, (1 - \varepsilon_{m+1}) \cdots (1 - \varepsilon_n) = 0 \tag{5.26}$$

since for each element $\gamma \in \Gamma(n) \setminus \Gamma(m, n)$ there exists $i > m$ such that $\gamma\, \varepsilon_i = \gamma$ or $\varepsilon_i\, \gamma = \gamma$. This implies

$$x = (1 - \varepsilon_{m+1}) \cdots (1 - \varepsilon_n)\, y\, (1 - \varepsilon_{m+1}) \cdots (1 - \varepsilon_n). \tag{5.27}$$

Finally, averaging over the group $S_m(n)$ transforms y into an element of $Z_m(n)$; cf. the proof of Lemma 4.7. $\square$

For $\sigma \in \Gamma(n)$, set

$$Q(\sigma) = \{ i \in \mathbb{N}_{mn} \mid \sigma_{ii} = 1 \}. \tag{5.28}$$

Lemma 5.12 *The mapping*

$$\sigma \mapsto \gamma, \qquad \gamma = \sigma\, \varepsilon_{Q(\sigma)} = \varepsilon_{Q(\sigma)}\, \sigma \tag{5.29}$$

defines a bijection of $\Gamma(m, n)$ *onto the set of all* $\gamma \in \Gamma(n)$ *satisfying the conditions*

$$\operatorname{dom} \gamma \cap \mathbb{N}_{mn} = \operatorname{range} \gamma \cap \mathbb{N}_{mn}, \tag{5.30}$$

$$\deg_m \gamma = n - m. \tag{5.31}$$

Proof. The effect of the multiplication of σ by $\varepsilon_{Q(\sigma)}$ from the left or from the right consists of replacing all the diagonal entries $\sigma_{ii} = 1$ with $i > m$ by zeros. Therefore γ satisfies (5.30). Relation (5.31) follows from this observation and the fact that both $\operatorname{dom} \sigma$ and $\operatorname{range} \sigma$ contain $\mathbb{N}_{mn}$.

Conversely, let $\gamma \in \Gamma(n)$ satisfy (5.30) and (5.31). Note that (5.30) can be reformulated as follows: for any $i = m+1, \dots, n$ the i-th row and the i-th column are zero or non zero at the same time, whereas (5.31) means that all the diagonal entries γ_{ii} with $i > m+1$ vanish. Now, let σ be defined by

$$\begin{aligned} &\sigma_{ij} = \gamma_{ij} && \text{if either } i \neq j \text{ or } \min\{i,j\} \leq m, \\ &\sigma_{ii} = 1 && \text{if } i \in \mathbb{N}_{mn} \end{aligned}$$

and the i-th row (or the i-th column) of γ is zero. Then it is easy to see that $\sigma \in \Gamma(m,n)$ and that γ is the image of σ under the mapping (5.29). □

Lemma 5.13 *For $m < n$ the restriction of the projection $\theta_n : A_m(n) \to A_m(n-1)$ to the subspace $F_m^{n-m-1}(A_m(n))$ is injective.*

Proof. Suppose that $x \in A_m(n)$ and $\theta_n(x) = 0$. We will show that then $\deg_m x = n - m$ unless $x = 0$.

By Lemma 5.11, x can be written as a linear combination of the elements of type

$$\begin{aligned} (1-\varepsilon_{m+1}) \cdots (1-\varepsilon_n)\, \sigma\, (1-\varepsilon_{m+1}) \cdots (1-\varepsilon_n) \\ = \sum_{R,S \subseteq N_{mn}} (-1)^{|R|+|S|} \varepsilon_R\, \sigma\, \varepsilon_S, \qquad \sigma \in \Gamma(m,n). \end{aligned} \tag{5.32}$$

Let us divide the terms in the sum (5.32) into two groups depending on whether $R \cup S$ contains $Q(\sigma)$ or not. Then the terms of the first group are of m-degree $n-m$ whereas those of the second group are of m-degree $< n-m$. So, it suffices to prove that the elements

$$\sum_{R \cup S \supseteq Q(\sigma)} (-1)^{|R|+|S|} \varepsilon_R\, \sigma\, \varepsilon_S, \qquad \sigma \in \Gamma(m,n), \tag{5.33}$$

are linearly independent. Note that, in the case $R \cup S = Q(\sigma)$,

$$\varepsilon_R\, \sigma\, \varepsilon_S = \sigma\, \varepsilon_{Q(\sigma)} \qquad \text{and} \qquad \operatorname{rank} \sigma\, \varepsilon_{Q(\sigma)} = \operatorname{rank} \sigma - |Q(\sigma)|, \tag{5.34}$$

whereas, in the case $R \cup S$ strictly contains $Q(\sigma)$,

$$\operatorname{rank} \sigma\, \varepsilon_{Q(\sigma)} < \operatorname{rank} \sigma - |Q(\sigma)|. \tag{5.35}$$

Therefore, we now need to show that for any fixed k the elements

$$\sum_{R \cup S = Q(\sigma)} (-1)^{|R|+|S|} \sigma\, \varepsilon_{Q(\sigma)}, \tag{5.36}$$

where σ runs over the subset of the elements in $\Gamma(m,n)$ with $\operatorname{rank}\sigma - |\,Q(\sigma)| = k$, are linearly independent.

Lemma 5.12 implies that the elements $\sigma\,\varepsilon_{Q(\sigma)} \in \Gamma(n)$ are pairwise distinct. Hence it remains to prove that all the coefficients in (5.36) are non-vanishing. This is implied by the following general fact: if Q is an arbitrary finite set, then

$$\sum_{R,S\subseteq Q,\ R\cup S=Q} (-1)^{|\,R|+|\,S|} \neq 0. \tag{5.37}$$

We will prove that the sum in (5.37) equals $(-1)^q$ where $q = |\,Q|$. Indeed, for any $r = 0,1,\dots,q$, there are $\binom{n}{r}$ subsets $R \subseteq Q$ with $|\,R| = r$. Given R, for any $t = 0,1,\dots,r$, there are $\binom{r}{t}$ subsets $S \subseteq Q$ such that $R\cup S = Q$ and $|\,R\cap S| = t$. Since

$$|\,R| + |\,S| = r + t + (q-r) = q + t, \tag{5.38}$$

the sum in (5.37) equals

$$(-1)^q \sum_{r=0}^{q} \binom{n}{r} \sum_{t=0}^{r} (-1)^t \binom{r}{t}.$$

If $r = 0$ then the interior sum is equal to 1, otherwise it is zero. Therefore the entire sum is $(-1)^q$. □

Proposition 5.14 *The elements $\Delta_n^{\alpha,\mathcal{M}}$ with*

$$\operatorname{ord}\alpha + |\mathcal{M}| \leq n - m \tag{5.39}$$

form a basis of $A_m(n)$. Moreover, for any M with $0 \leq M \leq n-m$ the elements $\Delta_n^{\alpha,\mathcal{M}}$ satisfying

$$\operatorname{ord}\alpha + |\mathcal{M}| \leq M \tag{5.40}$$

form a basis of $F_m^M(A_m(n))$.

Proof. It suffices to prove the second claim. We use induction on n and follow the argument of the proof of Proposition 4.10. The claim is obviously true for $n = m$. Assume that $n \geq m+1$ and $M \leq n-m-1$. Lemma 5.13 implies that the elements $\Delta_n^{\alpha,\mathcal{M}}$ with $\operatorname{ord}\alpha + |\mathcal{M}| \leq M$ form a basis of $F_m^M(A_m(n))$.

To show that the elements $\Delta_n^{\alpha,\mathcal{M}}$ with $\operatorname{ord}\alpha + |\mathcal{M}| = n - m$ form a basis of $I(n) \cap A_m(n)$ note that

$$\Delta_n^{\alpha,\mathcal{M}} = (1 - \varepsilon_{m+1}) \cdots (1 - \varepsilon_n)\, c_n^{\alpha,\mathcal{M}}\, (1 - \varepsilon_{m+1}) \cdots (1 - \varepsilon_n); \tag{5.41}$$

see (5.21). Now the claim follows from Proposition 5.7 and the fact that the elements $c_n^{\alpha,\mathcal{M}}$, being multiplied by $(1 - \varepsilon_{m+1}) \cdots (1 - \varepsilon_n)$, remain linearly independent; cf. (4.38). □

Using Proposition 5.10 we can introduce the elements $\Delta^{\alpha,\mathcal{M}} \in A_m$ as sequences $\Delta^{\alpha,\mathcal{M}} = (\Delta_n^{\alpha,\mathcal{M}} \mid n \geq m)$.

Remark. We can regard $\Delta^{\alpha,\mathcal{M}}$ as a formal series given by (5.21) where the sum is taken over all disjoint subsets P and Q in $\{m+1, m+2, \dots\}$ satisfying (5.18). □

Theorem 5.15 *The elements $\Delta^{\alpha,\mathcal{M}}$ with $\alpha \in \Gamma(m, \mathbb{Z}_+)$ and $\mathcal{M} \in \mathbb{P}$ form a basis of the algebra A_m. Moreover, for any $M \geq 0$, the elements $\Delta^{\alpha,\mathcal{M}}$ with* $\operatorname{ord}\alpha + |\mathcal{M}| \leq M$ *form a basis of the M-th subspace $F_m^M(A_m)$ in A_m.*

Proof. The first claim follows from the second one. The second claim follows from Proposition 5.14 and the definition of $F_m^M(A_m)$ as the projective limit of the spaces $F_m^M(A_m(n))$. □

Corollary 5.16 *For $n > M$, the mapping*

$$\theta_n : F_m^M(A_m(n)) \to F_m^M(A_m(n-1)) \tag{5.42}$$

is an isomorphism of vector spaces and so is the mapping

$$\theta^{(n)} : F_m^M(A_m) \to F_m^M(A_m(n)), \qquad n \geq M. \tag{5.43}$$

In particular, $\dim F_m^M(A_m) < \infty$. □

For each $k = 1, \dots, m$ consider the following elements of $A_m(n)$

$$u_{k|n} = \sum_{i=k+1}^{n} (ki)(1 - \varepsilon_k)(1 - \varepsilon_i) = \sum_{i=k+1}^{n} (1 - \varepsilon_i)(ki)(1 - \varepsilon_i). \tag{5.44}$$

The image of $u_{k|n}$ under the retraction homomorphism (3.9) is the *Jucys–Murphy element* for $S(n)$; cf. [9], [19]. We obviously have $\theta_n(u_{k|n}) = u_{k|n-1}$ and so, for each k the element $u_k \in A_m$ can be defined as the sequence $u_k = (u_{k|n} \mid n \geq m)$. Recall that the algebra $A(m)$ is naturally embedded in A_m; see Proposition 3.7.

Proposition 5.17 *The following relations hold in the algebra A_m:*

$$s_k\,u_k = u_{k+1}\,s_k + (1-\varepsilon_k)(1-\varepsilon_{k+1}), \qquad s_k\,u_l = u_l\,s_k, \quad l \neq k, k+1; \tag{5.45}$$

$$u_k\,u_l = u_l\,u_k, \qquad \varepsilon_k\,u_k = u_k\,\varepsilon_k = 0, \qquad \varepsilon_i\,u_k = u_k\,\varepsilon_i, \quad i \neq k; \tag{5.46}$$

where $s_k = (k, k+1)$.

Proof. For $n > m$ we have $u_{1|n} = \Delta_n^{(2)} - \Delta'^{(2)}_{n-1}$, where $\Delta'^{(2)}_{n-1}$ is the element of the center of $\mathbb{C}[\Gamma_1(n)]$ given by (4.13) with the sum taken over the indices from $\{2, \dots, n\}$. Now an easy induction proves that the elements $u_{1|n}, \dots, u_{m|n}$ pairwise commute, and so do the elements $u_1, \dots, u_m$. The remaining relations easily follow from (5.44) and the relations in the algebra $A(n)$. □

We shall denote by $\widetilde{\mathcal{H}}_m$ the subalgebra of A_m generated by $A(m)$ and the elements $u_1, \dots, u_m$. The following is our main result. The theorem describes the structure of the algebra A_m.

Theorem 5.18 *We have an algebra isomorphism*

$$A_m \simeq A_0 \otimes \widetilde{\mathcal{H}}_m. \tag{5.47}$$

Moreover, the algebra $\widetilde{\mathcal{H}}_m$ is isomorphic to an abstract algebra with generators $s_1, \dots, s_{m-1}$, $\varepsilon_1, \dots, \varepsilon_m$, $u_1, \dots, u_m$ and the defining relations given by (3.2)–(3.4) *and* (5.45)–(5.46).

Recall that by Theorem 4.21 A_0 is isomorphic to the algebra of shifted symmetric functions Λ^*.

Proof. For any $\alpha \in \Gamma(m, \mathbb{Z}_+)$ and $\mathcal{M} \in \mathbb{P}$ such that $\operatorname{ord}\alpha + |\mathcal{M}| \leq n - m$ we have the equality in the algebra $A_m(n)$,

$$\Delta_n^{\alpha,\emptyset}\Delta_n^{1,\mathcal{M}} = \Delta_n^{\alpha,\mathcal{M}} + \text{ lower } m\text{-degree terms}, \tag{5.48}$$

where $\emptyset$ stands for the empty partition while $1 \in \Gamma(m, \mathbb{Z}_+)$ is the $m \times m$ identity matrix; cf. the proofs of Proposition 4.2 and Corollary 4.11. On the other hand, we have

$$\deg_m(\Delta_n^{\mathcal{M}} - \Delta_n^{1,\mathcal{M}}) < |\mathcal{M}|; \tag{5.49}$$

see (4.13) for the definition of $\Delta_n^{\mathcal{M}}$. Now (5.48) and Proposition 5.14 imply that the elements $\Delta_n^{\alpha,\emptyset}\Delta_n^{\mathcal{M}}$ with $\operatorname{ord}\alpha + |\mathcal{M}| \leq n - m$ form a basis of $A_m(n)$. Hence the elements $\Delta^{\alpha,\emptyset}\Delta^{\mathcal{M}}$ with $\alpha \in \Gamma(m, \mathbb{Z}_+)$ and $\mathcal{M} \in \mathbb{P}$ form a basis of the algebra A_m. In other words, A_m is a free A_0-module with the basis $\{\Delta^{\alpha,\emptyset} \mid \alpha \in \Gamma(m, \mathbb{Z}_+)\}$.

Further, if $\alpha \in \Gamma(m, \mathbb{Z}_+)$ has zero rows $i_1, \ldots, i_r$ then

$$\Delta_n^{\alpha,\emptyset} = \varepsilon_{i_1} \cdots \varepsilon_{i_r} \, \Delta_n^{\alpha',\emptyset} \tag{5.50}$$

for some element $\alpha' \in S(m, \mathbb{Z}_+)$. Observe now that every element $\alpha' \in S(m, \mathbb{Z}_+)$ can be written as a product of the form

$$\alpha' = \sigma \, \alpha_1^{k_1} \cdots \alpha_m^{k_m}, \qquad k_i \geq 0, \tag{5.51}$$

where $\sigma \in S(m)$ and $\alpha_i \in \Gamma(m, \mathbb{Z}_+)$ is the diagonal matrix whose ii-th entry is z and all other diagonal entries are equal to 1. This implies that modulo lower m-degree terms, the element $\Delta_n^{\alpha',\emptyset}$ coincides with the product

$$\Delta_n^{\sigma,\emptyset} (\Delta_n^{\alpha_1,\emptyset})^{k_1} \cdots (\Delta_n^{\alpha_m,\emptyset})^{k_m}; \tag{5.52}$$

cf. the proof of (4.44). The claim remains valid if we replace each $\Delta_n^{\alpha_k,\emptyset}$ with the element $u_{k|n}$. Indeed, this follows from the equality

$$\Delta_n^{\alpha_k,\emptyset} = u_{k|n} + \text{ elements of } m\text{-degree zero.} \tag{5.53}$$

Note also that the element $\Delta_n^{\sigma,\emptyset}$ can be identified with σ. Thus, modulo lower m-degree terms, the element (5.52) coincides with the product $\sigma \, u_{1|n}^{k_1} \cdots u_{m|n}^{k_m}$. Using an obvious induction on the m-degree we may conclude that the A_0-module A_m is generated by the subspace $\widetilde{\mathcal{H}}_m$.

To prove that $\widetilde{\mathcal{H}}_m$ generates the A_0-module A_m freely, we check that for any $M > 0$ the dimension of the subspace $F_m^M(\widetilde{\mathcal{H}}_m)$ is less or equal to the number of elements $\alpha \in \Gamma(m, \mathbb{Z}_+)$ with $\operatorname{ord} \alpha \leq M$. Indeed, by Proposition 5.17 the subalgebra $\widetilde{\mathcal{H}}_m$ is spanned by the elements of the form $\gamma \, u_1^{k_1} \cdots u_m^{k_m}$ with $\gamma \in \Gamma(m)$. The relation $\varepsilon_k \, u_k = 0$ ensures that such a product is zero unless $k_j = 0$ for each zero column j in γ. To each of the nonzero products associate the element $\alpha \in \Gamma(m, \mathbb{Z}_+)$ which has the ij-entry z^{k_j} where the j-th column of γ is nonzero with $\gamma_{ij} = 1$. This shows that the cardinality of a basis of $F_m^M(\widetilde{\mathcal{H}}_m)$ can be at most the number of elements $\alpha \in \Gamma(m, \mathbb{Z}_+)$ with $\operatorname{ord} \alpha \leq M$, proving (5.47).

To prove the second claim of the theorem note that by Proposition 5.17 there is an algebra epimorphism from the abstract algebra in question to $\widetilde{\mathcal{H}}_m$. The above argument implies that the nonzero products $\gamma \, u_1^{k_1} \cdots u_m^{k_m}$ with $\gamma \in \Gamma(m)$ form a basis of $\widetilde{\mathcal{H}}_m$. □

Corollary 5.19 *The mapping*

$$s_k \mapsto s_k, \qquad \varepsilon_k \mapsto \varepsilon_k, \qquad u_k \mapsto u_{k|n} \tag{5.54}$$

defines an algebra homomorphism $\psi : \widetilde{\mathcal{H}}_m \to A_m(n)$. The algebra $A_m(n)$ is generated by $A_0(n)$ and the image of ψ. $\square$

The *degenerate affine Hecke algebra* $\mathcal{H}_m$ (see [6], [12]) is defined to be generated by elements $s_1, \dots, s_{m-1}$ and $u_1, \dots, u_m$ with the defining relations (3.2) and

$$s_k\, u_k = u_{k+1}\, s_k + 1, \qquad s_k\, u_l = u_l\, s_k, \quad l \neq k, k+1; \tag{5.55}$$

$$u_k\, u_l = u_l\, u_k. \tag{5.56}$$

As a linear space, $\mathcal{H}_m$ is isomorphic to the tensor product $\mathbb{C}[S(m)] \otimes \mathbb{C}[u_1, \dots, u_m]$. The following corollary is implied by Theorem 5.18 and provides an analog of the retraction homomorphism (3.9).

Corollary 5.20 *The mapping*

$$s_k \mapsto s_k, \qquad u_k \mapsto u_k, \qquad \varepsilon_k \to 0 \tag{5.57}$$

defines an algebra epimorphism $\widetilde{\mathcal{H}}_m \to \mathcal{H}_m$. $\square$

It can be seen from the proof of Theorem 5.18 that the retraction homomorphisms (3.9) and (5.57) "respect" the homomorphism $\psi : \widetilde{\mathcal{H}}_m \to A_m(n)$ defined in Corollary 5.19. More precisely, the following result takes place. It was announced in [29, Theorem 11], and a proof was given in [25]. We denote by $B_m(n)$ the centralizer of $S_m(n)$ in the group algebra $\mathbb{C}[S(n)]$; see Introduction.

Corollary 5.21 *The mapping*

$$s_k \mapsto s_k, \qquad u_k \mapsto \sum_{i=k+1}^{n} (ki) \tag{5.58}$$

defines an algebra homomorphism $\varphi : \mathcal{H}_m \to B_m(n)$. The algebra $B_m(n)$ is generated by $B_0(n)$ and the image of φ. $\square$

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