CONSTRUCTIONS FOR OCTONION AND EXCEPTIONAL JORDAN ALGEBRAS

L. J. RYLANDS AND D. E. TAYLOR

Abstract. In this note we reverse the usual process of constructing the Lie algebras of types G_2 and F_4 as algebras of derivations of the split octonions or the exceptional Jordan algebra and instead begin with their Dynkin diagrams and then construct the algebras together with an action of the Lie algebras and associated Chevalley groups. This is shown to be a variation on a general construction of all standard modules for simple Lie algebras and it is well suited for use in computational algebra systems. All the structure constants which occur are integral and hence the construction specialises to all fields, without restriction on the characteristic, avoiding the usual problems with characteristics 2 and 3.

1. Introduction

Traditionally, the Lie algebras of types G_2 and F_4 have been constructed as the algebra of derivations of the Cayley-Dickson (octonion) algebra and the exceptional Jordan algebra, respectively (see [9] and [3]). In this note it is shown that from an algorithmic and computational point of view there is an advantage in reversing the process and beginning with the Dynkin diagram defining the Lie algebra. As a result we have a uniform construction of the algebras over arbitrary fields together with explicit actions of the Lie algebra and associated Chevalley groups. For a history of other approaches see [10, Chapters VIII and IX] and the references given there, particularly [6].

2. Standard modules for semisimple Lie algebras

Where possible we choose our notation to be consistent with Humphreys [8]. Let $\mathcal{L}$ denote a complex semisimple Lie algebra with root system Φ, simple roots Δ, Cartan subalgebra H and Cartan decomposition

$$\mathcal{L} = H \oplus \bigoplus_{\alpha\in\Phi} \mathcal{L}_\alpha.$$

For each $\alpha \in \Phi$ choose $e_\alpha \in \mathcal{L}_\alpha$ so that $\{ e_\alpha \mid \alpha \in \Phi \}$ together with the elements $h_\alpha = [e_\alpha\, e_{-\alpha}]$ for $\alpha \in \Delta$ forms a Chevalley basis for $\mathcal{L}$.

Theorem 2.1. *Given a subset Δ_1 of Δ, let Φ_1 be the root system generated by Δ_1 and let $\mathcal{L}_1$ be the corresponding Lie subalgebra of $\mathcal{L}$. For $\gamma \in \Phi \setminus \Phi_1$, let*

$$X = \{ \alpha \in \Phi \mid \alpha - \gamma \text{ is in the span of } \Delta_1 \}.$$

Then the subspace V of $\mathcal{L}$ spanned by the set $\{e_\alpha \mid \alpha \in X\}$ is an $\mathcal{L}_1$-module of dimension $|X|$.

Proof. The restriction of the adjoint action of $\mathcal{L}$ to $\mathcal{L}_1$ allows us to view $\mathcal{L}$ as an $\mathcal{L}_1$-module. For $\alpha \in \Phi_1$ and $\beta \in X$ it is clear that $\alpha + \beta - \gamma$ is a linear combination of elements of Δ_1. Therefore, if $\alpha + \beta \in \Phi$, then $\alpha + \beta \in X$ and hence $\operatorname{ad}(e_\alpha)e_\beta \in V$. To complete the proof that V is an

Date: 6 January 2000.
1991 *Mathematics Subject Classification.* 17B25,17C40,17D05.

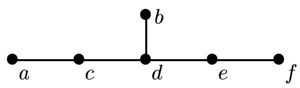
$\mathcal{L}_1$-submodule of $\mathcal{L}$ we note that we cannot have $\alpha + \beta = 0$, otherwise $-\gamma$ would belong to the span of Δ_1, contrary to the choice of γ. $\square$

The standard (lowest dimensional) representation of every simple Lie algebra of type A_n, B_n, C_n, D_n or E_n, except that of type E_8 (for which the standard module is the adjoint representation), can be obtained by applying this theorem to an embedding in an algebra of the same type but of rank $n+1$. This observation leads to a direct and uniform means of identifying the simple Lie algebras of type A_n, B_n, C_n and D_n with the matrix algebras $\mathfrak{sl}(n+1,\mathbb{C})$, $\mathfrak{o}(2n+1,\mathbb{C})$, $\mathfrak{sp}(2n,\mathbb{C})$ and $\mathfrak{o}(2n,\mathbb{C})$. It is our purpose to describe an extension of this theorem which, in types F_4 and G_2, leads to algebra structures on the modules.

The practical application of this method depends on two basic algorithms: (1) the construction of a root system from a Dynkin diagram (or Cartan matrix), and (2) the computation of structure constants for the Chevalley basis. For the root system we use the algorithm given in Humphreys [8, p. 56]. Given an ordered set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ this algorithm produces an ordered set Φ^+ of positive roots in which the roots of height i appear before the roots of height j whenever $i < j$. For each $\alpha \in \Phi^+ \setminus \Delta$, let i be the least index such that $\alpha - \alpha_i$ is a root. We choose the structure constant for $[e_{\alpha_i}\, e_\alpha]$ to be positive and then all other structure constants are uniquely determined by the algorithm given in Carter [2, §4.2]. (For further details see [7].)

3. The octonions

Let Φ be the root system of type E_6 and let $\Delta = \{a, b, c, d, e, f\}$ be a set of simple roots arranged according to the Dynkin diagram



The octonions will be constructed from the root system inclusion $D_4 \subset E_6$, where D_4 is the subsystem spanned by b, c, d and e. Let X_{ij} denote the set of roots in which a occurs with coefficient i and f occurs with coefficient j. The sets X_{11}, X_{10} and X_{01} each have 8 elements and they partition the set of 24 positive roots of E_6 not in the D_4 subsystem.

Let $\mathfrak{d}_4$ be the Lie algebra generated by the root vectors $e_{\pm\alpha}$, $\alpha \in \{b, c, d, e\}$. Then by Theorem 2.1 the vectors e_α for $\alpha \in X_{ij}$ span a $\mathfrak{d}_4$-module V_{ij} and the Lie bracket induces a $\mathfrak{d}_4$-invariant bilinear pairing $V_{10} \times V_{01} \to V_{11}$. The entries in the matrices representing the action of $\mathrm{ad}(e_\alpha)$ on V_{ij} with respect to the basis indexed by X_{ij} are the structure constants for $[e_\alpha\, e_\beta]$, $\beta \in X_{ij}$. The weights of the module V_{ij} are just the restrictions of the roots $\alpha \in X_{ij}$ to the Cartan subalgebra of $\mathfrak{d}_4$ spanned by h_b, h_c, h_d and h_e. Furthermore, if we identify $\alpha \in X_{ij}$ with its row vector of coefficients and if C is the Cartan matrix of Δ, then αC gives the coefficients of α as a sum of fundamental weights. From this we see that the highest weights of the $\mathfrak{d}_4$-modules V_{11}, V_{10} and V_{01} have multiplicity one and correspond to the end nodes of the D_4 Dynkin diagram. In particular, the modules are irreducible and pairwise non-isomorphic.

On the other hand, these modules are related by triality automorphisms of $\mathfrak{d}_4$. In general, if π is a permutation of the simple roots of $\mathfrak{d}_4$ we may extend π to an automorphism of the D_4 root system and then to an automorphism of $\mathfrak{d}_4$ itself such that $\pi(h_\alpha) = h_{\pi(\alpha)}$ and $\pi(e_\alpha) = \pm e_{\pi(\alpha)}$ for all roots α of $\mathfrak{d}_4$. In particular, let σ and τ denote the automorphisms induced by the permutations (b, c) and (b, e).

From a $\mathfrak{d}_4$-module V and an automorphism π we obtain a new $\mathfrak{d}_4$-module V^π by declaring the action of $x \in \mathfrak{d}_4$ on $v \in V$ to be $\pi(x)v$. By considering highest weights we see that there are isomorphisms $\varphi : V_{11}^\sigma \to V_{10}$ and $\psi : V_{11}^\tau \to V_{01}$. Similarly, the vectors $e_{-\alpha}$ for $\alpha \in X_{11}$ span a $\mathfrak{d}_4$-module V_{11}^- and we have an isomorphism $\mu : V_{11} \to V_{11}^-$. As all the modules V_{ij} are irreducible, these isomorphisms are unique (up to a scalar multiple).

For computational purposes we require explicit isomorphisms. To obtain these we use the fact that each module is spanned by the images of a maximal vector under the $\mathfrak{d}_4$-action ([8, §20.2]). In order to describe this we shall write the elements of Φ in an abbreviated form. For example, $a+b+c+2d+e$ will be written $abc2de$ and $-a-c-d$ will be written $-acd$.

As maximal vectors for V_{11}, V_{10}, V_{01} and V_{11}^- we take $e_{a2b2c3d2ef}$, $e_{ab2c2de}$, $e_{bc2d2ef}$ and e_{-acdef}, respectively. These vectors are unique (up to a scalar multiple). The non-zero structure constants for the actions of e_b, e_c, e_d and e_e on the elements of X_{ij} are all 1 except that $[e_c\, e_a] = -e_{ac}$, $[e_d\, e_{ac}] = -e_{acd}$, $[e_e\, e_{acd}] = -e_{acde}$ and $[e_e\, e_{abcd}] = -e_{abcde}$. Thus the effect of the isomorphisms φ, ψ and μ on a basis for V_{11} is given in the following table.

$w \in V_{11}$	$\varphi(w) \in V_{10}$	$\psi(w) \in V_{01}$	$\mu(w) \in V_{11}^-$
$w_1 = e_{a2b2c3d2ef}$	$e_{ab2c2de}$	$e_{bc2d2ef}$	e_{-acdef}
$w_2 = e_{ab2c3d2ef}$	e_{abc2de}	e_{bc2def}	$-e_{-abcdef}$
$w_3 = e_{ab2c2d2ef}$	e_{abcde}	e_{bcdef}	$e_{-abc2def}$
$w_4 = e_{abc2d2ef}$	e_{acde}	e_{bdef}	$-e_{-ab2c2def}$
$w_5 = e_{ab2c2def}$	$-e_{abcd}$	e_{cdef}	$-e_{-abc2d2ef}$
$w_6 = e_{abc2def}$	$-e_{acd}$	e_{def}	$e_{-ab2c2d2ef}$
$w_7 = e_{abcdef}$	e_{ac}	e_{ef}	$-e_{-ab2c3d2ef}$
$w_8 = e_{acdef}$	$-e_a$	e_f	$e_{-a2b2c3d2ef}$

From the isomorphism $\mu : V_{11} \to V_{11}^-$ and the Killing form κ for $\mathfrak{e}_6$ we obtain a bilinear form

$$(u,v) = \frac{1}{24}\kappa(u,\mu(v))$$

on V_{11} which is preserved by the $\mathfrak{d}_4$-action; that is, $(xu,v)+(u,xv)=0$ for all $x \in \mathfrak{d}_4$.

Theorem 3.1. $\mathfrak{d}_4 \simeq \mathfrak{o}(8,\mathbb{C})$.

Proof. In general, the root spaces $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ are orthogonal with respect to κ if and only if $\alpha+\beta \neq 0$. Thus for $\alpha,\beta \in X_{11}$ we have $(e_\alpha, e_\beta) \neq 0$ if and only if $\alpha+\beta = 2a+2b+3c+4d+3e+2f$. Then a direct calculation shows that the form $(-,-)$ is symmetric and that the only non-zero values of (w_i,w_j) are $(w_i,w_{9-i}) = (w_{9-i},w_i) = (-1)^{i+1}$, for $1 \le i \le 4$. (It suffices to determine (w_1,w_8) and then to use the $\mathfrak{d}_4$-action to calculate the other values.) Thus the matrices representing $\mathfrak{d}_4$ on V_{11} belong to $\mathfrak{o}(8,\mathbb{C})$ and on comparing dimensions we see that we have identified $\mathfrak{d}_4$ with the matrix algebra $\mathfrak{o}(8,\mathbb{C})$. □

Using the isomorphisms φ and ψ we define a bilinear product on V_{11} by

$$uv = [\varphi(u)\,\psi(v)].$$

In anticipation of its identification with the split octonions we shall denote the algebra V_{11} by $\mathbb{O}$. The table of products of basis elements is readily obtained from the description of φ and ψ.

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	0	0	0	w_1	0	w_2	w_3	w_5
w_2	0	0	$-w_1$	0	$-w_2$	0	w_4	w_6
w_3	0	w_1	0	0	$-w_3$	$-w_4$	0	w_7
w_4	0	w_2	w_3	w_4	0	0	0	w_8
w_5	$-w_1$	0	0	0	$-w_5$	$-w_6$	$-w_7$	0
w_6	$-w_2$	0	w_5	w_6	0	0	$-w_8$	0
w_7	$-w_3$	$-w_5$	0	w_7	0	w_8	0	0
w_8	$-w_4$	$-w_6$	$-w_7$	0	$-w_8$	0	0	0

An immediate consequence of the definition of the product is the so-called Principle of Local Triality:

$$x(uv) = (\sigma(x)u)v + u(\tau(x)v) \quad \text{for all } x \in \mathfrak{d}_4.$$

For $\alpha \in X_{ij}$ define the degree of e_α to be $\deg(e_\alpha) = \operatorname{ht}(\alpha) - 8$, where $\operatorname{ht}(\alpha)$ is the sum of the coefficients of α with respect to the basis Δ. Let A_i be the subspace of $\mathbb{O}$ spanned by the elements of X_{11} of degree i. Then $\mathbb{O} = \bigoplus_{j=-3}^{3} A_i$ is a graded algebra.

For $u = \sum_{i=1}^{8} c_i w_i \in \mathbb{O}$, define the *trace* of u to be $\operatorname{tr}(u) = c_4 - c_5$ and the *conjugate* of u to be $\overline{u} = \operatorname{tr}(u)\iota - u$, where $\iota = w_4 - w_5$. It is clear from the multiplication table that ι is the identity element of $\mathbb{O}$.

The polar form of the quadratic form $n(u) = c_1c_8 - c_2c_7 + c_3c_6 - c_4c_5$ is $n(u+v) - n(u) - n(v) = (u, v)$. If $\mathbb{O}(\mathbb{Z})$ denotes the $\mathbb{Z}$-span of the basis $w_1, w_2, \ldots, w_8$, then n takes integer values on $\mathbb{O}(\mathbb{Z})$ and it induces a non-degenerate quadratic form on $\mathbb{O}(F) = \mathbb{O}(\mathbb{Z}) \otimes_{\mathbb{Z}} F$ for all fields F. If the characteristic of F is not 2, then $n(u) = \frac{1}{2}(u, u)$. We call n the *norm form* of $\mathbb{O}$.

Lemma 3.2. *For all $u, v \in \mathbb{O}$ we have $\overline{\overline{u}} = u$, $\overline{uv} = \overline{v}\,\overline{u}$, $n(u)\iota = u\,\overline{u}$ and $n(uv) = n(u)n(v)$.*

Proof. The orthogonal complement of ι with respect to the bilinear form introduced above is spanned by w_1, w_2, w_3, $w_4 + w_5$, w_6, w_7 and w_8. The multiplication table for these elements is a skew matrix except for the elements on the secondary diagonal. There are two consequences of this observation. Firstly, that the form can be written $(u, v)\iota = u\overline{v} + v\overline{u}$ and secondly that $\overline{uv} = \overline{v}\,\overline{u}$ for all $u, v \in \mathbb{O}$. From the definitions we have $\operatorname{tr}(\overline{u}) = \operatorname{tr}(u)$ and hence $\overline{\overline{u}} = u$. Finally, the multiplicative property of n can be checked directly from the multiplication table of the basis elements. □

Theorem 3.3. *The algebra $\mathbb{O}$ is alternative, not associative and isomorphic to the* split octonion *algebra over $\mathbb{C}$.*

Proof. This result is a consequence of the preceding lemma and the Theorem of Hurwitz [12, Theorem 3.25] but it can be established directly as follows.

Let M be the subalgebra of $\mathbb{O}$ spanned by w_1, w_4, w_5 and w_8. It is easily seen that M is associative and that the map given by $w_1 \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $w_4 \mapsto \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $w_5 \mapsto \left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $w_8 \mapsto \left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right)$ identifies M with the algebra of 2×2 matrices over $\mathbb{C}$. The element $p = w_2 - w_3 + w_6 + w_7$ is orthogonal to M and therefore $pu = \overline{u}p$ for all $u \in M$. We have $p^2 = 2\iota$ and every element of $\mathbb{O}$ can be written in the form $u + vp$ for some $u, v \in M$. Furthermore, direct calculation with a basis for M shows that for $x, y, u, v \in M$ we have $(x + yp)(u + vp) = (ux + 2\overline{y}v) + (yu + v\overline{x})p$ and thus $\mathbb{O}$ arises from M via the *Cayley-Dickson* process. Thus $\mathbb{O}$ is uniquely determined as the split octonion algebra over $\mathbb{C}$ (cf. [5, Section 3]). □

Of course there are many ways to establish this result. In a computational context such as Lie [11], Magma [1] or GAP [13] it is a simple matter to establish that $\mathbb{O}$ is alternative by checking the linearised form of the alternative law on a basis. Another approach is to begin with a vector space V of dimension three over $\mathbb{C}$ with symmetric inner product (u, v), orthonormal basis v_1, v_2, v_3 and the usual cross product $u \times v$, then use Zorn's construction for $\mathbb{O}$ as the algebra of matrices $\left(\begin{smallmatrix} a & v \\ w & b \end{smallmatrix}\right)$, with $a, b \in \mathbb{C}$, $v, w \in V$ and product

$$\begin{pmatrix} a & v \\ w & b \end{pmatrix} \begin{pmatrix} a' & v' \\ w' & b' \end{pmatrix} = \begin{pmatrix} aa' - (v, w') & av' + b'v + w \times w' \\ a'w + bw' + v \times v' & bb' - (w, v') \end{pmatrix}.$$

The multiplication table given in Humphreys [8, §19.3] shows that the map defined by

$$w_1 \mapsto \begin{pmatrix} 0 & 0 \\ -v_3 & 0 \end{pmatrix}, \quad w_2 \mapsto \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix}, \quad w_3 \mapsto \begin{pmatrix} 0 & v_2 \\ 0 & 0 \end{pmatrix}, \quad w_4 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$w_5 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad w_6 \mapsto \begin{pmatrix} 0 & 0 \\ v_2 & 0 \end{pmatrix}, \quad w_7 \mapsto \begin{pmatrix} 0 & 0 \\ -v_1 & 0 \end{pmatrix}, \quad w_8 \mapsto \begin{pmatrix} 0 & -v_3 \\ 0 & 0 \end{pmatrix}$$

identifies $\mathbb{O}$ with Zorn's algebra.

It is equally easy to obtain the multiplication table given in Coxeter [4, page 561]. To this end, let $e_1 = w_1 + w_8$, $e_2 = w_2 - w_7$, $e_3 = i(w_1 - w_8)$, $e_4 = -w_3 - w_6$, $e_5 = i(w_3 - w_6)$, $e_6 = -i(w_2 + w_7)$ and $e_7 = -i(w_4 + w_5)$. Each of these elements squares to $-\iota$ and together with ι they form an orthonormal basis for $\mathbb{O}$; hence $e_i\, e_j = -e_j\, e_i$ for $i \neq j$. Their $\mathbb{R}$-span is a division algebra and a real form for $\mathbb{O}$. Coxeter's relations are: $e_r\, e_s = e_t$ for the triples $(r, s, t) = (1, 2, 4)$, $(2, 3, 5)$, $(3, 4, 6)$, $(4, 5, 7)$, $(5, 6, 1)$, $(6, 7, 2)$ and $(7, 1, 3)$.

The Lie algebra $\mathfrak{g}_2 = \mathfrak{g}_2(\mathbb{C})$ of type G_2 may be regarded as the subalgebra of $\mathfrak{d}_4$ generated by $e_A = e_b + e_c + e_e$, $e_B = e_d$, $e_{-A} = e_{-b} + e_{-c} + e_{-e}$ and $e_{-B} = e_{-d}$. Its Dynkin diagram is $\underset{A}{\bullet}\!\Lleftarrow\!\underset{B}{\bullet}$. As a Chevalley basis for $\mathfrak{g}_2$ we take the fixed points of the graph automorphisms σ and τ on a Chevalley basis for $\mathfrak{d}_4$. As $\mathfrak{g}_2$-modules the spaces V_{10}, V_{01} and V_{11} are isomorphic but no longer irreducible. We have $x\iota = 0$ for all $x \in \mathfrak{g}_2$ and $\mathfrak{g}_2$ acts irreducibly on the 7-dimensional space $\mathbb{O}_0 = \iota^\perp$ of elements whose trace is 0. The Principle of Local Triality shows that every element of $\mathfrak{g}_2$ acts on $\mathbb{O}$ as a derivation.

The elements of the Chevalley basis for $\mathfrak{g}_2$ preserve the $\mathbb{Z}$-form $\mathbb{O}(\mathbb{Z})$ spanned by the w_i. Thus our construction of $\mathbb{O}$ and the associated action of $\mathfrak{g}_2$ may be tensored with an arbitrary field F to give an action of $\mathfrak{g}_2(F)$ on the split octonions $\mathbb{O}(F)$ over F.

Theorem 3.4. *$\mathfrak{g}_2(F)$ is the algebra of derivations of $\mathbb{O}(F)$ for all fields F.*

Proof. For all fields, except those of characteristic 2 or 3, the argument of [3] shows that $\mathfrak{g}_2(F)$ is the algebra of derivations of $\mathbb{O}(F)$. In characteristic 3, all maps $u \mapsto su - us$, for $s \in \mathbb{O}_0(F)$ are derivations and the proof breaks down. However, the condition that a matrix be a derivation translates to a collection of linear equations in the matrix coefficients and we shall see that the space of solutions has dimension 14 in all characteristics.

We shall use the same notation (and the same multiplication table) for the basis of $\mathbb{O}(F)$ as we have been using for $\mathbb{O}$. Then w_1, w_6, w_7 is a basis for the subspace $A_1 = w_5\mathbb{O}(F)w_4$ and w_2, w_3, w_8 is a basis for $A_2 = w_4\mathbb{O}(F)w_5$. (Even though $\mathbb{O}$ is not associative, A_1 and A_2 are well-defined.)

If D is a derivation of $\mathbb{O}(F)$, then $D(\iota) = 0$ and hence $D(w_4) = D(w_5)$. If $D(w_4) = 0$, then both A_1 and A_2 are D-invariant. The description given above of the isomorphism between $\mathbb{O}$ and Zorn's algebra based on the vector space V defines isomorphisms $\varphi_i : A_i \to V = F^3$ such that $\varphi_2(a_1b_1) = \varphi_1(a_1) \times \varphi_1(b_1)$ for $a_1, b_1 \in A_1$ and $\varphi_1(a_2b_2) = \varphi_2(a_2) \times \varphi_2(b_2)$ for $a_2, b_2 \in A_2$. The restrictions of D to A_1 and A_2 induce linear transformations D_1 and D_2 of V such that $D_i\varphi_i(a) = \varphi_i(D(a))$ for all $a \in A_i$. It follows that $D_2(u \times v) = D_1(u) \times v + u \times D_1(v)$ and $D_1(u \times v) = D_2(u) \times v + u \times D_2(v)$. This implies that the trace of D_1 is 0 and that $D = 0$ if and only if $D_1 = 0$. (The matrix of D_1 with respect to the basis v_1, v_2, v_3 is the negative transpose of the matrix of D_2.) Thus the dimension of the space of derivations D such that $D(w_4) = 0$ is at most 8.

Suppose that D is an arbitrary derivation and write $D(w_4) = aw_4 + w$ where $\mathrm{tr}(w) = 0$. On applying D to $w_4w_5 = 0$ and using the fact that $\overline{w}_4 = -w_5$ we find that $aw_4 - w\,\overline{w}_4 - w_4\,\overline{w} = 0$. Hence $a = 0$ and $D(w_4)$ is orthogonal to w_4 and to w_5. Thus the dimension of the image of the map $D \mapsto D(w_4)$ is at most 6 and the dimension of its kernel is at most 8. On the other hand we already have a space of derivations $\mathfrak{g}_2(F)$ of dimension 14 and this completes the proof. □

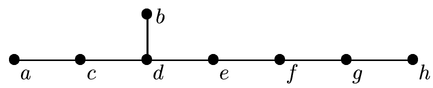
4. The exceptional Jordan algebra

A Jordan algebra is a nonassociative algebra that is commutative and satisfies the Jordan identity

$$a^2(ba) = (a^2b)a.$$

It is well known (see [5, 3.11]) that the exceptional Jordan algebras are associated with the Lie algebras of type F_4. In this section it is shown that the exceptional Jordan algebra over $\mathbb{C}$ can be constructed in an analogous way to the octonions in the previous section.

We begin with the root system inclusion $E_6 \subset E_8$. In this case, let Φ be the root system of type E_8, let $\Delta = \{a, b, c, d, e, f, g, h\}$ be a set of simple roots arranged according to the Dynkin diagram



and let $\mathfrak{e}_6$ be the Lie algebra generated by the root vectors $e_{\pm\alpha}$, $\alpha \in \{a, b, c, d, e, f\}$.

Let X_{ij} denote the set of roots in Φ which g occurs with coefficient i and h occurs with coefficient j. The sets X_{10}, X_{11} and X_{21} each have 27 elements and by Theorem 2.1, the vectors e_α for $\alpha \in X_{ij}$ span an $\mathfrak{e}_6$-module V_{ij}. The Lie bracket induces a bilinear pairing $V_{10} \times V_{11} \to V_{21}$, which ultimately will lead to a Jordan algebra structure on V_{21}.

The fundamental weights corresponding to the end nodes of the E_6 sub-diagram are $\lambda_1 = \frac{1}{3}(4a + 3b + 5c + 6d + 4e + 2f)$ and $\lambda_6 = \frac{1}{3}(2a + 3b + 4c + 6d + 5e + 4f)$. Given a weight λ we follow the notation of Humphreys [8, §20.3] and use $V(\lambda)$ to denote the irreducible module with highest weight λ. As maximal vectors for the modules V_{10}, V_{11} and V_{21} we have $e_{2a2b3c4d3e2fg}$, $e_{2a2b3c4d3e2fgh}$ and $e_{2a3b4c6d5e4f2gh}$, respectively. Thus, as in the previous section, consideration of highest weights shows that the three modules are irreducible and that V_{10} and V_{11} are isomorphic to $V(\lambda_1)$ whereas V_{21} is isomorphic to $V(\lambda_6)$. Note that $\mathrm{ad}(e_h) : V_{10} \to V_{11}$ is an $\mathfrak{e}_6$-isomorphism.

Instead of triality we now have a duality automorphism of $\mathfrak{e}_6$ induced by the permutation $\pi = (a, f)(c, e)$ of the simple roots. This automorphism interchanges the weights λ_1 and λ_6 and we have $\mathfrak{e}_6$-isomorphisms $\varphi : V_{21}^\pi \to V_{10}$ and $\psi : V_{21}^\pi \to V_{11}$, where V_{21}^π is the vector space V_{21} with the action of $x \in \mathfrak{e}_6$ on $v \in V_{21}$ given by $\pi(x)v$. Similarly, the vectors $e_{-\alpha}$ with $\alpha \in X_{21}$ span an $\mathfrak{e}_6$-module V_{21}^- with maximal vector $e_{-bc2d2e2f2gh}$ and highest weight λ_1. Hence there is an isomorphism $\mu : V_{21}^\pi \to V_{21}^-$. The Killing form κ for $\mathfrak{e}_8$ induces a non-degenerate pairing $V_{21}^- \times V_{21} \to \mathbb{C}$ and therefore V_{21}^- is the dual of V_{21}.

Providing a table to describe the isomorphisms φ, ψ and μ as we did for the octonions is too cumbersome and instead we simply exhibit a sequence of 26 pairs (i_k, α_k) where $1 \le i_k \le 26$ and $\alpha_k \in \{a, b, c, d, e, f\}$. We shall use these to produce compatible bases for V_{21}^π, V_{10} and V_{11}.

$$\begin{gathered}(1,a),(2,c),(3,d),(4,b),(4,e),(6,f),(6,b),(7,b),(8,d),\\(9,d),(10,c),(11,e),(11,c),(12,a),(13,c),(14,a),(16,d),(16,a),\\(18,b),(18,a),(20,a),(21,c),(22,c),(24,d),(25,e),(26,f).\end{gathered}$$

An ordered basis $z_1, z_2, \ldots, z_{27}$ for V_{21}^π can be obtained by taking z_1 to be the maximal vector in X_{21} and recursively setting $z_{k+1} = e_{-\alpha_k} \cdot z_k = \mathrm{ad}(e_{-\pi(\alpha_k)})z_{i_k}$, where (i_k, α_k) is the k-th pair from the above list. Given the basis $\{\, z_k \mid 1 \le k \le 27 \,\}$ for V_{21}^π, the basis $\{\, \varphi(z_k) \mid 1 \le k \le 27 \,\}$ for V_{10} is obtained by taking $\varphi(z_1)$ to be the maximal vector in X_{10} and carrying out the process described in the previous sentence. (Similarly we take $\psi(z_1)$ to be the maximal vector in X_{11} and apply the algorithm to obtain $\psi(z_k)$.) Our choice of structure constants means that the matrices representing $\mathrm{ad}(e_a), \ldots, \mathrm{ad}(e_f)$ with respect to any of the bases indexed by X_{10}, X_{11} or X_{21} have only six non-zero entries, all of which are 1, and the bases obtained here are permutations of the original bases.

On the other hand, the matrices representing $\mathrm{ad}(e_a), \ldots, \mathrm{ad}(e_f)$ on V_{21}^- are the negatives of the matrices representing the action on V_{10} and so the new basis for V_{21}^- is a signed permutation of the original basis. In fact, the sign attached to $e_{-\alpha}$ is $(-1)^{\deg(\alpha)}$, where in this case $\deg(\alpha) = \mathrm{ht}(\alpha) - 19$.

The obvious multiplication on V_{21} defined by $u \circ v = [\varphi(u)\,\psi(v)]$ is commutative and for $x \in \mathfrak{e}_6$ we have a "Principle of Local Duality":

$$x(u \circ v) = \pi(x)u \circ v + u \circ \pi(x)v.$$

But the algebra defined by this multiplication does not have an identity element, nor does it satisfy the Jordan identity. Therefore we modify our definition to the following:

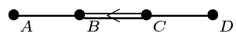
$$u \star v = [\varphi(u)\,\psi(v)] + \frac{1}{60}\kappa(u, \mu(v))\iota,$$

where κ is the Killing form for $\mathfrak{e}_8$ and $\iota = z_{13} - z_{14} + z_{15}$. (A direct calculation shows that $\kappa(e_\alpha, e_{-\alpha}) = 60$ for all $\alpha \in X_{21}$.)

From now on we use $\mathbb{J}$ to denote the algebra $(V_{21}, \star)$. If B_i is the subspace of $\mathbb{J}$ spanned by the elements of X_{21} of degree i, then $\mathbb{J} = \bigoplus_{j=-8}^{8} B_i$ exhibits $\mathbb{J}$ as a graded algebra.

The well-known $\mathfrak{e}_6$-invariant cubic form on V_{21}^π can be defined by

$$\tau(u, v, w) = \kappa([\varphi(u)\,\psi(v)], \mu(w)).$$

The Lie algebra $\mathfrak{f}_4$ may be regarded as the subalgebra of $\mathfrak{e}_6$ generated by $e_A = e_a + e_f$, $e_B = e_c + e_e$, $e_C = e_d$, $e_D = e_b$, $e_{-A} = e_{-a} + e_{-f}$, $e_{-B} = e_{-c} + e_{-e}$, $e_{-C} = e_{-d}$ and $e_{-D} = e_{-b}$. These elements are fixed by the graph automorphism π and their Dynkin diagram is .

As $\mathfrak{f}_4$-modules the spaces V_{10}, V_{11} and V_{21} are isomorphic but no longer irreducible.

The $\mathfrak{f}_4$-invariant bilinear form $(u, v) = \frac{1}{60}\kappa(u, \mu(v))$ on $\mathbb{J}$ is symmetric and takes only the values 0 and ± 1 on the basis $\{z_k\}$. On the other hand, $(\iota, \iota) = 3$ and so the splitting $\mathbb{J} = \mathbb{J}_0 \oplus \langle\, \iota \,\rangle$, where $\mathbb{J}_0$ is the orthogonal complement of ι, persists over all fields except those of characteristic 3. The elements of $\mathfrak{f}_4$ act as derivations of $\mathbb{J}$.

Theorem 4.1. *The algebra $\mathbb{J}$ is the exceptional Jordan algebra over $\mathbb{C}$; its identity element is ι.*

Proof. We shall identify $\mathbb{J}$ with the algebra $\mathfrak{H}(3, \mathbb{O})$ of 3×3 hermitian matrices over $\mathbb{O}$ described in Schafer [12, Chap. IV §2] except that we define the multiplication in $\mathfrak{H}(3, \mathbb{O})$ to be $x \cdot y = xy + yx$. This has the effect of making $\frac{1}{2}I$ the identity element (where I is the identity matrix) but it simplifies the description of the isomorphism.

The elements of $\mathfrak{H}(3, \mathbb{O})$ have the form

$$\Xi(\xi, \eta, \theta\,; a, b, c) = \begin{pmatrix} \xi & c & \overline{b} \\ \overline{c} & \eta & a \\ b & \overline{a} & \theta \end{pmatrix}$$

where ξ, η, $\theta \in \mathbb{C}$ and a, b, $c \in \mathbb{O}$.

If we put $e_1 = z_{13} - z_{14}$, $e_2 = z_{13} + z_{15}$ and $e_3 = -z_{14} + z_{15}$, then the elements $\frac{1}{2}e_i$ are orthogonal idempotents which sum to ι. The subspaces $A = e_2 \star (\mathbb{J} \star e_3)$, $B = e_3 \star (\mathbb{J} \star e_1)$ and $C = e_1 \star (\mathbb{J} \star e_2)$ are of dimension 8 and we shall identify their underlying vector spaces with $\mathbb{O}$. For example, we may take the basis z_1, z_7, z_9, z_{11}, z_{16}, z_{18}, z_{20}, z_{27} for A to correspond to the basis $w_1, w_2, \ldots, w_8$ for $\mathbb{O}$ obtained in Section 3. Similar identifications (with minor variations) apply to B and C. In fact it is a straightforward matter to use the multiplication tables given for the w_i and z_j to

$\star$	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}	z_{13}	z_{14}	z_{15}	z_{16}	z_{17}	z_{18}	z_{19}	z_{20}	z_{21}	z_{22}	z_{23}	z_{24}	z_{25}	z_{26}	z_{27}
z_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	z_1	0	z_2	0	z_3	0	z_4	z_5	z_6	z_8	z_{10}	z_{12}	$z_{15}+\iota$
z_2	0	0	0	0	0	0	0	0	0	0	0	$-z_1$	0	$-z_2$	0	$-z_3$	0	$-z_4$	0	$-z_5$	0	0	z_7	z_9	z_{11}	$z_{14}-\iota$	z_{17}
z_3	0	0	0	0	0	0	0	0	0	z_1	z_2	0	z_3	0	0	0	0	$-z_6$	0	$-z_8$	$-z_7$	$-z_9$	0	0	$z_{13}+\iota$	z_{16}	z_{19}
z_4	0	0	0	0	0	0	0	$-z_1$	$-z_2$	0	0	0	z_4	0	0	z_6	0	0	z_7	$-z_{10}$	0	$-z_{11}$	0	$-z_{13}-\iota$	0	z_{18}	z_{21}
z_5	0	0	0	0	0	z_1	z_2	0	0	0	0	0	z_5	0	0	z_8	0	z_{10}	z_9	0	z_{11}	0	$z_{13}+\iota$	0	0	z_{20}	z_{22}
z_6	0	0	0	0	z_1	0	0	0	$-z_3$	0	$-z_4$	0	0	$-z_6$	0	0	$-z_7$	0	0	$-z_{12}$	0	$-z_{14}+\iota$	0	$-z_{16}$	$-z_{18}$	0	z_{23}
z_7	0	0	0	0	z_2	0	0	z_3	0	z_4	0	z_6	0	0	z_7	0	0	0	0	$-z_{15}-\iota$	0	$-z_{17}$	0	$-z_{19}$	$-z_{21}$	$-z_{23}$	0
z_8	0	0	0	$-z_1$	0	0	z_3	0	0	0	$-z_5$	0	0	$-z_8$	0	0	$-z_9$	z_{12}	0	0	$z_{14}-\iota$	0	z_{16}	0	$-z_{20}$	0	z_{24}
z_9	0	0	0	$-z_2$	0	$-z_3$	0	0	0	z_5	0	z_8	0	0	z_9	0	0	$z_{15}+\iota$	0	0	z_{17}	0	z_{19}	0	$-z_{22}$	$-z_{24}$	0
z_{10}	0	0	z_1	0	0	0	z_4	0	z_5	0	0	0	0	$-z_{10}$	0	$-z_{12}$	$-z_{11}$	0	$-z_{14}+\iota$	0	0	0	z_{18}	z_{20}	0	0	z_{25}
z_{11}	0	0	z_2	0	0	$-z_4$	0	$-z_5$	0	0	0	z_{10}	0	0	z_{11}	$-z_{15}-\iota$	0	0	$-z_{17}$	0	0	0	z_{21}	z_{22}	0	$-z_{25}$	0
z_{12}	0	$-z_1$	0	0	0	0	z_6	0	z_8	0	z_{10}	0	z_{12}	0	0	0	$-z_{13}-\iota$	0	$-z_{16}$	0	$-z_{18}$	$-z_{20}$	0	0	0	0	z_{26}
z_{13}	0	0	z_3	z_4	z_5	0	0	0	0	0	0	z_{12}	ι	z_{15}	z_{14}	0	z_{17}	0	0	0	0	0	z_{23}	z_{24}	z_{25}	0	0
z_{14}	0	$-z_2$	0	0	0	$-z_6$	0	$-z_8$	0	$-z_{10}$	0	0	z_{15}	ι	z_{13}	0	0	0	$-z_{19}$	0	$-z_{21}$	$-z_{22}$	0	0	0	$-z_{26}$	0
z_{15}	z_1	0	0	0	0	0	z_7	0	z_9	0	z_{11}	0	z_{14}	z_{13}	ι	z_{16}	0	z_{18}	0	z_{20}	0	0	0	0	0	0	z_{27}
z_{16}	0	$-z_3$	0	z_6	z_8	0	0	0	0	$-z_{12}$	$-z_{15}-\iota$	0	0	0	z_{16}	0	z_{19}	0	0	0	$-z_{23}$	$-z_{24}$	0	0	z_{26}	0	0
z_{17}	z_2	0	0	0	0	$-z_7$	0	$-z_9$	0	$-z_{11}$	0	$-z_{13}-\iota$	z_{17}	0	0	z_{19}	0	z_{21}	0	z_{22}	0	0	0	0	0	$-z_{27}$	0
z_{18}	0	$-z_4$	$-z_6$	0	z_{10}	0	0	z_{12}	$z_{15}+\iota$	0	0	0	0	0	z_{18}	0	z_{21}	0	z_{23}	0	0	$-z_{25}$	0	$-z_{26}$	0	0	0
z_{19}	z_3	0	0	z_7	z_9	0	0	0	0	$-z_{14}+\iota$	$-z_{17}$	$-z_{16}$	0	$-z_{19}$	0	0	0	z_{23}	0	z_{24}	0	0	0	0	z_{27}	0	0
z_{20}	0	$-z_5$	$-z_8$	$-z_{10}$	0	$-z_{12}$	$-z_{15}-\iota$	0	0	0	0	0	0	0	z_{20}	0	z_{22}	0	z_{24}	0	z_{25}	0	z_{26}	0	0	0	0
z_{21}	z_4	0	$-z_7$	0	z_{11}	0	0	$z_{14}-\iota$	z_{17}	0	0	$-z_{18}$	0	$-z_{21}$	0	$-z_{23}$	0	0	0	z_{25}	0	0	0	$-z_{27}$	0	0	0
z_{22}	z_5	0	$-z_9$	$-z_{11}$	0	$-z_{14}+\iota$	$-z_{17}$	0	0	0	0	$-z_{20}$	0	$-z_{22}$	0	$-z_{24}$	0	$-z_{25}$	0	0	0	0	z_{27}	0	0	0	0
z_{23}	z_6	z_7	0	0	$z_{13}+\iota$	0	0	z_{16}	z_{19}	z_{18}	z_{21}	0	z_{23}	0	0	0	0	0	0	z_{26}	0	z_{27}	0	0	0	0	0
z_{24}	z_8	z_9	0	$-z_{13}-\iota$	0	$-z_{16}$	$-z_{19}$	0	0	z_{20}	z_{22}	0	z_{24}	0	0	0	0	$-z_{26}$	0	0	$-z_{27}$	0	0	0	0	0	0
z_{25}	z_{10}	z_{11}	$z_{13}+\iota$	0	0	$-z_{18}$	$-z_{21}$	$-z_{20}$	$-z_{22}$	0	0	0	z_{25}	0	0	z_{26}	0	0	z_{27}	0	0	0	0	0	0	0	0
z_{26}	z_{12}	$z_{14}-\iota$	z_{16}	z_{18}	z_{20}	0	$-z_{23}$	0	$-z_{24}$	0	$-z_{25}$	0	0	$-z_{26}$	0	0	$-z_{27}$	0	0	0	0	0	0	0	0	0	0
z_{27}	$z_{15}+\iota$	z_{17}	z_{19}	z_{21}	z_{22}	z_{23}	0	z_{24}	0	z_{25}	0	z_{26}	0	0	z_{27}	0	0	0	0	0	0	0	0	0	0	0	0

FIGURE 1. Multiplication table for a basis of the exceptional Jordan algebra

check that the following mapping defines an isomorphism $\mathbb{J} \to \mathfrak{H}(3,\mathbb{O})$.

$$
\begin{array}{lll}
z_1 \mapsto \Xi(0,0,0\,;w_1,0,0) & z_2 \mapsto \Xi(0,0,0\,;0,-w_1,0) & z_3 \mapsto \Xi(0,0,0\,;0,0,w_1) \\
z_4 \mapsto \Xi(0,0,0\,;0,0,w_2) & z_5 \mapsto \Xi(0,0,0\,;0,0,w_3) & z_6 \mapsto \Xi(0,0,0\,;0,w_2,0) \\
z_7 \mapsto \Xi(0,0,0\,;w_2,0,0) & z_8 \mapsto \Xi(0,0,0\,;0,w_3,0) & z_9 \mapsto \Xi(0,0,0\,;w_3,0,0) \\
z_{10} \mapsto \Xi(0,0,0\,;0,w_5,0) & z_{11} \mapsto \Xi(0,0,0\,;w_4,0,0) & z_{12} \mapsto \Xi(0,0,0\,;0,0,-w_4) \\
z_{13} \mapsto \Xi(\frac{1}{2},\frac{1}{2},-\frac{1}{2}\,;0,0,0) & z_{14} \mapsto \Xi(-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\,;0,0,0) & z_{15} \mapsto \Xi(-\frac{1}{2},\frac{1}{2},\frac{1}{2}\,;0,0,0) \\
z_{16} \mapsto \Xi(0,0,0\,;w_5,0,0) & z_{17} \mapsto \Xi(0,0,0\,;0,0,-w_5) & z_{18} \mapsto \Xi(0,0,0\,;w_6,0,0) \\
z_{19} \mapsto \Xi(0,0,0\,;0,-w_4,0) & z_{20} \mapsto \Xi(0,0,0\,;w_7,0,0) & z_{21} \mapsto \Xi(0,0,0\,;0,-w_6,0) \\
z_{22} \mapsto \Xi(0,0,0\,;0,-w_7,0) & z_{23} \mapsto \Xi(0,0,0\,;0,0,w_6) & z_{24} \mapsto \Xi(0,0,0\,;0,0,w_7) \\
z_{25} \mapsto \Xi(0,0,0\,;0,0,w_8) & z_{26} \mapsto \Xi(0,0,0\,;0,w_8,0) & z_{27} \mapsto \Xi(0,0,0\,;w_8,0,0)
\end{array}
$$

This completes the proof. □

As a result of this theorem we know that $\mathbb{J}$ satisfies the Jordan identity. On the other hand, the linearised form of this identity can be written

$$(x,y,w\star z)+(w,y,z\star x)+(z,y,x\star w)=0,$$

where $(x,y,z)=(x\star y)\star z - x\star(y\star z)$ is the associator (see [12, (4.3)]) and a computer algebra system such as Magma [1] can be used to check directly that this holds for the basis $\{z_k\}$.

We have seen that the $\mathfrak{f}_4$-module V_{21} has many multiplications preserved by $\mathfrak{f}_4$. In the next theorem we show that only one of these gives a Jordan algebra.

Theorem 4.2. *Up to a scalar multiple the multiplication defining $\mathbb{J}$ gives the unique Jordan algebra structure on V_{21}.*

Proof. Let μ_1, μ_2, μ_3 and μ_4 denote the fundamental weights for F_4. Then $\mu_1 = 2A+3B+2C+D$ and $\mathbb{J}_0 \simeq V(\mu_1)$. Using the formulae of Weyl and Freudenthal (Humphreys [8]) we find that the symmetric and exterior squares of $V(\mu_1)$ decompose as follows:

$$S^2V(\mu_1) = V(0)\oplus V(\mu_1)\oplus V(2\mu_1) \quad \text{and} \quad \Lambda^2 V(\mu_1) = V(\mu_2)\oplus V(\mu_4).$$

Thus (up to scalars) $\mathfrak{f}_4$ preserves a unique multiplication and a unique bilinear form on $\mathbb{J}_0$, both of which are symmetric. It follows that any $\mathfrak{f}_4$-invariant multiplication on $\mathbb{J}$ must have the form

$$u\circ v = a[\varphi(u)\,\psi(v)] + b\frac{1}{60}\kappa(u,\mu(v))\iota,$$

for some a and $b\in\mathbb{C}$. We may choose the scalars so that ι is the identity and then by checking the Jordan relation on a few basis elements we find that we must have $a=b=1$. □

Let $\mathbb{J}(\mathbb{Z})$ denote the $\mathbb{Z}$-form spanned by the basis z_1, z_2, … , z_{27}. The structure constants for this basis belong to $\mathbb{Z}$ and thus for any field F the space $\mathbb{J}(F)=\mathbb{J}(\mathbb{Z})\otimes_{\mathbb{Z}} F$ inherits an algebra structure from $\mathbb{J}$ on which $\mathfrak{f}_4(F)$ acts as derivations.

Theorem 4.3. *$\mathfrak{f}_4(F)$ is the algebra of derivations of $\mathbb{J}(F)$ for all fields F.*

Proof. We use the notation introduced in the proof of Theorem 4.1 but we carry out our calculations in F. In particular, let $A=e_2\star(\mathbb{J}\star e_3)$, $B=e_3\star(\mathbb{J}\star e_1)$ and $C=e_1\star(\mathbb{J}\star e_2)$.

Suppose at first that the characteristic of F is not 2 and consider the action of a derivation D on the elements $e_1=z_{13}-z_{14}$, $e_2=z_{13}+z_{15}$ and $e_3=-z_{14}+z_{15}$. We have $D(e_i)\star e_i = D(e_i)$ for all i. Furthermore, the elements $u\in\mathbb{J}(F)$ such that $u\star e_1 = u$ constitute the subspace $B\oplus C$ and for such elements u we have $u\star e_2\in C$. Similar observations apply to e_2 and e_3. It follows that the image of the map $\chi : D \mapsto D(e_1) \star e_2 + D(e_2) \star e_3 + D(e_3) \star e_1$ is contained in $A \oplus B \oplus C$, and thus its dimension is at most 24.

Suppose that $\chi(D) = 0$. Then $D(e_1) \star e_2 = D(e_1) \star e_3 = 0$ and $D(e_1) \star e_1 = D(e_1)$. Hence $2D(e_1) \star \iota = D(e_1)$ and so $D(e_1) = 0$. Similarly, $D(e_2) = D(e_3) = 0$. From its definition, it is clear that A is preserved by D. Moreover, for all $x, y \in A$ we can check from the multiplication table that $x \star y = (x, y)(e_2 + e_3)$. Thus $((Dx, y) + (x, Dy))(e_2 + e_3) = (Dx) \star y + x \star Dy = D(x \star y) = 0$. That is, D preserves the non-degenerate form (u, v) on A and hence the restriction of D to A belongs to $\mathfrak{d}_4(F)$. The identification of $\mathbb{J}$ with $\mathfrak{H}(3, \mathbb{O})$ used in Theorem 4.1 extends to any field of characteristic $\neq 2$ and identifies A, B and C with $\mathbb{O}(F)$. Furthermore, as shown in [3], the restrictions of D to A, B and C are related by triality. Thus if $D(a) = 0$ for all $a \in A$, then $D = 0$. Consequently the kernel of χ has dimension at most 28 and it follows from the previous paragraph that the space of derivations has dimension at most 52. On the other hand we already have a space of derivations $\mathfrak{f}_4(F)$ of dimension 52.

Now suppose that the characteristic of F is 2 and that D is a derivation. Then $z_1 \star z_{27} = z_{11} \star z_{16} = e_1$ and on applying D we see from the multiplication table that $D(e_1) \star e_2 \in C$. Similarly, $D(e_2) \star e_3 \in A$ and $D(e_3) \star e_1 \in B$. Defining χ as before we again deduce that the dimension of the image of χ is at most 24.

From now on assume that $\chi(D) = 0$. Then $D(e_1) \star e_2 = D(e_1) \star e_3 = 0$ and so $D(e_1) \in \langle z_{13}, z_{14}, z_{15} \rangle$. On applying D to the equality $e_1 \star z_2 = z_2$ we see that $D(z_2) \in B \oplus C$ and $D(e_1) \in \langle z_{13}, z_{15} \rangle$. Applying D to other equalities of the form $e_i \star z_j = z_j$ shows that D preserves A, B and C and that $D(e_1) \in \langle z_{15} \rangle$, $D(e_2) \in \langle z_{14} \rangle$ and $D(e_3) \in \langle z_{13} \rangle$. Further calculations of a similar nature show that $D(z_{13}) = D(z_{14}) = D(z_{15}) = 0$.

The bilinear form $(-, -)$ on $\mathbb{J}$ restricts to a non-degenerate symplectic form on A and, as shown above, $(Dx, y) + (x, Dy) = 0$ for all $x, y \in A$. That is, $D|_A$ belongs to the 36-dimensional symplectic Lie algebra $\mathfrak{sp}(8, F)$. Consideration of the action of D on equalities of the form $z_1 \star z_{10} = 0$ shows that $(Dx, x) = 0$ for all $x \in A$. This means that $D|_A \in \mathfrak{o}(8, F)$, the 28-dimensional orthogonal Lie algebra.

Consider a derivation T on $\mathbb{J}$ such that $\chi(T) = 0$, $T(z_1) = \alpha z_1$, $T(z_{27}) = \alpha z_{27}$ and $T(z_i) = 0$ for $i \in \{7, 9, 11, 16, 18, 20\}$. The action of T on equalities of the form $z_i \star z_j = z_k$ relating elements of A, B and C shows that $\alpha = 0$ and that T acts as a scalar on $B \oplus C$. There are two consequences of this calculation. Firstly, that not every element of $\mathfrak{o}(8, F)$ can be written in the form $D|_A$ where $\chi(D) = 0$ and secondly that the space of derivations D such that $\chi(D) = 0$ and which vanish on A has dimension at most 1.

This shows that the dimension of the space of derivations of $\mathbb{J}$ is at most $24 + 27 + 1 = 52$. As in the previous case, this proves that $\mathfrak{f}_4(F)$ is the space of derivations of $\mathbb{J}$. □

5. Implementation

Recent versions of Magma [1], GAP [13], Maple [14] and Lie [11] have the ability to construct root systems. Magma 2.5 has support for structure constants, making it an easy task to implement these algebras and to produce the tables used in this paper.

6. Acknowledgements

Some of the results in this paper were presented to the conference on Geometric and Algebraic Combinatorics, held in Oisterwijk, The Netherlands (August 1999), in honour of J. J. Seidel on the occasion of his 80th birthday.

References

1. W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system I: The user language*, J. Symbolic Comput. **24** (1997), 235–265.
2. R. W. Carter, *Simple groups of Lie type*, Pure and Applied Mathematics, vol. XXVIII, John Wiley & Sons, London New York Sydney Toronto, 1972.
3. C. Chevalley and R. D. Schafer, *The exceptional simple Lie algebras F_4 and E_6*, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 137–141.
4. H. S. M. Coxeter, *Integral Cayley numbers*, Duke Math. J. **13** (1946), 561–578.
5. J. R. Faulkner and J. C. Ferrar, *Exceptional Lie algebras and related algebraic and geometric structures*, Bull. London Math. Soc. **9** (1977), 1–35.
6. H. Freudenthal, *Oktaven, Ausnahmegruppen und Oktavengeometrie*, Geom. Dedicata **19** (1985), 1–63.
7. R. B. Howlett, L. J. Rylands, and D. E. Taylor, *Matrix generators for exceptional groups of Lie type*, submitted J. Symbolic Comput. **00** (2000), 000–000.
8. J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York Heidelberg Berlin, 1972.
9. N. Jacobson, *Cayley numbers and simple Lie algebras of type G_2*, Duke Math. J. **5** (1939), 775–783.
10. M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The Book of Involutions*, Colloquium Publications, vol. 44, Amer. Math. Soc., Providence, RI, 1998.
11. M. A. A. van Leeuwen, A. M. Cohen, and B. Lisser, *Lie Manual, version 2.1*, Computer Algebra Group, CWI, Amsterdam, 1993.
12. R. D. Schafer, *An Introduction to Nonassociative Algebras*, Pure and Applied Mathematics, vol. 22, Academic Press, New York London, 1966.
13. M. Schönert et al., *GAP — Groups, Algorithms and Programming*, Lehrstuhl D für Mathematik, RWTH Aachen, 1994.
14. J. R. Stembridge, *A Maple package for root systems and finite Coxeter groups (Version 2.1)*, Maple Share Library, 1998.

L. J. Rylands, School of Science, University of Western Sydney, Nepean, Australia
E-mail address: `L.Rylands@nepean.uws.edu.au`

D. E. Taylor, School of Mathematics and Statistics, University of Sydney, NSW 2006 Australia
E-mail address: `D.Taylor@maths.usyd.edu.au`
URL: `http://www.maths.usyd.edu.au:8000/u/don/`