Dirac Observables and the Phase Space of General Relativity

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Abstract

In the canonical approach to general relativity it is customary to parametrize the phase space by initial data on spacelike hypersurfaces. However, if one seeks a theory dealing with observations that can be made by a single localized observer, it is natural to use a different description of the phase space. This results in a different set of Dirac observables from that appearing in the conventional formulation. It also suggests a possible solution to the problem of time, which has been one of the obstacles to the development of a satisfactory quantum theory of gravity.

1 Introduction

The development of a satisfactory quantum theory of gravity has been hindered by both technical and conceptual difficulties. The latter include the problem of time and the closely related problem of identifying suitable observables in general relativity, both of of which arise from the general covariance of the classical theory and reflect the inherent ambiguity in the identification of points in space-time [1].

In a relativistic theory, an observer can only measure quantities inside his causal past. Unless the observer happens to live in a deterministic space-time [2], such measurements can never provide him with sufficient data to predict the future evolution of any system unless he assumes the fields are constrained by a suitable set of boundary conditions on some surface outside his causal past.

In the case of electromagnetism, such assumptions may be justifiable. Laboratory experiments can easily be shielded from incoming electromagnetic radiation, which effectively imposes a boundary condition on a surface outside the past light cone. While astrophysical and cosmological observations cannot be shielded in quite the same way, in fact a natural shield already exists; incoming electromagnetic radiation can safely be assumed to have the familiar thermal spectrum of the cosmic microwave background, since any radiation produced with the origin of the Universe would have subsequently been absorbed by the opaque plasma of protons and electrons that filled the Universe for about 10^5 years.

The situation is quite different for gravity, for the simple reason that it is a universally attractive force that cannot be shielded. Consequently, no laboratory experiment can be protected from the effects of incoming gravitational radiation. Similarly, in the context of astrophysical and cosmological observations, it is hard to justify any assumptions about the properties of gravitational radiation from the nascent Universe. In particular, there is little reason to suppose that such radiation will be confined to a narrow range of frequencies with a well-understood spectrum like electromagnetic radiation.

Our inability to justify assumptions about incoming radiation suggests that, for gravity in particular, we should develop a description of the classical that is not reliant on assumed boundary conditions, and which deals simply with observations that are accessible to a localized observer. Of course, there is a price to pay; when we abandon our assumptions about what goes on outside our causal past, we also surrender any possibility of solving the classical equations of motion to predict the future. It is tempting to conclude that determinism is lost, but in fact this is only half true; a local observer can reconstruct a unique solution to the classical equations of motion in his causal past using a subset of the observational data accessible to him. Thus, the abandonment of assumptions about what we cannot see prevents us from predicting the future, but still permits deductions about the past.

In this paper we propose an approach to canonical general relativity which explicitly incorporates the observer into the theory, and in which only those quantities that can be measured by this observer are regarded as physical. The remaining degrees of freedom, which are unphysical insofar as their values cannot be determined by any experiment, are treated in much the same way as unphysical gauge degrees of freedom.

While the analysis presented here is somewhat formal, it suggests a new definition of the phase space of general relativity and a novel approach to the development of a quantum theory of gravity. One of the main advantages of this approach is that (even in the case of pure gravity) it leads to a large family of Dirac observables, including one which may be interpreted as a time parameter and others associated with the results of localized measurements. In the traditional approach to canonical quantum gravity, some of the most profound conceptual problems are a direct consequence of the absence of observables [1].

2 Covariant Phase Space

Phase space is a fundamental concept in the canonical formulation of classical mechanics [3]. We begin with a summary of its geometric properties, and then discuss how the phase space is actually defined.

For a theory with no gauge symmetries, the phase space is generally a manifold $\mathcal{P}$ equipped with a symplectic form; i.e. a 2-form ω that is closed ($d\omega = 0$) and nondegenerate. The symplectic form is useful because it establishes a bijective linear correspondence between vector fields and 1-forms on $\mathcal{P}$; given any vector field X on $\mathcal{P}$, the associated 1-form is given by $i_X\omega$ (the contraction of ω with X).

The symplectic form ω also associates with each C^1 function $f : \mathcal{P} \mapsto \mathbf{R}$ a unique vector field X_f such that

$$i_{X_f}\omega = -df. \tag{1}$$

In particular, if $h : \mathcal{P} \to \mathbf{R}$ is the Hamiltonian function, then X_h generates the flow associated with time translation. The symplectic form is invariant under the action of this flow, since its Lie derivative with respect to X_h vanishes:

$$\mathcal{L}_{X_h}\omega \equiv i_{X_h}\, d\omega + d\, i_{X_h}\omega = 0 - ddh = 0. \tag{2}$$

(Here we used the closure of ω, as well as the standard identities $dd = 0$ and $\mathcal{L}_X = i_X d + d i_X$ [4].) Making further use of the symplectic form ω, we also define the Poisson bracket of two smooth functions f, g on $\mathcal{P}$ as

$$\{f, g\} \equiv -i_{X_f} dg = i_{X_f} i_{X_g}\omega = -i_{X_g} i_{X_f}\omega = i_{X_g} df. \tag{3}$$

In a gauge theory the phase space $\mathcal{P}$ is *not* equipped with a symplectic form, and hence there is no Poisson bracket on $\mathcal{P}$. Instead, $\mathcal{P}$ is equipped with a *presymplectic* form; i.e. a 2-form π that is closed but degenerate, in the sense that $\mathcal{P}$ admits a non-zero vector field V with $i_V\pi = 0$. In general, π can be regarded as the restriction to $\mathcal{P}$ of a symplectic 2-form ω defined on some *extended* phase space $\mathcal{E}$ in which $\mathcal{P}$ is embedded.

The phase space $\mathcal{P}$ may be described as the surface in $\mathcal{E}$ on which a set of constraint functions $\Phi_a : \mathcal{E} \mapsto \mathbf{R}$ vanish. For the sake of brevity, we will assume here that all these constraints are independent and first-class, which means that they form a closed algebra under the action of the Poisson brackets defined on $\mathcal{E}$ by ω; i.e.

$$\{\Phi_a, \Phi_b\} = f_{ab}{}^c \Phi_c \tag{4}$$

for a suitable choice of functions $f_{ab}{}^c = -f_{ba}{}^c$ on $\mathcal{E}$. (In general there may also be second-class constraints that cannot be included in such an algebra. These indicate the presence of redundant degrees of freedom, which - at least in principle - can be eliminated from the extended phase space $\mathcal{E}$. Here we will assume this has been done.)

Associated with each first-class constraint function Φ_a is a vector field V_a for which $i_{V_a}\omega = d\Phi_a$ on $\mathcal{E}$. These vector fields are tangential to the constraint surface $\mathcal{P}$ since

$$i_{V_a} d\Phi_b = -i_{V_a} i_{V_b}\omega = -\{\Phi_a, \Phi_b\} = -f_{ab}{}^c\Phi_c \quad = 0 \text{ on } \mathcal{P}. \tag{5}$$

and correspond to the null directions of the degenerate form π on $\mathcal{P}$;

$$i_{V_a}\pi|_{\mathcal{P}} = i_{V_a}\omega|_{\mathcal{P}} = -\, d\Phi_a|_{\mathcal{P}} = 0 \;\text{ since } \Phi_a = 0 \text{ on } \mathcal{P}. \tag{6}$$

The vector fields V_a can easily be shown to form a closed algebra

$$[V_a, V_b] = -f_{ab}{}^c V_c \tag{7}$$

and are in fact the generators of the gauge group $\mathcal{G}$. The corresponding directions in $\mathcal{P}$ are thus associated with purely gauge degrees of freedom.

These unphysical degrees of freedom can be eliminated by identifying points in $\mathcal{P}$ that can be mapped into each other by gauge transformations. One is left with the quotient space $\mathcal{R} = \mathcal{P}/\mathcal{G}$, whose elements are the orbits in $\mathcal{P}$ of the gauge group $\mathcal{G}$. This space inherits a non-degenerate symplectic form from $\mathcal{P}$, and is sometimes referred to as the *reduced* phase space of the theory [3]. It may be thought of as the phase space of just the physical degrees of freedom.

The above discussion provides a useful geometric insight into the canonical theory, but it leaves an important question unanswered: given the dynamical laws governing a physical system, how does one determine what the phase space is? While the answer is often straightforward, this is not always the case and so it is useful to give a definition that can be applied in any situation.

In fact the phase space $\mathcal{P}$ for a classical system can be defined in a fully covariant manner as the space of solutions to the equations of motion [5, 6, 7]. A single point in phase space is thus identified with an entire solution of the equations of motion, rather than with the state of the system at a particular instant[1]. The evolution of the system is therefore not represented by a Hamiltonian flow in phase space, as it is in the standard approach. Instead, the Hamiltonian generates transformations which map a given solution to a *distinct* solution, related to the first by a time translation.

As with any manifold, there are many possible coordinate systems that can be used to label points in $\mathcal{P}$; i.e. to identify particular solutions of the equations of motion. One natural approach is to label a particular solution by a set of initial data from which it can be uniquely determined. For example, in the familiar case of a particle moving in the configuration space $\mathbf{R}^N$, each point in phase space is represented by a classical trajectory; i.e. a C^1 mapping $\gamma : \mathbf{R} \to \mathbf{R}^N$ satisfying the Euler-Lagrange equations. (If symmetries are present there will be a family of physically indistinguishable trajectories representing the same point in phase space.)

[1] The perspective presented here differs from the conventional one in much the same way that the Heisenberg picture of quantum mechanics differs from the Schrödinger picture.

A natural way to identify the trajectory γ is by specifying an appropriate set of initial data. For example we could identify γ by specifying the instantaneous values of the N coordinates and their derivatives at the time $t = 0$;

$$(q^1, \ldots q^N) \equiv \gamma(0), \qquad (\dot{q}^1, \ldots, \dot{q}^N) \equiv \left.\frac{d\gamma}{dt}\right|_{t=0}. \tag{8}$$

In a typical theory, these initial data would be sufficient to uniquely identify the entire trajectory γ. Another approach is to eliminate the velocities $\dot{q}^i$ in favour of an equal number of functions $p_i(q, \dot{q})$ called momenta, defined so that

$$dp_i \wedge dq^i = \omega. \tag{9}$$

The resulting description appears completely standard, except that the 2N-tuple $(q^1, \ldots q^N, p_1, \ldots p_N)$ labels the entire classical trajectory γ rather than just a point on the trajectory. As a consequence, the evolution of the system is not described by motion in phase space.

When the theory admits gauge symmetries, the equations of motion will admit a number of distinct solutions that are physically indistinguishable. Each point in the reduced phase space $\mathcal{R}$ is then taken to represent a single equivalence class of such solutions. For example, suppose that two classical trajectories γ and $\bar{\gamma}$ can be mapped into each other by a gauge transformation, and hence are physically indistinguishable. Both γ and $\bar{\gamma}$ are represented by the same point in phase space, which is defined as the equivalence class $[\gamma]$ of all trajectories that are physically indistinguishable from γ.

A special kind of symmetry is that associated with reparametrization. In a theory of the kind described above, a trajectory $\bar{\gamma} : \mathbf{R} \to \mathbf{R}^N$ is said to be related to γ by a reparametrization if there exists an increasing C^∞ bijection $\tau : \mathbf{R} \mapsto \mathbf{R}$ such that

$$\bar{\gamma}(t) = \gamma(\tau(t)) \quad \forall t \in \mathbf{R}. \tag{10}$$

If such trajectories are physically indistinguishable, the theory is said to possess reparametrization symmetry. In such a theory, a point in phase space is an equivalence class $[\gamma]$ of classical trajectories related to each other by reparametrization, and is represented by a directed curve in $\mathbf{R}^N$ without any preferred parametrization.

The absence of a natural parametrization introduces additional ambiguity into any attempts to label this curve using initial data. In order to determine $(q^1 \ldots q^N, \dot{q}^1 \ldots \dot{q}^N)$, one must first choose a particular parametrization; i.e. a particular trajectory from the equivalence class $[\gamma]$. There are many ways to do this, and hence many different sets of initial data $(q^1 \dots q^N, \dot{q}^1 \dots \dot{q}^N)$ all denoting the same point $[\gamma]$ in phase space.

Having many different labels for the same point in phase space may be somewhat confusing. Given two sets of initial data, it may be difficult to determine whether they represent the same point in phase space or different points[2]. This problem also arises in general relativity, where it is difficult to tell if two different sets of initial data on a hypersurface Σ will generate space-times with the same 4-geometry.

The confusion described above arises when one uses initial data to parametrize phase space. It clears when one recalls that these initial data are merely labels, and focuses instead on the geometric picture of the phase space discussed above.

This approach is particularly helpful in the case of general relativity. Here $\mathcal{P}$ is the space of solutions to Einstein's field equations, and $\mathcal{R}$ is obtained by factoring out the gauge group; i.e. by identifying solutions that can be mapped into each other by space-time diffeomorphisms.

Each point in $\mathcal{R}$ therefore represents a space-time with a Lorentz metric satisfying Einstein's equations; for simplicity, we assume here that there are no matter fields, although these could easily be incorporated. (By "space-time", we mean here an equivalence class of isometric inextendible connected Hausdorff C^∞ 4-manifolds with C^2 Lorentz metrics [8]. Each space-time thus represents an orbit of the group of diffeomorphisms.) One could refine this definition by imposing additional conditions on the class of admissible space-times, such as strong causality or asymptotic flatness at spatial infinity, but such possibilities will not be considered here.

This definition of the phase space of general relativity is essentially that proposed by Witten and Crnković [5, 6]. Its primary advantage is that it is fully space-time covariant, and is formulated without reference to any particular coordinate system or preferred time coordinate.

Of course, in order to describe the physical properties of a given space-time one requires a system for describing a particular solution by a set of numerical labels. As remarked earlier, a convenient and popular way to do this is by specifying initial data on some spacelike initial hypersurface Σ. For example one might specify the induced 3-metric and the second fundamental form on Σ, as in the ADM formulation of canonical general relativity [9]. Alternatively, one might follow the approach of Ashtekar and specify the

[2]One would actually have to solve the equations of motion to decide.

self-dual part of the $SO(3)$ connection, along with the spatial triad density (with weight one) [10]. Either choice of initial data is generally sufficient to single out a particular solution of Einstein's equations (up to possible isometries).

However, one is not obliged to use initial data to identify points in the phase space of general relativity; indeed, solutions of Einstein's equations can be described in entirely different ways. For example, Landi and Rovelli propose using the eigenvalues of the Dirac operator to identify particular solutions of Einstein's equations, and thus to label points in the phase space of [7]. Undoubtedly, many other approaches also exist.

3 Observables in General Relativity

The central task in canonical quantization is to find operator representations for observables. Before attempting to quantize a theory, it is therefore important to know which quantities qualify as observables. An appropriate definition was given by Dirac [11], and is presented below in the terminology of the previous section.

According to Dirac's definition, a function $F : \mathcal{E} \mapsto \mathbf{R}$ on the extended phase space is an observable if its Poisson bracket with each first-class constraint Φ_a vanishes on the constraint surface $\mathcal{P} \subset \mathcal{E}$:

$$\{\Phi_a, F\}|_{\mathcal{P}} = 0. \tag{11}$$

This requirement ensures that F is gauge-invariant, since these constraints are the generators of the gauge transformations. Indeed we recall that $\{\Phi_a, F\} \equiv -i_{V_a} dF$ where V_a is the vector field associated with Φ_a, so if (11) holds then

$$i_{V_a} dF|_{\mathcal{P}} = 0 \tag{12}$$

and hence F is unaffected by gauge transformations.

An observable F therefore takes the same value at all points in a given orbit of the gauge group $\mathcal{G}$, and may be regarded as a function on the quotient space $\mathcal{P}/\mathcal{G}$; that is, on the reduced phase space $\mathcal{R}$. The converse is also true; any well-defined function on $\mathcal{R}$ can be viewed as a gauge-invariant function on $\mathcal{P}$ and hence as a Dirac observable. To put it another way, Dirac's criterion is trivially satisfied by any function $F : \mathcal{R} \mapsto \mathbf{R}$ as there are no constraints on $\mathcal{R}$.

If we adopt the definitions of $\mathcal{P}$ and $\mathcal{R}$ proposed in the previous section, then an observable F will assign a single real value to an entire solution γ of the equations of motion (and the same value to all other solutions obtained from it by gauge transformations). This value does not evolve in time, as the state of the system is always represented by the same point $[\gamma]$ in the phase space.

However, another kind of time evolution can be observed if the observable F is replaced by a family of observables $\{F_t | t \in \mathbf{R}\}$, each of which assigns to the entire classical solution γ some gauge-independent quantity associated with the instantaneous properties of this solution at time t. For example, if $\gamma : \mathbf{R} \mapsto \mathbf{R}^N$ represents some classical particle trajectory, then F_t might assign to this entire trajectory (and to gauge equivalent trajectories) a single value determined by the instantaneous position and velocity of the particle at time t;

$$F_t([\gamma]) = f(\gamma(t), \dot{\gamma}(t)). \tag{13}$$

Thus, for a fixed choice of t, $F_t([\gamma])$ is a single number associated with the entire trajectory γ. On the other hand, if the value of t is changed then F_t denotes a different observable and produces a different value when acting on γ. Thus it is the observable itself that changes with time, not merely its value.

This is precisely analogous to the Heisenberg picture of quantum mechanics, in which operators evolve rather than state vectors. On the other hand, the conventional approach corresponds to the Schrödinger picture, in which it is the states rather than the operators which evolve.

This approach is useful when considering general relativity, for which the gauge transformations are diffeomorphisms. In this case, an observable is a function on $\mathcal{R}$ that assigns to each space-time a value that is unaffected by diffeomorphisms; in other words, a geometric invariant of the space-time manifold.

In pure general relativity, in the absence of any preferred coordinate system, it is difficult to identify a particular space-time point at which local data is to collected. For this reason, geometric invariants of space-time must generally be defined in a global manner without reference to any special space-time points. These globally defined quantities may be expressible as integrals of local invariants such as R, R^2, $R^{\mu\nu}R_{\mu\nu}$ or $C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}$ (where $R, R_{\mu\nu}$ and $C_{\mu\nu\rho\sigma}$ denote the Ricci scalar and the Ricci and Weyl tensors). There are also a variety of globally defined geometric invariants (such as the eigenvalues of the Dirac or Klein-Gordon operators [7]) that cannot be expressed simply as integrals of local quantities.

While all of these are genuine Dirac observables, none can be evaluated without full knowledge of the future and past of the Universe. In particular, their values cannot be ascertained by a localized observer who only has access to data from his causal past (unless the observer inhabits a deterministic space-time [2]).

Real observations are made locally, and the things we can measure are *not* globally defined geometric invariants of the space-time manifold as described above. Observables of this type are therefore inappropriate quantities to consider in a theory which purports to relate to physical observation. What is clearly needed is a set of observables whose values are determined by local properties associated with particular points or regions in space-time. The difficulty is that any such local observables must be unaffected by arbitrary diffeomorphisms that map points to different points.

The only way around this problem would be to evaluate local invariants at space-time points that are identified in a diffeomorphism-invariant manner; for example, as the points at which certain local invariants take specified values. However, this approach does not appear to work in the case of pure gravity, since at each point there are only 4 algebraically independent local invariants (all obtained from the Weyl tensor $C_{\mu\nu\rho\sigma}$ [12]). Hence, even if a point could be identified as the unique location where these 4 quantities took specified values, there would be no remaining independent local invariants to measure there.

If matter is present then there are more possibilities. For example, Rovelli proposes labelling space-time points with reference to a "material reference system" provided by a space-filling cloud of particles carrying clocks [13]. He shows how this permits the definition of observables that are local but also invariant under space-time diffeomorphisms. However, he emphasizes that the material nature of the reference particles is a vital ingredient; not only will their trajectories in space-time be determined by appropriate equations of motion, but they will also have a non-vanishing effect on the geometry of space-time to which they must be coupled via Einstein's equations.

While this approach provides a large set of local Dirac observables for general relativity coupled to matter, it has some shortcomings. In the first place, it requires the existence of a collection of particles[3] with clocks and

[3]These particles must have non-zero rest mass, as the trajectories of massless particles

consequently sheds no light on the problem of identifying observables in a theory without matter.

More importantly, observation is ultimately a local process and data obtained from the various particles does not constitute an observation until it has been collected by a single localized observer. Attention must therefore be focussed on this observer and the set of observables to which he has access; data from the region outside his causal past cannot form part of any observation.

4 Space-Times with Localized Observers

In practice, all physical theories must deal with observations that can be collected by a single localized observer[4]. Indeed, the scientific method requires that a theory should be subjected to experimental tests that culminate in the collection and interpretation of data by a scientist or a localized team of scientists.

Although Rovelli's approach allows us to define a variety of local observables, many of these will be inaccessible to a localized observer. Moreover, such an observer will be unable to determine the value of any globally defined observables. This argument suggests that the observer should be incorporated into the definition of an observable. We therefore start with some definitions.

According to Hawking and Ellis [8], a space-time can be represented by a C^2-inextendible pair $(\mathcal{M}, \mathbf{g})$ where $\mathcal{M}$ is a connected 4-dimensional Hausdorff C^∞ manifold and $\mathbf{g}$ is a C^2 Lorentz metric on $\mathcal{M}$. We now modify this definition by requiring each space-time to contain a privileged point corresponding to the observer's "here and now"; a space-time will then be represented by a triple $(\mathcal{M}, \mathcal{O}, \mathbf{g})$ where $\mathcal{O} \in \mathcal{M}$ represents this special point. The same space-time can also be represented by any other triple related to $(\mathcal{M}, \mathcal{O}, \mathbf{g})$ by an isometry, but it is generally convenient to work with a single representative triple.

For the sake of clarity, we restrict the following analysis to pure gravity. However, it is a simple matter to accommodate matter fields if desired.

cannot be parametrized by proper time.

[4]This point is of critical importance in quantum mechanics, where the local nature of observation must be recognized if one is to avoid the causal paradoxes associated with the instantaneous collapse of the wave function.

For any $(\mathcal{M}, \mathcal{O}, \mathbf{g})$ the *observer's causal past* is defined as the set $\mathcal{J}^-(\mathcal{O}, \mathcal{M})$ of all points in $\mathcal{M}$ that can be connected to $\mathcal{O}$ by future-directed non-spacelike curves in $\mathcal{M}$. From the perspective of a localized observer performing an experiment that culminates at $\mathcal{O}$, the set $\mathcal{J}^-(\mathcal{O}, \mathcal{M})$ represents the visible part of space-time. This motivates us to focus on developing a classical theory that deals exclusively with the parts of space-time that are visible from $\mathcal{O}$ – without necessarily regarding them as part of some larger hypothetical[5] space-time.

Two space-times represented by triples $(\mathcal{M}, \mathcal{O}, \mathbf{g})$ and $(\mathcal{M}', \mathcal{O}', \mathbf{g}')$ will be said to be *indistinguishable* if the interior of $\mathcal{J}^-(\mathcal{O}, \mathcal{M})$ is isometric to the interior of $\mathcal{J}^-(\mathcal{O}', \mathcal{M}')$. This is clearly an equivalence relation, which we denote $\sim$. An equivalence class of indistinguishable space-times will be referred to as a *visible space-time*, and characterizes the geometry of the observer's causal past; it is what remains of a space-time when we discard what cannot be observed. If this equivalence class includes at least one space-time on which the metric satisfies Einstein's equations, then it will be referred to as a *classical* visible space-time.

In case the above definition seems rather formal, note that a visible space-time can also be represented by a pair $(\mathcal{J}, \mathbf{g})$ where

- $\mathcal{J}$ is a connected four-dimensional manifold-with-boundary, and is smooth everywhere except at a singular point $\mathcal{O} \in \mathcal{J}$;
- $\mathcal{O}$ has a neighbourhood $\mathcal{N} \subset \mathcal{J}$ on which is defined a homeomorphism $\phi : \mathcal{N} \to \{(x^1, x^2, x^3, x^4) \in \mathbf{R}^4 \,:\, (x^1)^2 + (x^2)^2 + (x^3)^2 \leq (x^4)^2,\ x^4 \leq 0\}$ which is smooth on $\mathcal{N} \backslash \mathcal{O}$ and also regarded as smooth[6] at $\mathcal{O}$, with $\phi(\mathcal{O}) = (0, 0, 0, 0)$;
- $\mathbf{g}$ is a C^2 Lorentz metric on $\mathcal{J}$;
- every point in $\mathcal{J} \backslash \{\mathcal{O}\}$ can be connected to $\mathcal{O}$ by a non-spacelike curve in $\mathcal{J}$;
- $(\mathcal{J}, \mathbf{g})$ is C^2-inextendible, in the sense that it is not isometric to a proper subset of another pair $(\mathcal{J}', \mathbf{g}')$ with the above properties.

[5] The work "hypothetical" is appropriate because nothing can be known about the region outside the observer's causal past.

[6] The homeomorphism ϕ defines a local coordinate system on $\mathcal{N}$, and extends the differential structure on $\mathcal{J} \backslash \{\mathcal{O}\}$ to cover the point $\mathcal{O}$. Thus, a function $f : \mathcal{N} \to \mathbf{R}$ is said to be C^k at $\mathcal{O}$ if $f \circ \phi^{-1}$ is C^k at $\phi(\mathcal{O}) = (0, 0, 0, 0)$.

Note that the causal past of any interior point of space-time has precisely these properties. If the metric $\mathbf{g}$ satisfies Einstein's equations everywhere on $\mathcal{J}$, the visible space-time is classical.

In section 2 we showed how the reduced phase space $\mathcal{R}$ is obtained by discarding the unphysical gauge degrees of freedom from a larger phase space. In particular, for general relativity we saw that elements of $\mathcal{R}$ are space-times with metrics satisfying Einstein's equations. However these space-times still contain unphysical degrees of freedom; those which the observer cannot measure because they are associated with points outside his causal past. To remove these unphysical degrees of freedom, we therefore define the reduced phase space $\bar{\mathcal{R}}$ of our observer-based theory as the quotient space $\bar{\mathcal{R}} = \mathcal{R}/\sim$ obtained from $\mathcal{R}$ by identifying indistinguishable space-times. This quotient space inherits a symplectic structure from $\mathcal{R}$. (The inherited symplectic form will be non-degenerate because there is no gauge group acting on $\bar{\mathcal{R}}$.) The elements of $\bar{\mathcal{R}}$ are then the classical visible space-times.

Unlike $\mathcal{R}$, the new reduced phase space $\bar{\mathcal{R}}$ is partially ordered; given two elements of $\bar{\mathcal{R}}$ represented by triples $(\mathcal{M}, \mathcal{O}, \mathbf{g})$ and $(\mathcal{M}', \mathcal{O}', \mathbf{g}')$ respectively, we say that the first *contains* the second if the interior of $\mathcal{J}^-(\mathcal{O}', \mathcal{M}')$ is isometric to an open subset of $\mathcal{J}^-(\mathcal{O}, \mathcal{M})$. It is easy to verify that the relation of containment is reflexive, antisymmetric and transitive, and therefore a partial ordering.

The physical meaning of this ordering relation is straightforward; if one visible space-time contains another, it means that the causal past of the first observer can be regarded as containing the causal past of the first, and so the first observer may be regarded as being in the casual future of the second. It follows that any future-directed causal curve in space-time correpsonds to a totally ordered subset of the reduced phase space $\bar{\mathcal{R}}$; conversely, any totally ordered subset of $\bar{\mathcal{R}}$ corresponds to a set of points along a future-directed causal curve.

We now investigate what observables the theory admits. These are defined as functions on the reduced phase space $\bar{\mathcal{R}}$. Thus, an observable is a rule assigning a real quantity to each classical visible space-time; i.e. a geometric invariant of the observer's causal past.

One such observable is the 4-volume of the observer's causal past (if this happens to be finite). In fact, the value of this observable is strictly increasing along the observer's world-line, and is therefore naturally regarded as a time parameter. Its status as a bona fide observable suggests that it may be used in constructing a time-dependent version of the quantum theory. (Note that this quantity cannot be defined in the conventional formulation of general relativity, in which there is no privileged point representing the observer's here and now.)

It also proves quite easy to find local observables in this theory, as one can readily locate space-time points with reference to the observer's here-and-now $\mathcal{O}$. For example, by constructing a Riemann normal coordinate system about $\mathcal{O}$, it is possible to attach a label to every point in a neighbourhood of $\mathcal{O}$ with a mimimum of ambiguity; the only arbitrariness in this procedure is that associated with the $SO(1,3)$ freedom available in choosing the directions of the coordinate axes. One can then take as observables the values of any local invariants at specified points in this neighbourhood.

The conclusion of this analysis is that any measurements of local invariants collected from points nearby the observer in his causal past qualify as observables. This coincides very closely with what most physicists mean by observables.

5 Boundary Data and Quantization

In order to put this theory into a more conventional form, we consider what boundary data must be specified in order to identify particular elements of the reduced phase space $\bar{\mathcal{R}}$; i.e. a classical visible spacetime.

A classical visible space-time corresponds to a possible geometry of the observer's causal past $\mathcal{J}^-(\mathcal{O})$. In fact the causal structure of general relativity ensures that this geometry can be reconstructed using Einstein's equations from final data on the observer's past light cone C^-, defined here as the null surface generated by past-directed null geodesics through $\mathcal{O}$. (We are only considering pure general relativity, and therefore don't need to worry about the possible formation of caustics.) Indeed, Dautcourt has shown that this can be achieved with knowledge of just two real function of the metric on C^- [14]. Suppose the metric is written in the form

$$ds^2 = m^2\, du^2 + 2h\, du\, dv\, + 2k_A w^A du\, + g_{AB} dw^A dw^B \qquad (14)$$

where u and v are null coordinates, and w^A $(A = 2,3)$ are constant along the null generators of C^-, and $m^2 = k_A k_B g^{AB}$. Then it is sufficient to specify two of the three independent components of g_{AB} everywhere on C^-; for example, one might specify the conformal part of g_{AB} (i.e. g_{AB} up to a conformal factor).

This also happens to be just enough data to specify the intrinsic geometry of C^-. A metric on a 3-surface generally has 6 independent components at each point; however, C^- is null so the metric is degenerate and only has 5. Of these, 3 can be removed by an appropriate choice of coordinates, leaving just 2 degrees of freedom at each point in C^- corresponding, for example, to the components of $(\det[g_{AB}])^{-1/2} g_{AB}$.

It therefore appears that the entire 4-geometry of the observer's causal past can be reconstructed from a knowledge of the intrinsic geometry of his past light cone. It is therefore natural to use this boundary data to label elements of the reduced phase space $\bar{\mathcal{R}}$.

It should also be possible to represent the symplectic form on phase space as an antisymmetric bilinear function

$$\omega(\tilde{\delta} g_{AB}, \tilde{\delta} g'_{CD}) = \int_{C^-} \tilde{\delta} g_{AB} \omega^{ABCD} \tilde{\delta} g'_{CD} \tag{15}$$

where the 3-form $\omega^{ABCD} = -\omega^{CDAB}$ scales like $(\det[g_{AB}])^{-1}$ under conformal transformation of g_{AB}, and

$$\tilde{\delta} g_{AB} \equiv \delta g_{AB} - \frac{1}{2} g_{AB} g^{CD} \delta g_{CD}. \tag{16}$$

In principle it should be possible to derive an expression for ω^{ABCD} using the results of [6], but this will not be attempted here.

We conclude with a brief discussion of the quantum theory. Prior to quantization, one must identify a set of generalized coordinates q^α and momenta p_α such that the symplectic form can be written as

$$\omega = dp_\alpha \wedge dq^\alpha. \tag{17}$$

The generalized coordinates q^α will contain just half the degrees of freedom needed to parametrize the reduced phase space, and thus represent a single degree of freedom of g_{AB} at each point in C^-. The momenta p_α contain the remaining degree of freedom at each point in C^-.

As there are no first-class constraints in the reduced phase space, the wave function may be taken as an arbitrary complex function $\Psi(q)$ of the generalized coordinates q^α, with the generalized momenta p_α represented as differential operators. In order to select a particular wave-function, it would be necessary to augment the canonical theory with an appropriate set of boundary conditions arising from non-dynamical considerations.

As remarked in the previous section, one of the observables is strictly increasing along any classical world line (i.e. any totally ordered sequence in the reduced phase space) and is therefore naturally regarded as a time parameter. In the quantum theory, this observable will be represented by a Hermitian operator on the space of wave functions. The eigenstates of this operator will represent quantum states in which the observed Universe has a definite age, and so projecting the wave function into these eigenstates will result in a time-dependent version of the quantum theory.

6 Summary and Discussion

By explicitly incorporating a localized observer into the canonical analsysis of general relativity, and eliminating the unobservable degrees of freedom, we have obtained a modified reduced phase space equipped with a natural partial ordering that encapsulates the notion of causality. The advantage of this approach is that the set of observables now includes the results of local measurements, as well as a natural time parameter.

The analysis suggests that elements of the reduced phase space can be identified by specifying the intrinsic geometry of the observer's past light cone C^-. Once suitable gauge conditions have been imposed, this means specifying two real quantities at each point of C^-. It is expected that these can be chosen so that one may be thought of as a generalised coordinate, and the other as a generalized momentum; in the quantum theory, the latter will be represented by a functional derivative.

Because the reduced phase space contains no gauge degrees of freedom, there are no constraints on the form of the wave function (except possibly for boundary conditions that might arise from non-dynamical considerations).

The motivation for this approach is the recognition that observation is a local phenomenon, and that an observable should therefore correspond to physical data that can collected at a definite time and place. The price that must be paid is a partial loss of determinism; the theory does not permit predictions about the future although it does allow deductions about the past. However, unless one can justify the imposition of suitable boundary conditions outside the observer's causal past, loss of determinism is inevitable and realistic.

The same approach can of course be applied to theories other than general relativity, but this leads to only very minor modifications. Even in the conventional approach these theories admit local observables and a natural time parameter, so no advantage is gained by using the approach described here. Moreover, non-gravitational fields can effectively be shielded and so it is generally reasonable to assume that they will satisfy boundary conditions on surfaces outside the observer's causal past. In such cases the observer effectively has information about the behaviour of the fields outside the region he can observe directly, and so the dynamics may be regarded as deterministic. It is only in the case of gravity that one is forced to address the indeterminacy of a theory dealing with the observations made by a localized observer.

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