Modeling the limit order book: analytic results and Monte Carlo simulations

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Abstract

We describe a model of a simple limit order market. Given the underlying supply and demand functions, an analysis of the model yields the stationary probability distributions for the best ask and bid prices in the order book, and for the transaction prices. It also predicts the existence of a clearly-defined price window within which all trades take place. These predictions are confirmed by Monte Carlo simulations.

1 Introduction

The primary function of an exchange is to match orders from buyers and sellers. In modern automated exchanges the matching mechanism generally involves an electronic limit order book, in which unexecuted limit buy orders (bids) and limit sell orders (asks) are stored while awaiting new orders with which they can be matched.

It has long been recognized that market microstructure plays an important role in price formation [1, 2]. However the microstructure of limit order markets is not yet well understood, despite several empirical studies (such as those presented in [3] and [4]) and various attempts to develop theoretical models.

So far most of these models have focussed on the game-theoretic problems faced by traders [5, 6, 7] and provide little insight into the statistical properties of the market as a whole. Here we consider a different and complementary approach recently proposed by Luckock [9], along the lines originally suggested by Garman [2]. (This approach is also similar in some respects to that of Domowitz and Wang [8], although it leads to very different results. A detailed critique of their model is given in [9].)

Instead of attempting to anticipate how rational traders will behave, we start by simply assuming that their combined effect is to generate a flow of orders with a known price distribution. The problem is then to determine the resulting distribution of bids and asks in the order book, and the distribution of transaction prices.

This problem would be mathematically intractable for a real limit order market, but considerable progress is possible in the case of the model market proposed in [9]. A mathematical analysis of this model yields the stationary probability distributions of the highest bid price and lowest ask price in the order book, as well as the frequency of transactions at any given price. All trades are found to take place in a clearly defined price window, between upper and lower limits that technical analysts would refer to as "resistance" and "support" levels. A description of the model market is presented in section 2, and its analysis is summarised in sections 3 and 4.

One of the advantages of considering a model market governed by simple dynamical rules is that, aside from the possibility of obtaining precise analytic results, one can generally obtain unlimited "empirical" data from Monte Carlo simulations. This method was used to test the theoretical predictions obtained from the mathematical analysis, for a variety of markets with price-independent supply and demand functions. The simulation algorithm is discussed in section 5, and a typical sample of the resulting distributions is displayed (along with the corresponding predicted distributions) in a series of figures.

2 A model market

We consider the simple limit order market proposed in [9]. This model market is characterised by the following properties:

L1: There are large numbers of potential buyers and sellers acting independently of one another, with each individual only occasionally submitting an order to the exchange. This allows us to regard each order as originating from a different source, and hence unrelated to any other order. The arrival of orders of any specified type, or within any specified price range, will then be a Poisson process.

L2: All orders are for a single unit. Hence, a buyer needs only to specify the maximum price she will pay (the "bid price"), while a seller needs only to specify the minimum price she will accept (the "ask price"). This assumption also allows us to regard market orders simply as limit orders with very high bid prices or very low ask prices.

L3: There is a continuous range of possible prices.

L4: The underlying supply and demand functions are time-independent.

L5: Market participants prepare and submit their orders without making use of detailed information about the current state of the order book. This prevents opportunistic traders from taking advantage of temporary anomalies in the order book, although it permits them to use their knowledge of its statistical properties when preparing their orders.

L6: Once submitted to the exchange, orders are never cancelled. This is reasonable here, since assumptions L4 and L5 preclude the possibility that a change in market conditions may motivate a trader to cancel an order.

L1, L2, L4 and L5 are similar to assumptions made in the seminal paper by Garman [2]. Properties L3 and L6 are assumed solely in order to simplify the mathematical analysis, and play an inessential role in the market dynamics. (Note however that continuous prices would be problematic in the absence of property L5, as traders with detailed information about the curent state of the order book could repeatedly outbid or undercut each other by arbitrarily small amounts in order to obtain priority. This cannot happen here, as traders do not know the exact prices of rival orders.)

These properties do not generally hold in real markets, although some may hold in certain markets at certain times. Nonetheless, most of them appear to be relatively innocuous and so it seems worthwhile to investigate the properties of the limit order book in the model market they define.

At a given time t, the order book will contain a queue of unexecuted asks at prices $(\alpha_1(t), \alpha_2(t), \ldots)$ and another queue of unexecuted bids at prices $(\beta_1(t), \beta_2(t), \ldots)$ waiting to be matched with incoming orders. These prices are indexed so that α_1 denotes the lowest (i.e. best) ask price, α_2 the second lowest, and so on. Similarly, β_1 denotes the highest (i.e. best) bid price, β_2 the second highest, etc. The prices will therefore satisfy the inequalities

$$\ldots \beta_3(t) \ \leq \beta_2(t) \ \leq \ \beta_1(t) \ < \ \alpha_1(t) \ \leq \ \alpha_2(t) \ \leq \ \alpha_3(t) \ldots . \qquad (1)$$

Whenever a new order is received by the exchange, the contents of the order book are modified according to the following rules:

R1: If the new order is an ask at the price α, then

- If $\alpha \leq \beta_1$, the new ask is matched with the best unexecuted bid, and the two are executed at the bid price β_1.
- If $\alpha > \beta_1$, no match is possible, and the new ask joins the queue of unexecuted asks in the order book.

R2: If the new order is a bid at the price β then

- If $\beta \geq \alpha_1$, the new bid is matched with the best unexecuted ask, and the two are executed at the ask price α_1.
- If $\beta < \alpha_1$, no match is possible, and the new bid joins the queue of unexecuted bids in the order book.

Given a sequence of orders, these rules completely determine the evolution of the market. However, in this model the arrivals of new bids and asks are regarded as stochastic processes. Assumptions L1 and L4 imply that the arrival of new asks at prices not exceeding x will be a Poisson process with parameter $\lambda_A(x)$, where λ_A is an increasing function. Similarly, the arrival of new bids at prices of at least x will be a Poisson process with parameter $\lambda_B(x)$, where λ_B is a decreasing function. We assume both of these functions are differentiable and strictly monotonic.

In fact $\lambda_A(x)$ and $\lambda_B(x)$ may be interpreted respectively as the supply and demand functions. Indeed, $\lambda_A(x)$ would be the frequency of sales if demand was unlimited at the price x, whereas $\lambda_B(x)$ would be the frequency of purchases if supply was unlimited at this price.

3 Deriving the stationary distributions

In this section and the next, we summarise some of the results obtained in [9].

We are primarily interested in determining the stationary probability distributions of the best ask price α_1 and best bid price β_1, for given supply and demand functions $\lambda_A(x)$ and $\lambda_B(x)$. Define $A(x)$ to be the probability that, at a random time, the best ask in the order book is executable at the price x;

$$A(x) \equiv P[\,\alpha_1 \leq x\,]. \tag{2}$$

Then $A(x)$ is the cumulative distribution function for α_1, and is necessarily increasing. Its derivative $A'(x)$, if it exists, is the density function for α_1. We assume that the order book always contains some unexecuted asks at prices above zero, and consequently

$$A(0) = 0, \qquad A(\infty) = 1. \tag{3}$$

Similarly, let $B(x)$ denote the probability that the order book contains at least one bid executable at the price x;

$$B(x) = P[\beta_1 \geq x] \tag{4}$$

Then $B(x)$ is known as the *survivor function* for the highest unexecuted bid price β_1, and must be decreasing. The density function for β_1 is $-B'(x)$, if this derivative exists. We assume also that the order book always contains unexecuted bids, which implies

$$B(0) = 1, \qquad B(\infty) = 0. \tag{5}$$

Being probabilities, the functions $A(x)$ and $B(x)$ are both restricted to the interval $[0, 1]$. It also transpires that

$$0 \leq A(x) + B(x) \leq 1. \qquad \forall x > 0, \tag{6}$$

since at any time t the sum $P[\alpha_1(t) \leq x] + P[\beta_1(t) \geq x]$ is just the probability that either $\alpha_1(t) \leq x$ *or* $\beta_1(t) \geq x$ (since these possibilities are mutually exclusive) and is therefore restricted to the interval $[0, 1]$.

It is useful now to define

$$x_{min} \equiv \inf\{x : B(x) < 1\} \tag{7}$$
$$x_{max} \equiv \sup\{x : A(x) < 1\}. \tag{8}$$

Assuming $A(x)$ and $B(x)$ are continuous, we then have

$$B(x_{min}) = 1, \qquad A(x_{max}) = 1 \tag{9}$$

and hence, on account of (6),

$$A(x_{min}) = 0, \qquad B(x_{max}) = 0. \tag{10}$$

As a consequence of these inequalities, the highest ask and lowest bid prices ay any time are certain to be found between x_{min} and x_{max};

$$\alpha_1, \beta_1 \in [x_{min}, x_{max}]. \tag{11}$$

We will refer to this interval as the *competitive window*, since unexecuted orders at prices outside this interval are inferior to those inside and are therefore ineligible to be matched with new orders. Conversely, only those unexecuted orders at prices within the competitive window have any chance of being executed against the next incoming order.

In principle, it should be possible to use the dynamical rules R1 and R2 to determine the stationary distribution functions $A(x)$ and $B(x)$ and also the prices x_{min} and x_{max}. This could provide probabilistic answers to many questions about the probable state of the order book, and to calculate expectations of functions of α_1 and β_1. As we shall see, knowledge of $A(x)$ and $B(x)$ also determines the frequency of trades in a given price range.

To determine the form of the functions $A(x)$ and $B(x)$, one might attempt to derive an appropriate set of evolution equations for these distributions, and then look for stationary solutions to these equations. However it is difficult to write down self-contained evolution equations for $A(x)$ and $B(x)$, because their evolution turns out to depend on the distributions of all the unexecuted orders in the order book. Fortunately, there is another approach.

Let $M(t, x)$ denote the number of queued asks at time t that are executable at the price x (i.e. awaiting execution at prices not exceeding x), and let $N(t, x)$ denote the number of queued bids executable at the this price (i.e. awaiting excution at prices of at least x). Assuming that the expectations of $M(t, x)$ and $N(t, x)$ are differentiable functions of t and x, we also define

$$u(t, x) \equiv \frac{\partial}{\partial x} E[\, M(t, x)\,], \qquad v(t, x) \equiv -\frac{\partial}{\partial x} E[\, N(t, x)\,]. \tag{12}$$

Then at time t the number of unexecuted asks in the price range $[x_1, x_2]$ and the number of unexecuted bids in the price range $[x_1, x_2]$ are random variables with expectations

$$E[M(t, x_2) - M(t, x_1)\,] \;=\; \int_{x_1}^{x_2} u(t, x)dx \tag{13}$$

$$E[N(t, x_1) - N(t, x_2)\,] \;=\; \int_{x_1}^{x_2} v(t, x)dx \tag{14}$$

respectively. We can think of $u(t, x)$ and $v(t, x)$ as the expected densities of asks and bids in the order book around the price x. In particular, if $[x, x+\Delta x]$ is some small price interval, then $u(t, x)\Delta x$ and $v(t, x)\Delta x$ are the expected numbers of unexecuted asks and bids to be found in this price range.

Suppose ΔM denotes the actual number of unexecuted asks in the price interval $[x, x + \Delta x]$, and let us determine the probabilities that ΔM may increase or decrease due to the arrival of a new ask or the execution of an existing ask.

Over a short time interval Δt, the probability of receiving a new ask in this price range is $\lambda'_A(x)\Delta t \Delta x$. Also, the probability that there are no higher priced bids with which it can be matched is $1 - B(x)$. As these are independent events, the probability of receiving a new ask in this price range that is not immediately matched with a higher-priced bid, and hence of ΔM increasing by 1, is just the product $[1 - B(x)]\lambda'_A(x)\Delta t \Delta x$.

During this time interval, there is also a probability $\lambda_B(x)\Delta t$ of receiving a new bid at a price higher than x. However, this will not result in the execution of any of the asks in the price range $[x, x + \Delta x]$ unless one of them happens to be the lowest priced ask, and the probability of this is $A'(x)\Delta x$.

The probability that an ask in the price range $[x, x + \Delta x]$ will be executed against a new bid, and hence of ΔM decreasing by 1 during this time interval, is therefore given by the product $\lambda_B(x)\Delta t\, A'(x)\Delta x$.

Thus, over the time interval Δt, the probability that ΔM will increase by 1 is $[1 - B(x)]\lambda'_A(x)\Delta t\Delta x$, while the probability that it will decrease by 1 is $A'(x)\lambda_B(x)\Delta t\Delta x$. Consequently the change in its expectation $E[\Delta M] = u(t, x)\Delta x$ is

$$[u(t+\Delta t, x) - u(t, x)]\Delta x = \lambda'_A(x)[1 - B(x)]\Delta t\Delta x - \lambda_B(x)A'(x)\Delta t\Delta x. \quad (15)$$

Dividing through by $\Delta t\Delta x$ and taking the $\Delta t \to 0$ limit then gives

$$\frac{\partial u}{\partial t} = \lambda'_A(x)[1 - B(x)] - \lambda_B(x)A'(x). \quad (16)$$

An exactly analogous argument leads to the equation

$$\frac{\partial v}{\partial t} = -\lambda'_B(x)[1 - A(x)] + \lambda_A(x)B'(x). \quad (17)$$

These equations imply that there is a build-up of uncompetitive bids and asks outside the competitive window. Indeed, for prices x exceeding x_{max} we have $A'(x) = B(x) = 0$, and so (16) gives

$$\frac{\partial u}{\partial t} = \lambda'_A(x) > 0 \quad (18)$$

indicating that unexecuted asks continue to accumulate. Similarly, for $x < x_{min}$ we have $A(x) = B'(x) = 0$, and so (17) gives

$$\frac{\partial v}{\partial t} = -\lambda'_B(x) > 0. \quad (19)$$

Thus, the expected densities of orders cannot remain stationary outside the competitive window $[x_{min}, x_{max}]$. At prices higher than x_{max}, asks accumulate faster than they can be matched with incoming bids, while at prices lower than x_{min} bids accumulate faster than they can be matched with incoming asks (as shown by equations (18) and (19) respectively). Such orders can be regarded as uncompetitive as they never result in trades, and are of little interest.

However, the order densities should be stationary within the competitive window or else unexecuted orders would accumulate there indefinitely. (This would be inconsistent with the stationarity of $A(x)$ and $B(x)$.) Requiring $\partial u/\partial t$ and $\partial v/\partial t$ to vanish within the competitive window, equations (16,17) then imply that $A(x)$ and $B(x)$ must satisfy the coupled first-order linear differential equations

$$A'(x) = \frac{\lambda'_A(x)}{\lambda_B(x)}[1 - B(x)] \tag{20}$$

$$B'(x) = \frac{\lambda'_B(x)}{\lambda_A(x)}[1 - A(x)]. \tag{21}$$

for all $x \in [x_{min}, x_{max}]$. They must also satisfy the boundary conditions (9) and (10), which we repeat here for emphasis;

$$A(x_{min}) = 0, \quad B(x_{min}) = 1, \quad A(x_{max}) = 1, \quad B(x_{max}) = 0. \tag{22}$$

Recall that the boundary points x_{min} and x_{max} are not yet known, but must be deduced along with $A(x)$ and $B(x)$ from these equations.

Before solving this free boundary problem, we note that the quantity

$$\kappa \equiv [1 - B(x)]\lambda_A(x) + \lambda_B(x)[1 - A(x)] \tag{23}$$

is independent of x over the competitive window, on account of equations (20) and (21). Putting $x = x_{min}$ and then $x = x_{max}$ therefore implies

$$\kappa = \lambda_B(x_{min}) = \lambda_A(x_{max}). \tag{24}$$

Note also that we can now obtain a simple expression for the expected frequency of trades in different price ranges. There are two types of trades; those in which a pre-existing ask is matched with a new bid, and those in which a pre-existing bid is matched with a new ask. The expected frequency of type 1 trades in the infinitesimal price interval $[x, x + dx]$ is easily seen to be $\lambda_B(x)A'(x)dx$, which can be rewritten as $\lambda'_A(x)[1 - B(x)]$ using (20). Similarly, the expected frequency of type 2 trades in this price interval is $-\lambda_A(x)B'(x)dx$. Adding these expressions and integrating, we find that the expected frequency of trades at prices not exceeding x is

$$\lambda_T(x) = \lambda_A(x)[1 - B(x)] \qquad x \in [x_{min}, x_{max}]. \tag{25}$$

(Recall that there are no trades at prices outside the competitive window $[x_{min}, x_{max}]$.) It follows that the overall frequency of trades is $\lambda_T(x_{max}) = \kappa$.

We conclude this section by observing that the distributions obtained in this manner will be unaffected by variations in supply and demand at prices outside the competitive window, as solutions to (20), (21) and (22) will continue to be solutions even if the form of the functions $\lambda_A(x)$ and $\lambda_B(x)$ changes outside $[x_{min}, x_{max}]$. Indeed since no trades are ever executed outside this window, variations in $\lambda_A(x)$ and $\lambda_B(x)$ for $x \notin [x_{min}, x_{max}]$ can have no effect whatsoever on the statistical properties of bids, asks and trades within the window.

It follows that the stationary distributions of bid, ask and transaction prices inside the competitive window $[x_{min}, x_{max}]$ will depend only on the form of the functions $\lambda_A(x)$ and $\lambda_B(x)$ inside this window.

4 Application to a market with fixed elasticities of supply and demand

Real markets are characterised by narrow bid-ask spreads – typically less than 1% of the equilibrium price. An application of the model to such markets should therefore be tailored to produce a narrow competitive window, within which the elasticity of supply and the elasticity of demand will hardly vary and may reasonably be assumed to be constant.

Let us therefore consider a market in which the elasticity of supply and elasticity of demand have fixed values $\eta_a > 0$ and $\eta_b > 0$ (at least within the competitive window). The supply and demand functions will then have the form

$$\lambda_A(x) = kx^{\eta_a}, \qquad \lambda_B(x) = kx^{-\eta_b} \tag{26}$$

for some positive constant k. (We suppose the numeraire has been chosen so that the equilibrium price x_E is exactly 1, and hence $\lambda_A(1) = \lambda_B(1)$.)

If $\eta_b \neq \eta_a$ then the solution of (20) and (21) subject to the free boundary conditions (22) is found to be

$$A(x) = 1 - \frac{1}{\eta_a - \eta_b}\left[\eta_a\left(\frac{x}{x_{min}}\right)^{\eta_b} - \eta_b\left(\frac{x}{x_{min}}\right)^{\eta_a}\right] \tag{27}$$

$$B(x) = 1 - \frac{1}{\eta_a - \eta_b}\left[\eta_a\left(\frac{x}{x_{max}}\right)^{-\eta_b} - \eta_b\left(\frac{x}{x_{max}}\right)^{-\eta_a}\right] \tag{28}$$

for $x \in [x_{min}, x_{max}]$, where

$$x_{min} = \left(\frac{\eta_a}{\eta_b}\right)^{-\frac{\eta_a}{{\eta_a}^2 - {\eta_b}^2}} \qquad x_{max} = \left(\frac{\eta_a}{\eta_b}\right)^{\frac{\eta_b}{{\eta_a}^2 - {\eta_b}^2}} \qquad \kappa = k\left(\frac{\eta_a}{\eta_b}\right)^{\frac{\eta_a \eta_b}{{\eta_a}^2 - {\eta_b}^2}}. \tag{29}$$

If $\eta_b = \eta_a = \eta$, we instead find that

$$A(x) = 1 - e^{1/2} x^{\eta}\left(\frac{1}{2} - \eta \ln x\right) \tag{30}$$

$$B(x) = 1 - e^{1/2} x^{-\eta}\left(\frac{1}{2} + \eta \ln x\right) \tag{31}$$

for $x \in [x_{min}, x_{max}]$, where now

$$x_{min} = e^{-1/2\eta}, \qquad x_{max} = e^{1/2\eta}, \qquad \kappa = k e^{1/2}. \tag{32}$$

It is a simple exercise to verify that these expressions do satisfy the differential equations (20,21) on this interval, as well as the boundary conditions (22).

The lowest ask price α_1 and highest bid price β_1 have probability density functions $A'(x)$ and $-B'(x)$, both of which are supported on the competitive window $[x_{min}, x_{max}]$. These density functions can be used to calculate expectations of any functions of α_1 or β_1.

For example, if $\eta_a \neq \eta_b$ then the lowest ask and highest bid prices in the order book are found to have expectations

$$E[\alpha_1] = \frac{\eta_a \eta_b x_{min} + \sigma x_{max}}{(\eta_a + 1)(\eta_b + 1)} \tag{33}$$

$$E[\beta_1] = \frac{\eta_a \eta_b x_{max} - \sigma x_{min}}{(\eta_a - 1)(\eta_b - 1)} \qquad (\eta_a \neq 1 \neq \eta_b) \tag{34}$$

where $\sigma \equiv \left({\eta_a}^{\eta_a} {\eta_b}^{-\eta_b}\right)^{\frac{1}{\eta_a - \eta_b}}$. However, if $\eta_a = \eta_b = \eta$ then the corresponding expressions are found to be

$$E[\alpha_1] = \frac{\eta^2}{(\eta+1)^2} e^{-\frac{1}{2\eta}} + \frac{\eta e}{(\eta+1)^2} e^{\frac{1}{2\eta}} \tag{35}$$

$$E[\beta_1] = \begin{cases} \frac{\eta^2}{(\eta-1)^2} e^{\frac{1}{2\eta}} - \frac{\eta e}{(\eta-1)^2} e^{-\frac{1}{2\eta}} & (\eta \neq 1) \\ \frac{1}{2} e^{\frac{1}{2}} & (\eta = 1) \end{cases}. \tag{36}$$

We can also use (25) to determine the frequency of actual trades at different prices. Given any $x \in [x_{min}, x_{max}]$, the frequency of trades at prices less than x is found to be

$$\lambda_T(x) = \begin{cases} \frac{\kappa \eta_b}{\eta_a - \eta_b} \left[\left(\frac{x}{x_{min}} \right)^{\eta_a - \eta_b} - 1 \right] & (\eta_a \neq \eta_b) \\ \kappa \left(\frac{1}{2} + \eta \ln x \right) & (\eta_a = \eta_b = \eta) \end{cases} . \tag{37}$$

Since $\kappa = \lambda_T(x_{max})$ is the overall frequency of trades, the density function for the price of an arbitrary trade is

$$\frac{\lambda_T'(x)}{\kappa} = \begin{cases} \left({\eta_a}^{\eta_a} {\eta_b}^{\eta_b} \right)^{\frac{1}{\eta_a + \eta_b}} x^{\eta_a - \eta_b - 1} & (\eta_a \neq \eta_b) \\ \frac{\eta}{x} & (\eta_a = \eta_b = \eta) \end{cases} . \tag{38}$$

The transaction price X_T of a randomly chosen trade is then found to have expectation

$$E[X_T] = \frac{\eta_a x_{max} - \eta_b x_{min}}{1 + \eta_a - \eta_b}. \tag{39}$$

This expression is valid whether $\eta_a \neq \eta_b$ or $\eta_a = \eta_b$, but in the latter case it can also be written as

$$E[X_T] = 2\eta \sinh(\frac{1}{2\eta}) = 1 + \frac{1}{3!(2\eta)^2} + \frac{1}{5!(2\eta)^4} \cdots \tag{40}$$

which always exceeds the equilibrium price $x_E = 1$.

Figures 1-9 show the predicted distributions of transaction prices and best ask and bid prices (together with actual distributions produced by Monte Carlo simulations) for various values of η_a and η_b between 40 and 400. These distributions are typical of those obtained for any η_a and η_b. Note that it is the ratio of these parameters that primarily determines the shapes of the distributions within the competitive window, whereas their average magnitude determines the approximate size of the window [9]. For figures 1-9, η_a and η_b were chosen to give plausible bid-ask spreads, ranging from 0.3% to 1.1% of the equilibrium price. (The expectations of the best ask and bid prices, spread, and transaction price for each choice of η_a and η_b are given beneath the corresponding figure caption.)

5 Monte Carlo simulation

The simple dynamical rules governing this model market make it well suited to numerical simulation. Using this appoach, it is posible to determine the statistical properties of the order book for arbitrary (and possibly time-dependent) supply and demand functions.

We used a simple Monte Carlo algorithm to simulate markets characterised by supply and demand functions (i.e. price-dependent order arrival rates) having the form

$$\lambda_A(x) = kx^{\eta_a}, \qquad \lambda_B(x) = kx^{-\eta_b}. \tag{41}$$

Our main purpose was to test the predictions of the previous sections and to confirm the existence of a clearly defined competitive window for various choices of η_a and η_b.

In a simulation of this kind, the order book can be represented in various ways. Computationally, the simplest approach is to approximate the continuous range of possible prices by a finite set of discrete prices (as in a real market). The state of the order book is then represented by two integer arrays of fixed size, giving the number of bids or asks at each price. However, as the primary purpose of the simulation was to test the theoretical model developed in the previous section, and we therefore wished to preclude extraneous features related to tick size, this approach was not employed.

Instead we represented the order book by two real arrays in which were stored the exact prices of a number of unexecuted asks and bids. As this approach does not impose any tick size, it is the more accurate realisation of the model.

However in this approach the fixed size of each array presents a minor difficulty. If the number of unexecuted bids or asks at some time happens to exceed the size of the array in which they are being stored, then the least competitive of these orders must be discarded. This is not a concern in itself; however, execution of any of the remaining orders stored in the array will then create vacancies that will be quickly filled by whatever new orders are received, even if these happen to be less competitively priced than those previously discarded. As a result, the least competitive orders in the array may occasionally be unworthy pretenders that have got there by "jumping the queue". If the array is large enough, it is unlikely that any of these unworthy orders will ever reach the top of its queue and affect the distributions of α_1, β_1 and X_T, although it is difficult to anticipate what is "large enough". For arrays containing fewer than about 100 bids and 100 asks, we found that this effect produced a small overspill from the competitive window, while for arrays containing fewer than 10 bids and 10 asks the overspill was quite pronounced.

The results shown in figures 1-9 were obtained with arrays of 200 bids and 200 asks, for which the overspill effect was sporadic and barely discernible. (The effect would have been completely invisible if the arrays had been made 5 or 10 times larger; but as it cannot be eliminated completely we preferred to acknowledge its presence.)

Another issue of potential significance involves the random generation of bids and asks. The assumed form (41) of the supply and demand functions appears to imply infinite arrival rates for very high-priced bids and very low-priced asks. In fact such orders are entirely irrelevant; asks at prices higher than x_{max} and bids at prices lower than x_{min} never result in trades and can safely be ignored. Provided x_{min} and x_{max} are known accurately, one may therefore disregard all bids at prices less than some fixed $X_L < x_{min}$, and all asks at prices exceeding some fixed $X_U > x_{max}$, without affecting the distribution of transaction prices or of bids and asks within the competitive window.

However, the choice of the cutoff prices X_L, X_U is a delicate matter as the values of x_{min} and x_{max} are assumed to be unknown. (As the purpose of the simulation is to test the predictions of sections 3 and 4, the values given by (29) and (32) cannot be used.) The cutoff prices must therefore be chosen very conservatively, with X_L small enough and X_U large enough to ensure that $X_L < x_{min} < x_{max} < X_U$ for any plausible values of x_{min} and x_{max}. A Monte Carlo simulation, with randomly generated asks above X_L and bids below X_U, then reveals the location of the competitive window. This procedure must be repeated several times with different choices of X_L and X_U to verify that similar results are obtained each time, regardless of the values of X_L and X_U.

We used this approach to locate the competitive window and then to determine the stationary distributions of the best ask and bid prices α_1 and β_1, and the execution price X_T, for various values of η_a and η_b. Some typical results are displayed in figures 1-9, where they are compared with the distributions predicted in the previous section.

It is evident from these figures that the Monte Carlo data fit the predictions almost perfectly. This was true of all the simulations we performed, not just those displayed in the figures. (In fact with larger numbers of trials, and larger arrays in which to store to unexecuted orders, the simulated distributions began to resemble the predicted distributions so closely that it was difficult to tell them apart. For this reason, results of simulations involving more than 200,000 trials have not been shown.)

The competitive window is clearly evident in all the simulations, and in each case has precisely the predicted properties. The window's sharp boundaries are a striking feature of the observed distributions, and it should be emphasised that they were not programmed into the simulation in any way. They emerged entirely as a consequence of the market microstructure, exactly as predicted by the mathematical analysis of sections 3 and 4.

6 Conclusions

The main result of this paper can be summarised as follows. We have shown analytically, and confirmed numerically, that all trades in this model market take place between two limiting prices x_{min} and x_{max} that are determined by the form of the supply and demand functions.

Clearly, x_{min} and x_{max} are what technical analysts refer to as the "support" and "resistance" levels. At prices below the support level x_{min}, buy orders accumulate faster than they can be executed (as shown by (19)) and build up into a solid wall. This wall of unfilled bids prevents any new sell orders from being executed at prices less than x_{min}. Similarly, above the resistance level x_{max}, sell orders accumulate (as shown by (18)) to form an impenetrable wall, ensuring that no buy orders are executed at prices above x_{max}.

It is natural to ask to what extent the competitive window is an artefact of the properties L1-L6 assumed in the construction of the model. Although insufficient work has been done to answer this question decisively, a strong argument can be made that the competitive window should be a generic feature of sufficiently liquid limit-order markets in the absence of changes in the underlying supply and demand. Indeed, while all of the properties L1-L6 are required for the derivation of the mathematical results presented in sections 3 and 4, only L4 and perhaps L1 appear to play an essential role in the process which leads to the formation of the competitive window. Moreover, evidence for the existence of competitive windows in real limit-order markets is provided by technical analysts, who rely extensively on the existence of support and resistance levels between which price fluctuations are temporarily contained.

Further work is clearly required to determine the effect of each of the assumptions L1-L6 on the statistical properties of the limit order book, and to ascertain whether some generalisation of this simple model might provide a useful description of real limit order markets. However the innate complexity of markets limits the power of purely mathematical analysis and suggests that the Monte Carlo approach may be ultimately more fruitful.

In conclusion we note that this model provides a clear demonstration of the way in which market microstructural effects produce unpredictable price fluctuations, in addition to any which might result from variations in supply and demand. This suggests that the price fluctuations we observe in any real market should be regarded as the sum of two distinct components; one purely microstructural, the other economic. We believe that any attempt to distinguish between these two components will require a better quantitative understanding of market microstructure than currently exists, and the best hope of achieving this is by a systematic analysis of model markets of increasing complexity and realism.

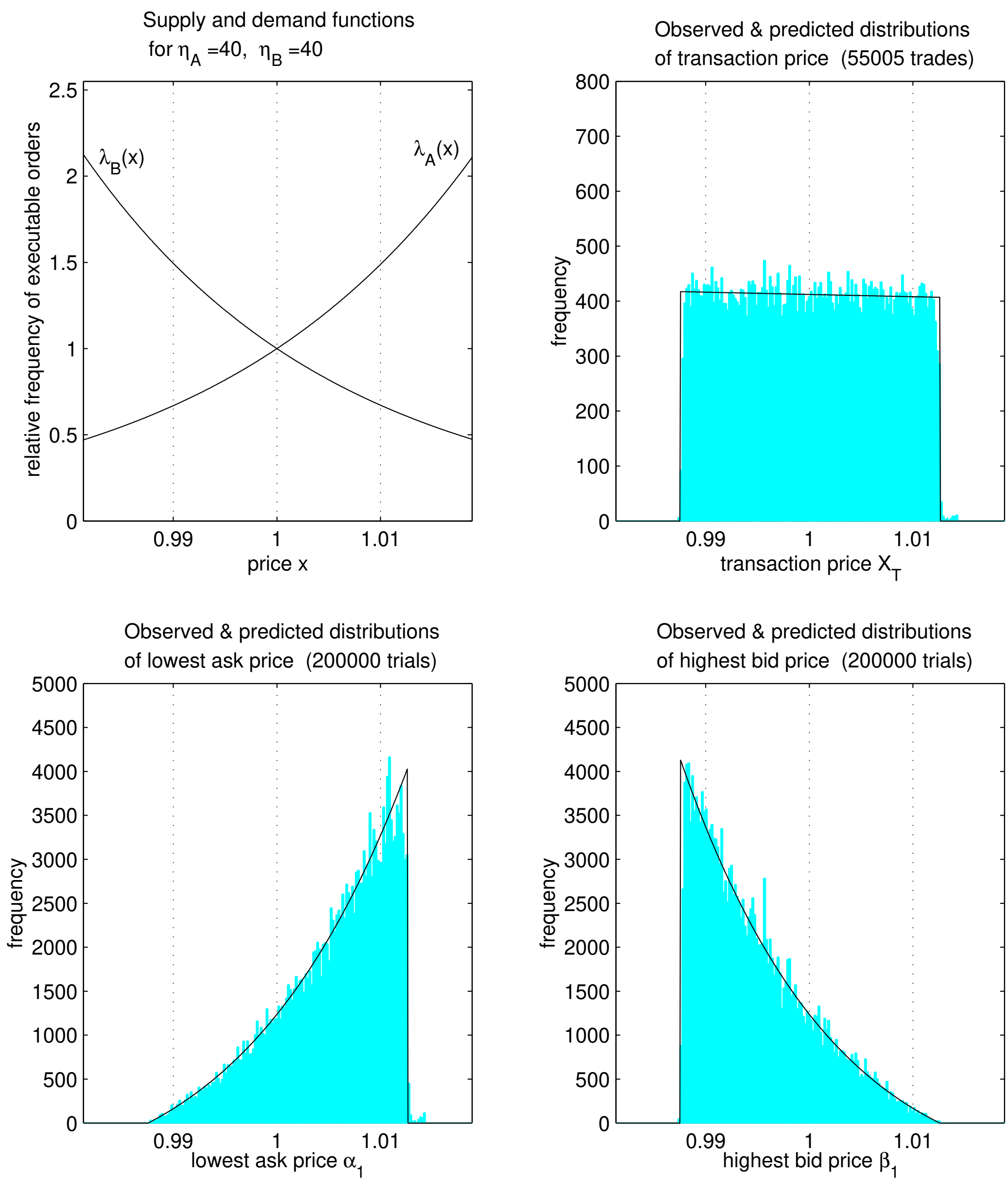

Figure 1: Supply & demand functions, and distributions of transaction price, lowest ask price & highest bid price, for $\eta_a = 40$, $\eta_b = 40$.

$E[\alpha_1] = 1.00549$, $E[\beta_1] = 0.99457$, $E[\alpha_1 - \beta_1] = 0.01091$, $E[X_T] = 1.00003$.

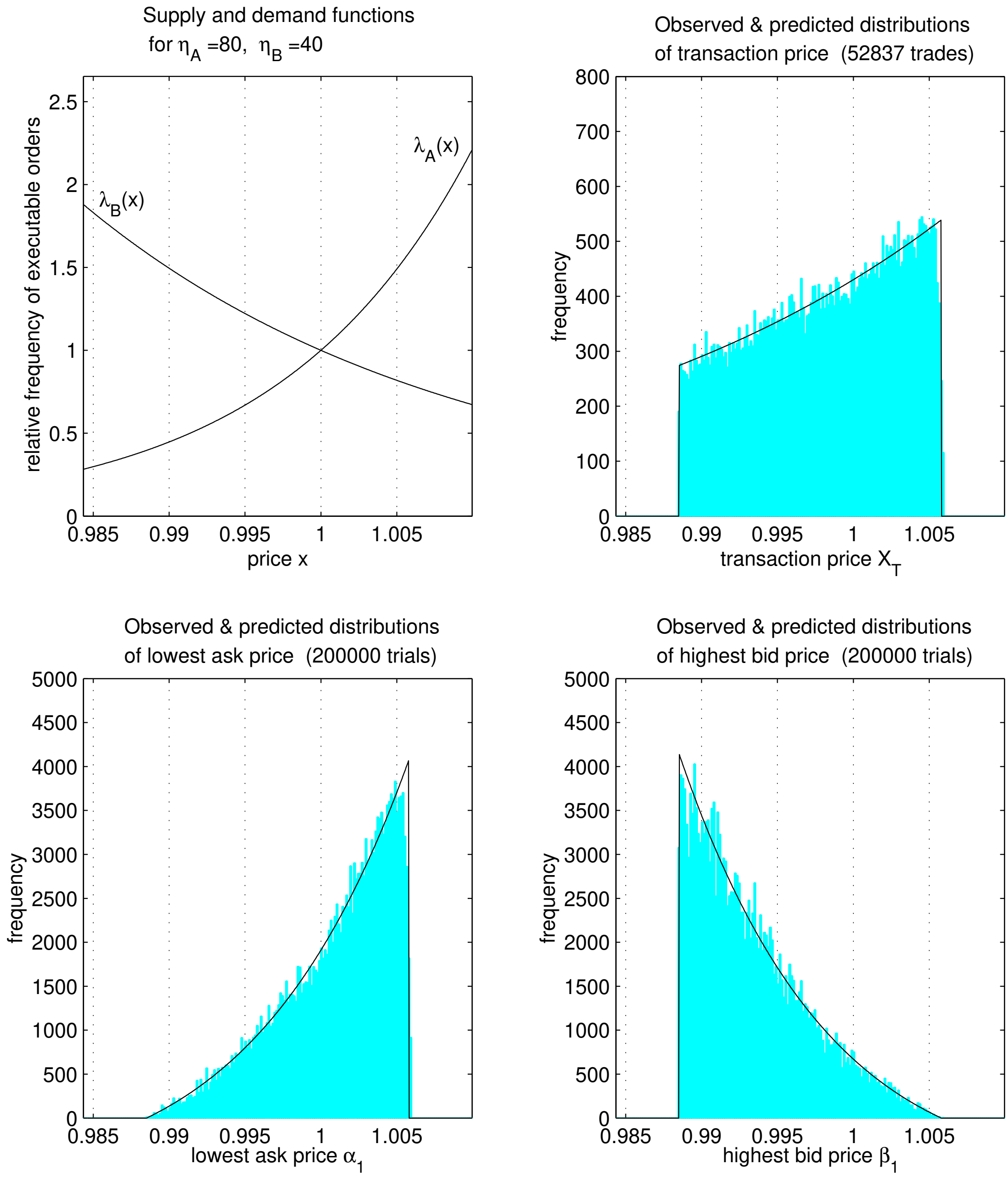

Figure 2: Supply & demand functions, and distributions of transaction price, lowest ask price & highest bid price, for $\eta_a = 80$, $\eta_b = 40$.

$E[\alpha_1] = 1.00096$, $E[\beta_1] = 0.99331$, $E[\alpha_1 - \beta_1] = 0.00765$, $E[X_T] = 0.99812$.

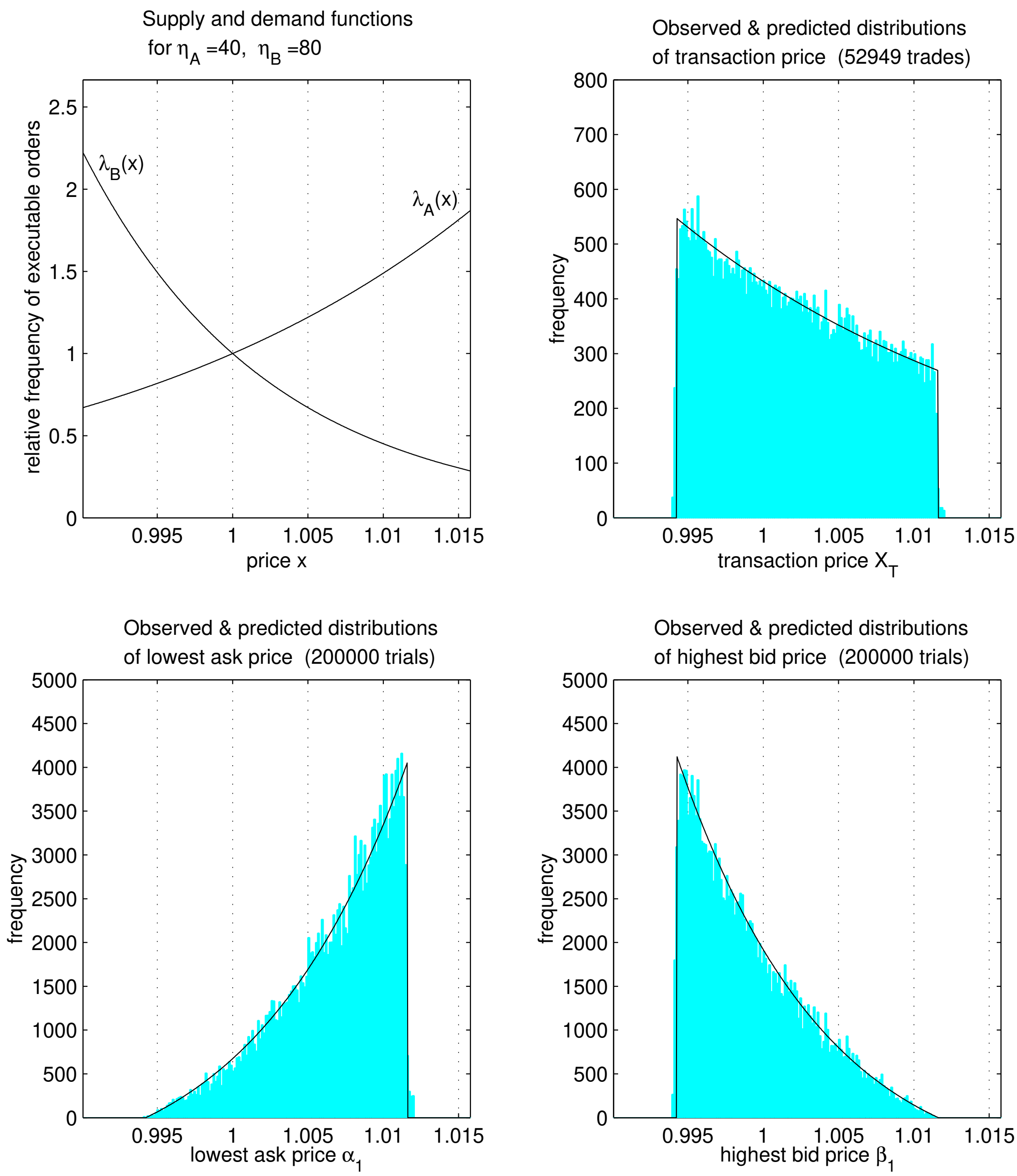

Figure 3: Supply & demand functions, and distributions of transaction price, lowest ask price & highest bid price, for $\eta_a = 40$, $\eta_b = 80$.

$E[\alpha_1] = 1.00675$, $E[\beta_1] = 0.99906$, $E[\alpha_1 - \beta_1] = 0.00769$, $E[X_T] = 1.00191$.

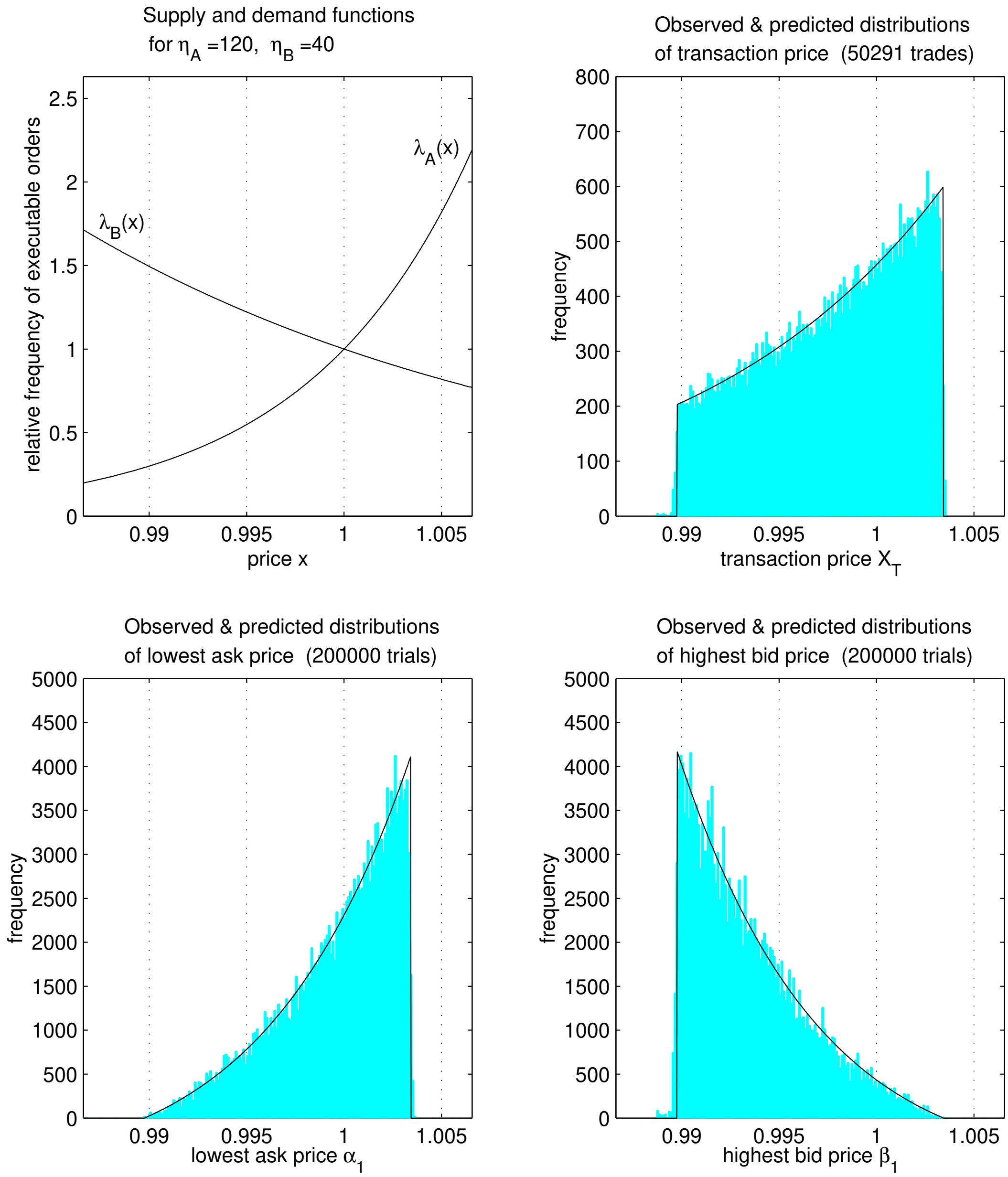

Figure 4: Supply & demand functions, and distributions of transaction price, lowest ask price & highest bid price, for $\eta_a = 120$, $\eta_b = 40$.

$E[\alpha_1] = 0.99967$, $E[\beta_1] = 0.99349$, $E[\alpha_1 - \beta_1] = 0.00618$, $E[X_T] = 0.99781$.

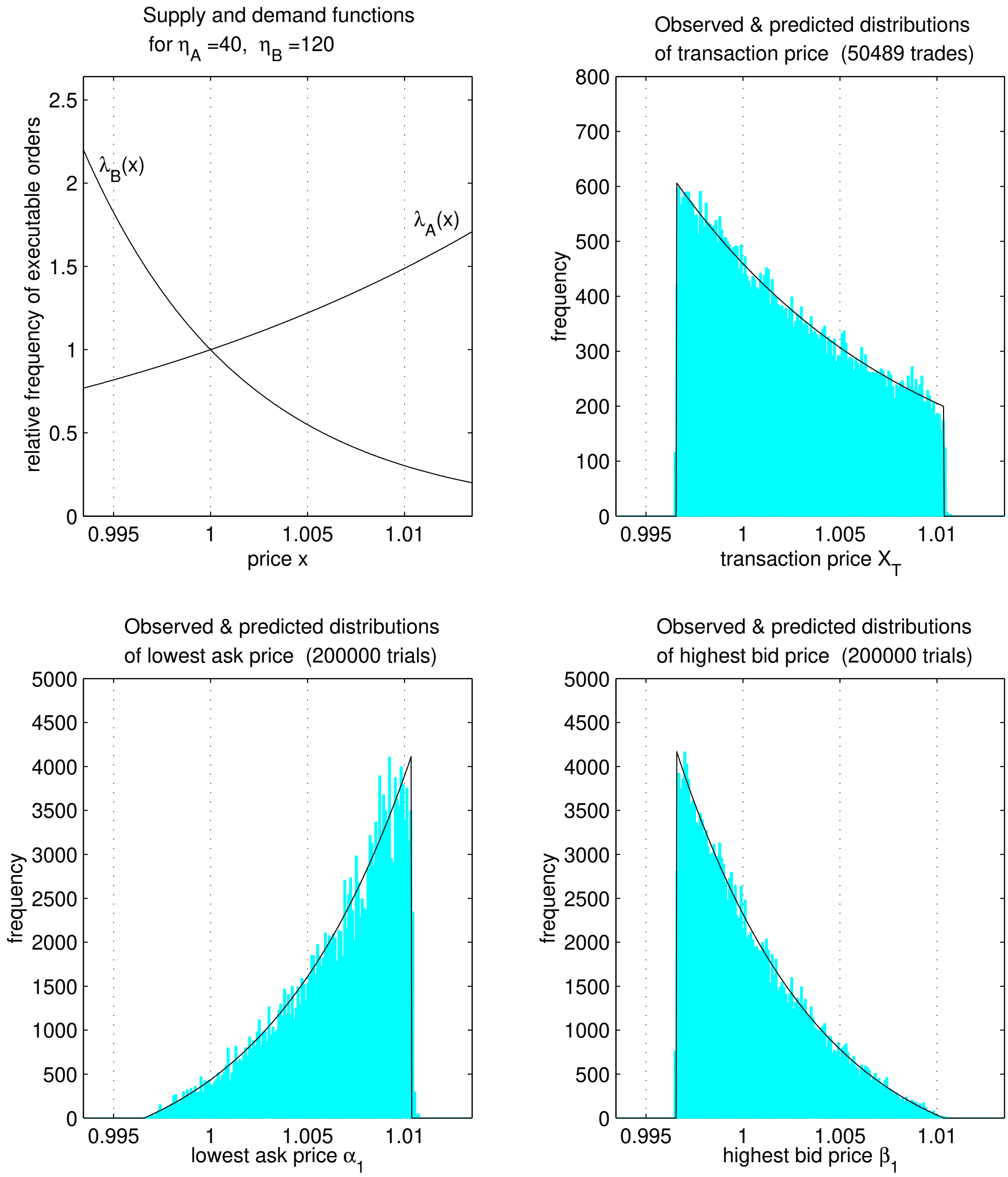

Figure 5: Supply & demand functions, and distributions of transaction price, lowest ask price & highest bid price, for $\eta_a = 40$, $\eta_b = 120$.

$E[\alpha_1] = 1.00656,\ \ E[\beta_1] = 1.00034,\ \ E[\alpha_1 - \beta_1] = 0.00622,\ \ E[X_T] = 1.00221.$

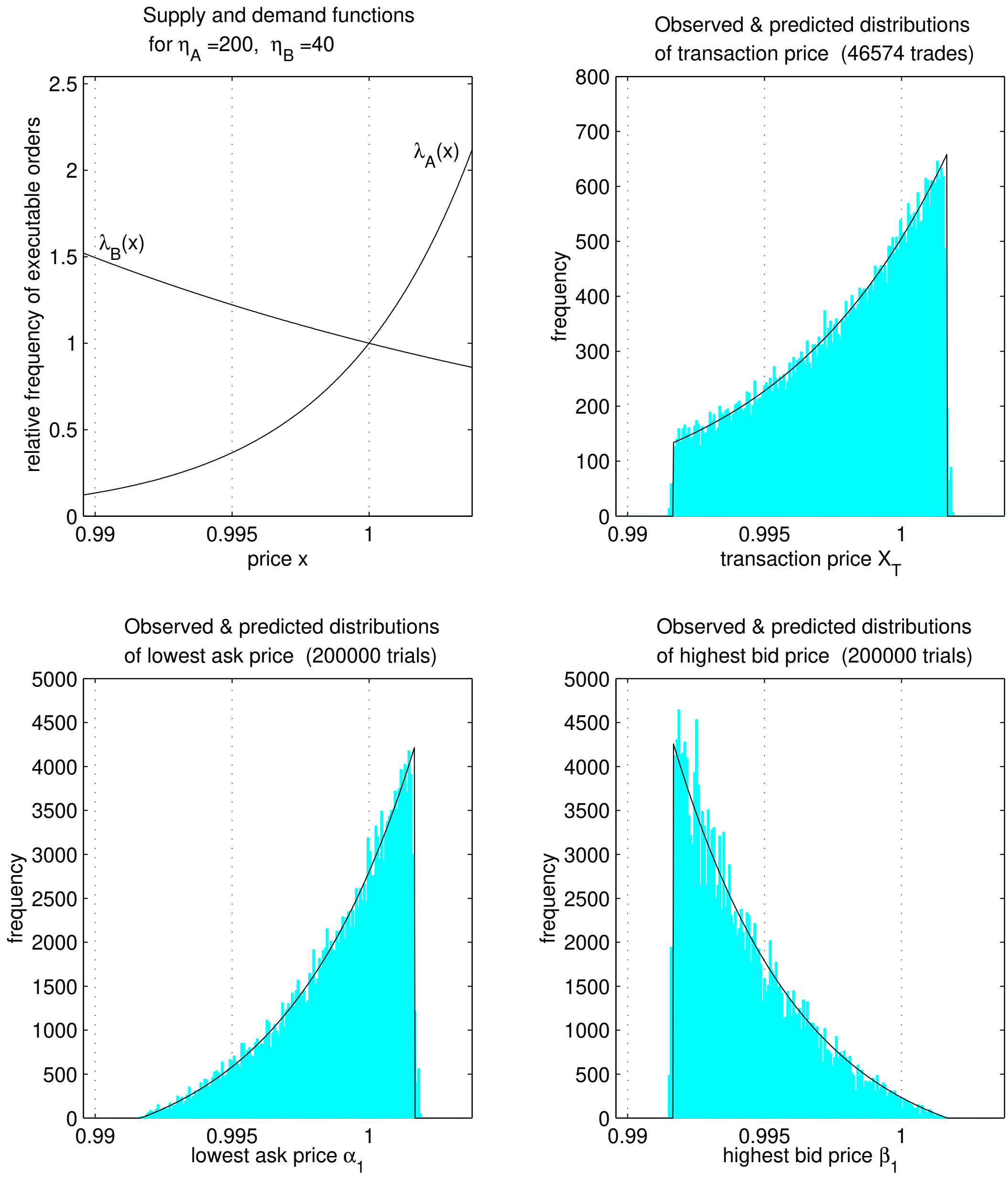

Figure 6: Supply & demand functions, and distributions of transaction price, lowest ask price & highest bid price, for $\eta_a = 200$, $\eta_b = 40$.

$E[\alpha_1] = 0.99900,\ \ E[\beta_1] = 0.99431,\ \ E[\alpha_1 - \beta_1] = 0.00469,\ \ E[X_T] = 0.99795.$

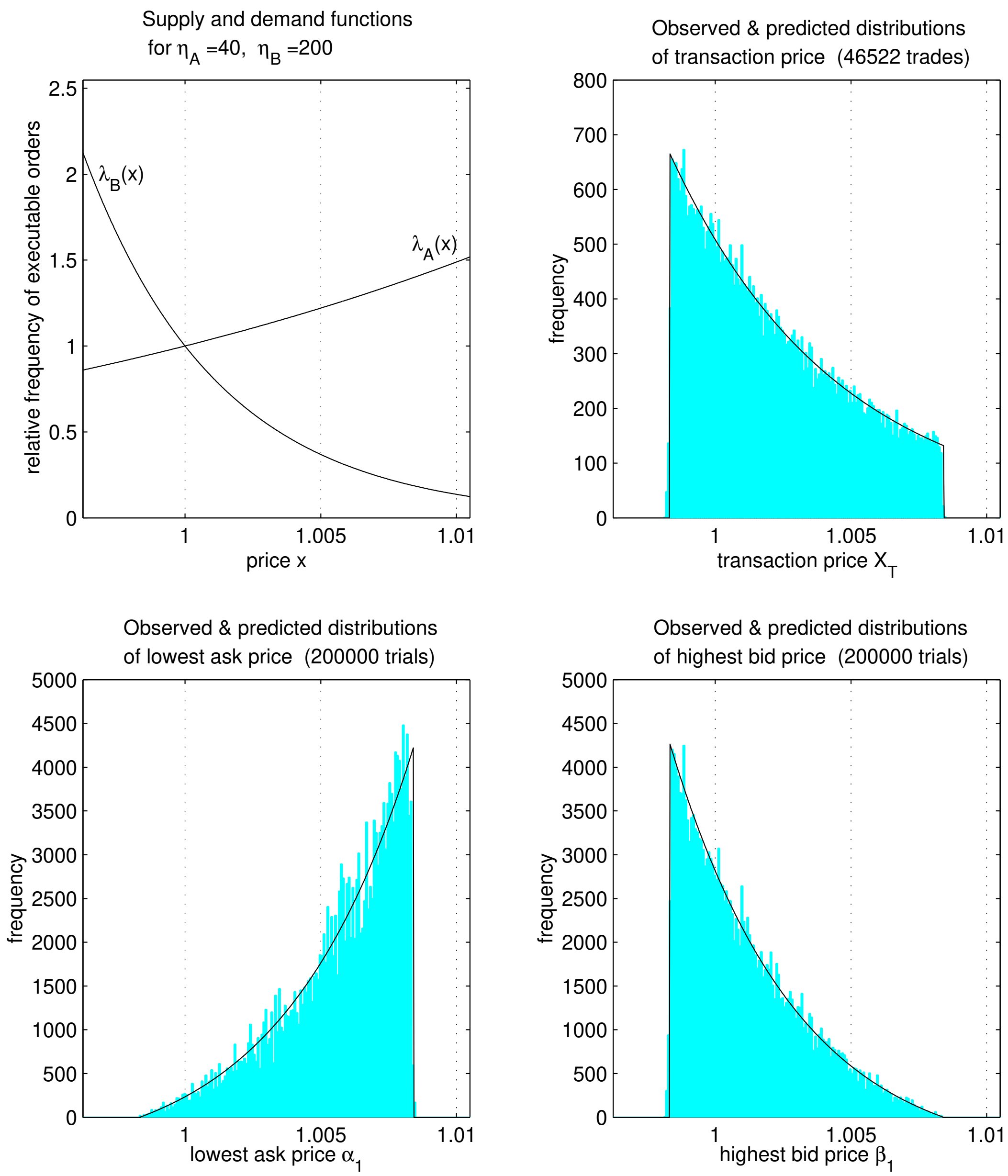

Figure 7: Supply & demand functions, and distributions of transaction price, lowest ask price & highest bid price, for $\eta_a = 40$, $\eta_b = 200$.

$E[\alpha_1] = 1.00573,\ \ E[\beta_1] = 1.00100,\ \ E[\alpha_1 - \beta_1] = 0.00472,\ \ E[X_T] = 1.00206.$

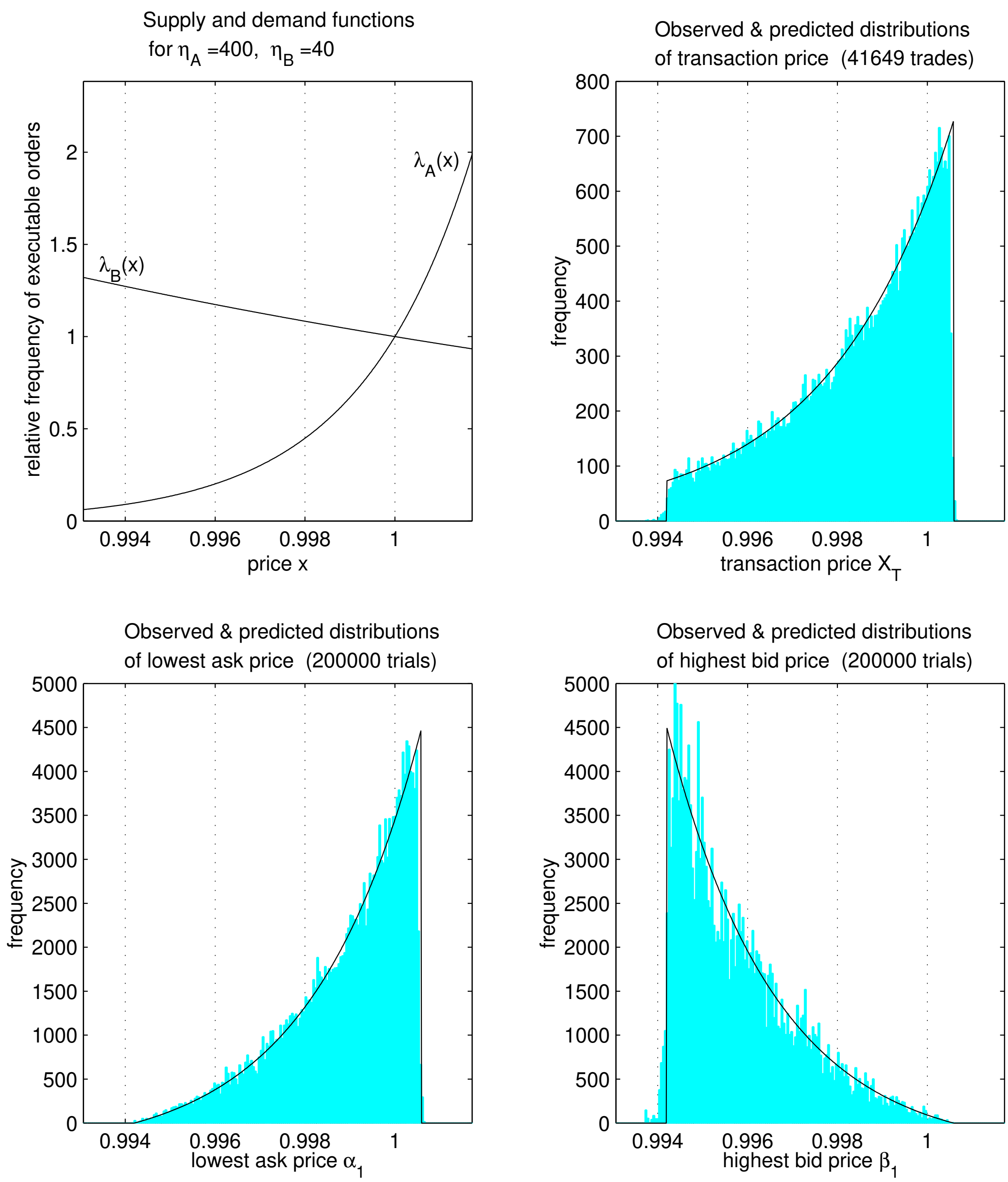

Figure 8: Supply & demand functions, and distributions of transaction price, lowest ask price & highest bid price, for $\eta_a = 400$, $\eta_b = 40$.

$E[\alpha_1] = 0.99898,\ \ E[\beta_1] = 0.99580,\ \ E[\alpha_1 - \beta_1] = 0.00317,\ \ E[X_T] = 0.99852.$

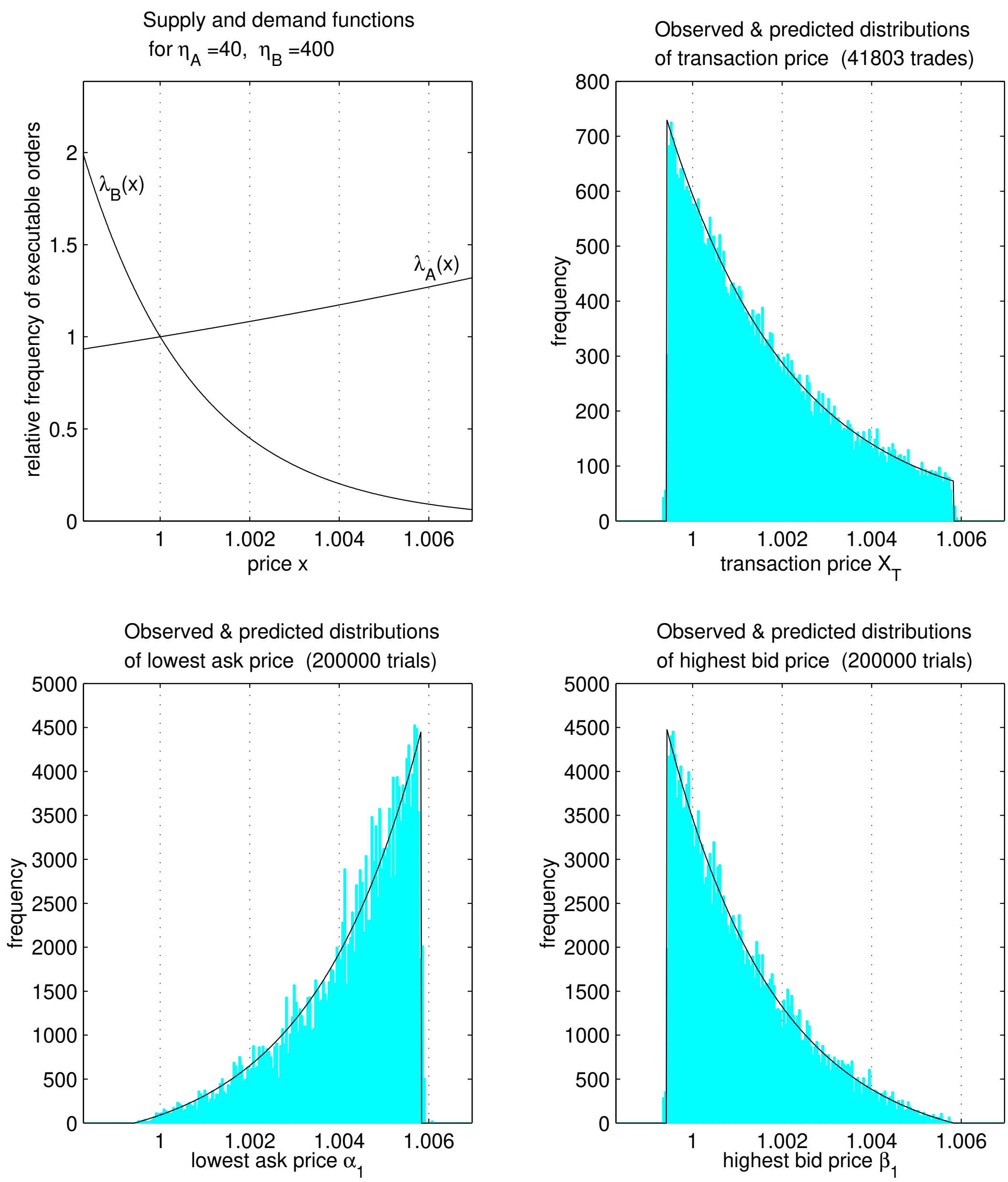

Figure 9: Supply & demand functions, and distributions of transaction price, lowest ask price & highest bid price, for $\eta_a = 40$, $\eta_b = 400$.

$E[\alpha_1] = 1.00422$, $E[\beta_1] = 1.00103$, $E[\alpha_1 - \beta_1] = 0.00319$, $E[X_T] = 1.00149$.

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