AN INVERSE MAPPING THEOREM FOR ARC-ANALYTIC HOMEOMORPHISMS

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Abstract. We show that a subanalytic map-germ $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ which is arc-analytic and bilipschitz has an arc-analytic inverse.

Let U be an open subset of $\mathbf{R}^n$ or more generally a smooth real analytic variety. Following [8] we say that a map $f : U \to \mathbf{R}^k$ is *arc-analytic* if $f \circ \alpha$ is analytic for any analytic arc $\alpha : (-\varepsilon, \varepsilon) \to U$. In general these functions are very far from being analytic, in particular there are arc-analytic functions which are not subanalytic [9], not continuous [2], with a nondiscrete singular set [10]. Hence it is natural to consider only arc-analytic maps with subanalytic graphs. In the late seventies T.-C. Kuo introduced the notion of *blow-analytic* mappings, i.e., mappings which become analytic after a composition with appropriate proper bimeromorphic maps (e.g. composition of blowings-up with smooth centers). He suggested that the equivalence relation for germs of analytic functions, defined by blow-analytic homeomorphisms, should give "the canonical" stratification of the singular set of an analytic function. Actually he proved it for analytic families with isolated singularities [7].

Clearly any blow-analytic mapping is arc-analytic and subanalytic. The second author suggested that also the converse holds (published in [1], see also [11]). To be more precise one should assume that the function is arc-analytic and its graph is semialgebraic with respect to the target variables.

Blow-analytic maps have been studied by several authors (see the survey [5]). One disappointing fact is that blow-analytic homeomorphisms are not necessarily bilipschitz (locally) and therefore may not preserve the multiplicity of analytic curves [5], [13]. Clearly bilipschitz arc-analytic homeomorphisms preserve the multiplicity of analytic curves (for convenience we prove it in Section 3). So now it becomes quite clear that something close to the bilipschitz assumption must be added to the definition of the equivalence relation which is supposed to give the "the canonical" stratification of the singular sets of an analytic functions.

In this note we prove Implicit and Inverse Mapping Theorems in the bilipschitz arc-analytic category. A version of an Inverse Mapping Theorem is given in [6] for some special blow-analytic analytic mappings (without assuming the bilipschitz condition), see also [12] for similar results.

In Section 3 we discuss the problem of preserving multiplicities of curves by arc-analytic homeomorphisms. We also give an example which shows that it is hard to avoid the bilipschitz assumption in the Inverse Mapping Theorem. Moreover we propose a conjecture,

Date: July 18, 2000.

1991 *Mathematics Subject Classification.* Primary 32B20; Secondary 14Pxx.

Key words and phrases. arc-analytic, lipschitz, sub-analytic, homeomorphism.

The authors thank JSPS for making possible their colaboration.

namely that the bilipschitz arc-analytic homeomorphisms preserve the multiplicities of the real analytic sets. Finally, in Section 4, we give an effective characterization of the bilipschitz map-germs.

1. Local version

We say that a map-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ is *bilipschitz* if there exist positive constants c_1, c_2 such that

$$c_1|y - y'| \leq |f(y) - f(y')| \leq c_2|y - y'|, \quad \forall y,\ y' \in (\mathbf{R}^n, 0). \tag{1.1}$$

We recall now a notion which is a set-theoretic counter part of arc-analyticity. Let M be an analytic manifold, we say that $E \subset M$ is *arcwise symmetric* in M (cf. [8]), if one of the following two equivalent conditions holds:

i) $\operatorname{Int} \gamma^{-1}(E) \neq \emptyset \Longrightarrow \gamma(-1,1) \subset E$, for every analytic arc $\gamma : (-1,1) \to V \subset M$;

ii) $\gamma(-1,0) \subset E \Longrightarrow \gamma(-1,1) \subset E$, for every analytic arc $\gamma : (-1,1) \to V \subset M$.

Recall also that any arc-analytic mapping has arcwise symmetric graph (cf. [8]), but of course the converse is not true. In fact, take for example $f(x) = x^{5/3}$, $x \in \mathbf{R}$, which is a C^1 function with algebraic graph $y^3 = x^5$ and obviously f is not arc-analytic.

We state now the local version of our Implicit Function Theorem.

Theorem 1.1. *Let $F : (\mathbf{R}^m \times \mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be a subanalytic map-germ which is arc-analytic. Assume that there exists positive constants c_1, c_2 such that*

$$c_1|y - y'| \leq |F(x,y) - F(x,y')| \leq c_2|y - y'|, \quad \forall x \in (\mathbf{R}^m, 0), \forall y, y' \in (\mathbf{R}^n, 0).$$

Then there exists an arc-analytic and subanalytic map $\tau : (\mathbf{R}^m, 0) \to (\mathbf{R}^n, 0)$ such that

$$\{F(x,y) = 0\} = \{y = \tau(x)\} \tag{1.2}$$

near the origin of $\mathbf{R}^m \times \mathbf{R}^n$.

Let us put $f_x(y) = F(x,y)$, by (1.1) we know that each f_x is continuous and injective in a small neighborhood of the origin. This fact and our assumption that $F(0,0) = 0$ will imply the existence of a unique continuous solution $\tau(x)$ of $f_x(y) = 0$. This is a purely topological property (it involves only Brower's theorem on the invariance of domain). For completeness, we give a proof of this fact here.

Lemma 1.2. *Let $F : (\mathbf{R}^m \times \mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be a continuous map. Assume that there exists a positive constant c such that*

$$c|y - y'| \leq |F(x,y) - F(x,y')|, \quad \forall x \in (\mathbf{R}^m, 0), \forall y, y' \in (\mathbf{R}^n, 0). \tag{1.3}$$

Then there exists a unique continuous map $\tau : (\mathbf{R}^m, 0) \to (\mathbf{R}^n, 0)$ satisfying (1.2).

Proof. We choose D an open neighborhood of the origin in $\mathbf{R}^{m+n}$. We denote by D_x the intersection $D \cap (\{x\} \times \mathbf{R}^n)$ for x in a small neighborhood of the origin. We define

$$\varepsilon(x) = \operatorname{dist}(f_x(0), \partial f_x(D_x)),$$

where $\partial f_x(D_x)$ means the boundary of $f_x(D_x)$. This is clearly a positive number. We claim that there is no sequence x_n tending to the origin such that $\varepsilon(x_n)$ tends to zero. Indeed we may choose $y_0 \in \mathbf{R}^n$ small enough ($\neq 0$) such that D_x is a neighborhood of (x, y_0) for all x small enough. We have $|f_x(y_0) - f_x(0)| \leq \varepsilon(x)$. Because when x tends toward the origin, $|f_x(y_0) - f_x(0)|$ tends to $|f_0(y_0) - f_0(0)| > 0$. It follows that $\varepsilon(x)$ cannot tend to zero as x tends toward the origin of $\mathbf{R}^m$. This shows that our function satisfies $\varepsilon(x) \geq c > 0$ in a neighborhood of the origin. Because F is continuous, we can choose $\delta > 0$ such that $|f_x(0) - f_0(0)| \leq \frac{c}{2} < \varepsilon(x)$ for all x with $|x| < \delta$. Thus we obtain that $0 = f_0(0) \in f_x(D_x)$.

Therefore we obtain a unique continuous map $\tau : (\mathbf{R}^m, 0) \to (\mathbf{R}^n, 0)$ such that $F(x, \tau(x)) \equiv 0$ near the origin. $\square$

Proof of Theorem 1.1

The map τ is subanalytic, because its graph coincides with the zero set of F. We have to show that τ is arc-analytic.

Let $\gamma : (\mathbf{R}, 0) \to (\mathbf{R}^m, 0)$ be a germ of an analytic arc. By the definition of arc-analyticity we have to prove that $\tau \circ \gamma$ is analytic. Replacing $F(x, y)$ by $F(\gamma(t), y)$ we see that it is enough to show the following.

Proposition 1.3. *Let $F : (\mathbf{R}^m \times \mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be as in Theorem 1.1 and assume $m = 1$. Then the corresponding τ is analytic near the origin.*

Proof. Because F is arc-analytic and subanalytic, its graph is arcwise symmetric. So it follows that $\{F(x, y) = 0\}$ is arcwise symmetric and subanalytic (see [8]). Since $\{F(x, y) = 0\}$ is the graph of a subanalytic function in one variable its dimension is 1. Recall that any arcwise symmetric subanalytic set of dimension 1 is actually analytic ([8]). Hence $\{F(x, y) = 0\}$ is analytic in a neighborhood of $(0, 0) \in \mathbf{R} \times \mathbf{R}^n$. Moreover its germ at the origin is irreducible since it is a germ of the graph of function τ. Let us write $\tau = (\tau_1, ..., \tau_n)$. Each component has a Newton-Puiseux expansion

$$\tau_i(x) = \sum_{\nu \in \mathbf{N}} a^i_\nu x^{\frac{\nu}{q}}, \quad i = 1, ..., n, \tag{1.4}$$

for some positive integer q and $x > 0$.

Because the germ of $\{F(x, y) = 0\}$ is analytic and irreducible if we show that $\tau(x)$, $x > 0$, can be extended to an analytic map in a neighborhood of $0 \in \mathbf{R}$, this extension must coincide with τ itself. Hence we have to prove that there are no terms with non integer exponents in the expansion (1.4).

Let $p_i(x)$, $i = 1, ..., n$, be the analytic part of the expansion (1.4). Performing the following analytic change of variable $(x, y_1, ..., y_n) \to (x, y_1 + p_1(x), ..., y_n + p_n(x))$ we may assume that all the expansions in (1.4) start with fractional exponents if not identical zero. If all of them are zero we are done. Suppose that they are not identically zero. By assumption, there exist positive numbers c_1, c_2 (independent on x) such that

$$c_1|y| \leq |F(x, y) - F(x, 0)| \leq c_2|y|, \quad \forall y \in (\mathbf{R}^n, 0), \ \forall x \in (\mathbf{R}, 0).$$

If we replace y by $\tau(x)$ we obtain

$$c_1|\tau(x)| \leq |F(x,\tau(x)) - F(x,0)| = |F(x,0)| \leq c_2|\tau(x)|. \tag{1.5}$$

By assumption, $F(x,0)$ is analytic in x and vanishes at $x=0$. These two inequalities show that τ cannot start with a non integer power and this finishes the proof. $\square$

Theorem 1.4. *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be a subanalytic map which is arc-analytic. If f is bilipschitz, then f has an inverse which is arc-analytic (and subanalytic).*

Proof. We apply Theorem 1.1 to the map $F(x,y) = x - f(y)$. $\square$

Since, by [1], every semialgebraic and arc-analytic mapping is actually blow-analytic, we obtain

Theorem 1.5. *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be a semialgebraic map which is blow-analytic. If f is bilipschitz, then f has an inverse which is blow-analytic.*

2. Global version

The same proofs show the following global versions of Theorems 1.1, 1.4, 1.5.

Theorem 2.1. *Let $F : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^n$ be a subanalytic map which is arc-analytic. If there are positive numbers c_1, c_2 such that*

$$c_1|y - y'| \leq |F(x,y) - F(x,y')| \leq c_2|y - y'|, \quad \forall x \in \mathbf{R}^m,\ \forall y, y' \in \mathbf{R}^n,$$

then there exists an arc-analytic and subanalytic map $\tau : \mathbf{R}^m \to \mathbf{R}^n$ such that

$$\{F(x,y) = 0\} = \{y = \tau(x)\} \quad \text{in } \mathbf{R}^m \times \mathbf{R}^n.$$

Theorem 2.2. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a subanalytic map which is arc-analytic. If f is bilipschitz, then f has an inverse which is arc-analytic (and subanalytic).*

Similarly in the semialgebraic case we have

Theorem 2.3. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a semialgebraic map which is blow-analytic. If f is bilipschitz, then f has an inverse which is blow-analytic (and semialgebraic).*

3. Preserving orders and multiplicities

The role of the bilipschitz condition in the above theorems is crucial, as it shows an example given at the end of the section. Actually one could try to weaken the bilipschitz condition (1.1) in the Inverse Mapping Theorem 1.4 by fixing one variable. More precisely we say that a map-germ $f : (\mathbf{R}^n, y) \to (\mathbf{R}^k, f(y))$ is *semi-bilipschitz at the point y* if there exist positive constants c_1, c_2, such that

$$c_1|y - y'| \leq |f(y) - f(y')| \leq c_2|y - y'|, \quad \forall\ y' \in (\mathbf{R}^n, y). \tag{3.1}$$

What is interesting about semi-bilipschitz and arc-analytic maps is that they preserve the multiplicities of the real analytic curves. As a piece of terminology we recall that if $\gamma : (\mathbf{R}, 0) \to (\mathbf{R}^n, 0)$ is an analytic (non-constant) arc then there exists a unique $m \in \mathbf{N}$ such that

$$at^m \leq |\gamma(t)| \leq bt^m \tag{3.2}$$

for some $0 < a \leq b$. We call the integer $m = \operatorname{ord}_0 \gamma$ the *order* of γ (at 0). Assume moreover that γ is injective, then Γ, the image of γ, is a well defined irreducible germ of an analytic set of dimension 1, and we shall say for short that Γ is a germ of an analytic curve. So Γ has a well defined *multiplicity* at $0 \in \mathbf{R}^n$, and we denote it by $\operatorname{mult}_0 \Gamma$. Recall that

$$\text{(3.3)} \qquad \operatorname{mult}_0 \Gamma = \inf\{\operatorname{ord}_0 \gamma : \gamma \text{ is a parametrization of } \Gamma\}.$$

Moreover, $\operatorname{mult}_0 \Gamma = \operatorname{ord}_0 \gamma$ iff the complexification of γ is injective.

Clearly we have:

Proposition 3.1. *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^k, 0)$ be the germ of a semi-bilipschitz, arc-analytic mapping. Then f preserves the orders of analytic arcs, i.e., $\operatorname{ord}_0(f \circ \gamma) = \operatorname{ord}_0 \gamma$ for any analytic arc $\gamma : (\mathbf{R}, 0) \to (\mathbf{R}^n, 0)$.*

As a consequence we obtain:

Proposition 3.2. *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be the germ of a subanalytic homeomorphism. Assume that f and f^{-1} are arc-analytic. Then f is semi-bilipschitz if and only if for any germ of irreducible analytic curve Γ, the multiplicities of Γ and $f(\Gamma)$ are the same.*

Proof. Let $\gamma(t)$ be an analytic parametrisation of Γ such that its complexification is injective. Suppose that f is semi-bilipschitz, so by Proposition 3.1 and (3.3)

$$\operatorname{mult}_0 \Gamma = \operatorname{ord}_0 \gamma = \operatorname{ord}_0(f \circ \gamma) \geq \operatorname{mult}_0 f(\Gamma).$$

The same argument applied to f^{-1} gives the inverse inequality, so $\operatorname{mult}_0 \Gamma = \operatorname{mult}_0 f(\Gamma)$.

To prove the converse let us assume that f preserves the multiplicities of analytic curves but it is not semi-bilipschitz at the origin. By the curve selection lemma there exists an analytic arc $\gamma(t)$, $\gamma(0) = 0$, such that

$$\lim_{t \to 0} \frac{|f(\gamma(t))|}{|\gamma(t)|} = 0 \quad \text{or} \quad \infty.$$

We may suppose that the limit equals ∞, otherwise we replace f by f^{-1}. We may also choose γ such that $\operatorname{mult}_0 \Gamma = \operatorname{ord}_0 \gamma$. So, as above, we get

$$\operatorname{mult}_0 \Gamma = \operatorname{ord}_0 \gamma > \operatorname{ord}_0(f \circ \gamma) \geq \operatorname{mult}_0 f(\Gamma),$$

which is a contradiction. ☐

Remark. Recently G.Comte, P.D. Milman, and D.Trotman [4] observed that under a strong extra condition, semi-bilipschitz homeomorphisms preserve multiplicities of the complex hypersurfaces, which is a special case of the Zariski Conjecture about topological invariance of the multiplicities of the complex hypersurfaces.

Proposition 3.2 led us to the following conjecture.

Conjecture 3.3. *Bilipschitz arc-analytic homeomorphisms preserve the multiplicities of the real analytic sets. More precisely; let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be the germ of a subanalytic, arc-analytic, bilipschitz homeomorphism, and let X, Y be two irreducible analytic germs. Suppose that $Y = f(X)$, then $\operatorname{mult}_0 X = \operatorname{mult}_0 Y$.*

Note that it is not even known that the multiplicity is a bilipschitz invariant of the complex hypersurfaces. On the other hand in the real case clearly multiplicity is not a bilipschitz invariant. The curve $x^7 = y^3$ is bilipschitz equivalent to the line $y = 0$ but not arc-analytic bilipschitz equivalent.

As the following example shows the Inverse Mapping Theorem 1.4 does not hold if we replace the bilipschitz condition (1.1) by the semi-bilipschitz condition (3.1). Hence in some sense our results are optimal.

Example 3.4. *Let $h : \mathbf{R}^2 \to \mathbf{R}^2$ be the homeomorphism defined by*

$$h(x,y) = \left(\frac{x^2+9y^2}{x^2+y^2}x, y\right).$$

Then h is lipschitz in a neighborhood of the origin, and it satisfies also the following global inequality,

$$|(x,y)-(0,y')| \leq |h(x,y)-h(0,y')|$$

but the inverse of h is not arc-analytic.

Indeed consider the curve $a(t) = \big(t\sqrt{3(9-t)}, t\big)$. We can see that $h^{-1}(a(t))$ is not analytic (it comes to show that the equation $X^3 + Xt^3 + 2\frac{\sqrt{3(9-t)}}{9}t^4 = 0$ has no analytic solutions, and this can be done easily by inspecting the possible orders which may appear). ¿From this last example we can see that the multiplicity of the analytic set $h^{-1}(a(t))$ is not preserved under h (the image being a smooth curve).

4. Discussion of the bilipschitz condition

In this section we shall give a simple and efficient characterization of subanalytic bilipschitz map germs.

Let U be an open and convex subset of $\mathbf{R}^n$ and let $f : U \to \mathbf{R}$ be a continuous and subanalytic function. Recall that f is differentiable (even analytic) outside a closed nowhere dense subanalytic set $Z \subset U$. By the Mean Value Theorem and continuity of f we obtain

Lemma 4.1. *The function f is lipschitz if and only if all the partial derivatives of f are bounded in $U \setminus Z$, that is, there exists a constant $M > 0$ such that $|\frac{\partial f}{\partial x_i}(x)| \leq M$ for all $x \in U \setminus Z$ and $i = 1, \ldots, n$.*

We deduce now a characterization of the bilipschitz map-germs:

Proposition 4.2. *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be a continuous subanalytic map-germ, and let Z denote the set of points where f is not differentiable (recall that Z is subanalytic and nowhere dense). Then f is bilipschitz if and only if it verifies the following three conditions:*

(i) *f is injective,*
(ii) *the partial derivatives of f are bounded,*
(iii) *there exists a constant $c > 0$ such that $c \leq |\operatorname{Jac} f(x)|$, for any $x \notin Z$ close to the origin, where $\operatorname{Jac} f(x)$ is the jacobian of f at x.*

Proof. Assume that f is bilipschitz, then conditions (i), (ii) follow immediately. Let A be the jacobian matrix of f at x. Since f^{-1} is lipschitz there exists a constant $b > 0$, independent on x, such that $\|A^{-1}\| \leq b$. By elementary linear algebra, writing the module of the determinant as the volume of the image of the unit ball, we get

$$|\det A| \geq \|A^{-1}\|^{-n} \geq b^{-n}$$

(here it is convenient to take the "maximum" norm on $\mathbf{R}^n$). So condition (iii) follows with $c = b^{-n}$.

To prove the converse assume that f veryfies the above three conditions. It is enough to check that f^{-1} is lipschitz, that is, by Lemma 4.1 (f^{-1} is already continuous subanalytic) that the partial derivatives of f^{-1} are bounded. But this follows easily from the formula for the inverse matrix, since $|\operatorname{Jac} f(x)|$ is bounded away from zero and also the partial derivatives of f are bounded from above. □

Remark. If f is as above and bilipschitz it follows that there exist constants $c_1 > 0, c_2 > 0$ such that $c_1 \leq |\operatorname{Jac} f(x)| \leq c_2$, for any $x \notin Z$ close to the origin, where $\operatorname{Jac} f(x)$ is the jacobian of f at x.

We also remark that our results above hold under the weaker assumption that f is continuous and admits the Mean Value Theorem outside a thin set in the domain (in the subanalytic case this is automatically satisfied). To compare it is worth mentioning an interesting sufficient condition, for a map-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ to be bilipschitz, given in [3] (Theorem 3.12). It claims that if the closure of the convex hull of the set $\{df(x) : x \in U \setminus Z\}$, where U is a sufficiently small neighborhood of the origin, is compact and contains no singular matrix, then f is bilipschitz.

It is worth noticing that all assumptions in Proposition 4.2 are necessary, as the following example and remark show.

Example 4.3. *Let n be a non-negative integer. Let $f : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ be the map defined by $z \mapsto z^{n+1}/\bar{z}^n$. We identify $\mathbf{C}$ with $\mathbf{R}^2$. The map f is blow-analytic and hence arc-analytic and its graph is subanalytic. Moreover its jacobian is constant:*

$$\operatorname{Jac}(f) = \begin{vmatrix} (n+1)z^n/\bar{z}^n & -nz^{n+1}/\bar{z}^{n+1} \\ -n\bar{z}^{n+1}/z^{n+1} & (n+1)\bar{z}^n/z^n \end{vmatrix} = (n+1)^2 - n^2 = 2n + 1.$$

Lemma 4.1 shows that f is lipschitz. Since the mapping degree of f is $2n+1$, f does not have an inverse and hence it cannot be bilipschitz.

Remark. Also our Example 3.4 shows that it is not enough to have only $f = h$ injective and lipschitz (here again f is blow-analytic).

If we consider again our Example 3.4 and take as $f = h^{-1}$, we can see that it does not suffice that f is injective and $1/9 \leq |\operatorname{Jac} f(x)|$ (note in this case f is not arc-analytic).

However in the setting of arc-analytic maps, it is not clear for us if all the conditions in the Proposition 4.2 are necessary, in particular we have the following question.

Question 4.4. *Does there exist an arc-analytic and subanalytic map-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ with the following properties?*

(i) *There exists a positive number c such that*

$$c|y - y'| \leq |f(y) - f(y')|, \quad \forall y, y' \in (\mathbf{R}^n, 0).$$

(ii) f *is not lipschitz.*

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