Deficiencies of lattices in connected Lie groups

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ABSTRACT

We complete the determination of the groups of deficiency >0 *which occur as lattices in connected Lie groups. The torsion free groups among them are 3-manifold groups. We show that any other torsion free 3-manifold group which is such a lattice is the group of an aspherical closed geometric 3-manifold.*

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Lott showed in [Lo99] that if Γ is a lattice in a connected Lie group G and the deficiency of Γ is positive ($def(\Gamma)>0$) then either

(i) Γ has a finite normal subgroup F such that Γ/F is a lattice in $PSL(2,\mathbb{R})$; or

(ii) $def(\Gamma)=1$ and Γ is isomorphic to a torsion-free nonuniform lattice in $R\times PSL(2,\mathbb{R})$ or $PSL(2,\mathbb{C})$; or

(iii) $\Gamma\cong Z$, Z^2 or $Z\times_{-1}Z$ (the fundamental group of the Klein bottle).

This was an improvement upon earlier work of Lubotzky, who assumed G simple and either of rank ≥ 2 or $G=Sp(n,1)$ or F_4, in which cases $def(\Gamma)\leq 0$, or $G=SO(n,1)$ (for $n\geq 3$) or $SU(n,1)$ (for $n\geq 2$), in which cases $def(\Gamma)\leq 1$ [Lu83]. We shall show that in case (i) the subgroup F must be trivial, and exclude the Klein bottle group. Excepting the lattices in $PSL(2,\mathbb{R})$ with finite abelianization, all the remaining possibilities have positive deficiency.

Theorem 1. *Let Γ be a finitely presentable group with a nontrivial finite normal subgroup F such that Γ/F is a lattice in $PSL(2,\mathbb{R})$. Then $def(\Gamma) \leq 0$.*

Proof. A group has a presentation of deficiency > 0 if and only if it is the fundamental group of a finite 2-complex with Euler characteristic ≤ 0. The latter property is clearly inherited by subgroups of finite index.

Let P be a cyclic subgroup of F of prime order p. The centralizer $C_\Gamma(P)$ has finite index in Γ. Let E be a subgroup of $C_\Gamma(P)$ containing P and such that E/P is torsion free. Then E/P is either a free group or is the fundamental group of an aspherical closed surface. Let D be a subgroup of E containing P and of index divisible by p, and let $H = D/P$. Then the class in $H^2(E/P;P) \cong H^2(E/P;\mathbb{F}_p)$ corresponding to the central extension $1 \to P \to E \to E/P \to 1$ has image 0 in $H^2(H;\mathbb{F}_p)$, and so $D \cong H \times P$.

Now the deficiency of D is at most $\beta_1(D;\mathbb{F}_p) - \beta_2(D;\mathbb{F}_p) = -\beta_2(H;\mathbb{F}_p)$, and so is nonpositive. Therefore $def(\Gamma) \leq 0$, by the observation in the first paragraph of this proof. $\square$

The estimate is best possible, in general. If $H \cong F(r)$ (the free group of rank $r \geq 2$) or is a PD_2^+-group (the fundamental group of an aspherical closed orientable surface) and C is a nontrivial finite cyclic group then $H \times C$ is a lattice in $PSL(2,\mathbb{R}) \times SO(2)$, and has deficiency 0 or -1, respectively. (When $C = Z/2Z$ there are such lattices in $SL(2,\mathbb{R})$).

Theorem 2. *Let G be a connected Lie group with a virtually abelian lattice*

Γ. *Then* G/Γ *is compact.*

Proof. Let $R = Rad(G)$ and $p : G \to G/R$ the natural epimorphism. Since Γ is amenable and G/Γ has finite volume G is amenable, by Proposition 4.1.11 of [Zi]. (This follows easily from the definition of amenability). Hence G/R is compact, by Corollary 4.1.9 of [Zi]. The closure of $p(\Gamma)$ in G/R is a compact Lie subgroup, and so has finitely many components. On replacing Γ by a subgroup of finite index, if necessary, we may assume that $p(\Gamma)$ is abelian, and hence that $\overline{p(\Gamma)}$ is abelian. Let $H = p^{-1}(\overline{p(\Gamma)})$ and $\Gamma_o = H_o \cap \Gamma$. Then Γ_o is a lattice in the connected solvable Lie group H_o, and therefore is cocompact, by Theorem 3.1 of [Ra]. It follows easily that Γ is cocompact. $\square$

A crystallographic group is a discrete cocompact subgroup of $Isom(\mathbb{R}^n) = R^n \rtimes O(n)$. It is *orientable* if it is a subgroup of $R^n \rtimes SO(n)$.

Corollary. *Let* Γ *be a crystallographic group. Then* Γ *is a lattice in a connected Lie group if and only if it is orientable.*

Proof. Suppose that Γ is a lattice in the connected Lie group G. Let K be a maximal compact subgroup of G. Then Γ acts discretely and cocompactly by left multiplication on the symmetric space $X = G/K$, which is diffeomorphic to R^d for some d. If Γ is torsion free the action is free, and so $d = v.c.d.\Gamma$. Since G is connected every element of G must preserve the orientation of X. The converse is clear. $\square$

In particular, the Klein bottle group is not isomorphic to such a lattice.

Lott's list can be revised in the light of these results as follows. Either

(i) Γ is a lattice in $PSL(2,\mathbb{R})$; or

(ii) $\Gamma \cong Z \times F(r)$ for some $r \geq 0$; or

(iii) Γ is isomorphic to a torsion-free nonuniform lattice in $PSL(2,\mathbb{C})$.

The only lattices in $PSL(2,\mathbb{R})$ which do *not* have deficiency ≥ 1 are those with signature $(0; e_1, \dots e_k; t)$ where $t = 0$ or 1. (See page 99 of [Ka]).

The torsion free groups of type (i) are the free groups of finite rank ≥ 2 and the PD_2^+-groups other than Z^2, and have deficiency > 1, while the groups of types (ii) and (iii) have deficiency 1, as observed in [Lo99]. All these groups are also fundamental groups of compact orientable 3-manifolds with nonempty boundary. (A torsion free nonuniform lattice in $PSL(2,\mathbb{C})$ is the fundamental group of a compact orientable $\mathbb{H}^3$-manifold whose boundary is a nonempty union of tori. Every such manifold is homotopy equivalent to a finite aspherical 2-complex with Euler characteristic 0).

The multiplicativity of the Euler characteristic in finite coverings appears to be of little use when the deficiency is ≤ 0, and the range of examples is much greater. There are already many examples in dimension 3. If M is a compact 3-manifold then $\pi = \pi_1(M)$ has a presentation of deficiency 0, and has deficiency > 0 if and only if either ∂M has a component S with $\chi(S) \leq 0$ or M has $S^2 \times S^1$ or $S^2 \tilde{\times} S^1$ as a summand. The cocompact lattices in connected 3-dimensional Lie groups were determined in [RV81]. As $\pi_2(G) = 0$ for any Lie group 3-dimensional coset spaces G/Γ have no $S^2 \times S^1$ summands, and so such lattices have deficiency 0. More generally, we have the following result. We shall say that a compact manifold is *geometric* if its interior is homeomorphic to a manifold with a complete geometry of finite volume in the sense of Thurston [Th82].

Theorem 3. *Let M be a compact 3-manifold with fundamental group π. Then M is aspherical and geometric if and only if it has no fake 3-cells and π*

is torsion free but not free and is a lattice in a Lie group with finitely many components.

Proof. The conditions are clearly necessary. Suppose that they hold. On passing to a subgroup of finite index, if necessary, we may assume that M is orientable and π is a lattice in a connected Lie group G. Let $G_1 = Rad(G)K$, where K is the maximal compact connected normal subgroup of a Levi subgroup of G, and let $p : G \to G_2 = G/G_1$ be the canonical epimorphism. Then $G_1 \cap \pi$ and $p(\pi)$ are lattices in G_1 and G_2, respectively [Au63].

Suppose first that $G_1 \cap \pi \neq 1$. Since G_1 is an extension of a compact group by a solvable group $G_1 \cap \pi$ is amenable, by Proposition 4.1.11 of [Zi]. Since the compact quotient $G_1/Rad(G) \cong K$ is semisimple $G_1 \cap \pi$ is virtually solvable, by Tits' theorem. Hence M is either Seifert fibred or is a $\mathbb{S}ol^3$-manifold [Ga92]. Thus either M is a (closed) $\mathbb{R}^3$-, $\mathbb{N}il^3$- or $\mathbb{S}ol^3$-manifold, or its interior is a $\mathbb{H}^2 \times \mathbb{E}^1$- or $\widetilde{SL}$-manifold.

If π has no nontrivial solvable normal subgroup then we may assume that $G = G_2$, which is a semisimple Lie group whose Lie algebra has no compact factors. The group π therefore acts discretely with finite covolume on the symmetric space X of compact subgroups of G. Moreover π is irreducible. Therefore X must be typewise homogeneous, and has no Euclidean factor. (See §7 of Chapter IX of [Ma]).

If X has rank $r(X) \geq 2$ then π is arithmetic, by Theorem IX.1.11 of [Ma]. Hence $c.d.\pi = dim(X) - r_{\mathbb{Q}}(G)$, where $r_{\mathbb{Q}}(G) \leq r(X)$ is the $\mathbb{Q}$-rank of G [BS73]. If moreover $r_{\mathbb{Q}}(G) > 0$ then π is not a Poincaré duality group, and so $c.d.\pi \leq 2$. However there are no symmetric spaces of rank ≥ 2 and dimension ≤ 4. (See §6 of Chapter X of [He]). Therefore we may assume that X has rank 1. If now X/π is compact then $c.d.\pi = dim(X)$. If X/π is not compact then $c.d.\pi = dim(X) - 1$, and π contains parabolic subgroups of the same cohomological dimension. Since $c.d.\pi \leq 3$ we may conclude that $X = \mathbb{H}^2$ or $\mathbb{H}^3$, in either case. If $X = \mathbb{H}^2$ then π is a PD_2^+-group (since it is not free) and the interior of M is a $\mathbb{H}^2 \times \mathbb{E}^1$-manifold. If $X = \mathbb{H}^3$ then M is homotopy equivalent to an $\mathbb{H}^3$-manifold, and therefore is homeomorphic to an $\mathbb{H}^3$-manifold, by [GMT96]. $\Box$

The fundamental group $\pi = \pi_1(M)$ of any compact 3-manifold M is virtually torsion free. If π has nontrivial torsion then M is not aspherical. Therefore if moreover π is isomorphic to a lattice in a Lie group with finitely many components it must be virtually free, by the theorem. Hence it is a free product of finite 3-manifold groups and a free group. Finite cyclic groups are lattices in $SO(2)$. The group $\pi_1(RP^3 \sharp RP^3) \cong (Z/2Z) * (Z/2Z) = Z \rtimes (Z/2Z)$ is virtually Z and is a lattice in $Isom(\mathbb{R}) = R \rtimes (Z/2Z)$. However, it is not a lattice in a connected Lie group, by the Corollary to Theorem 2. All other free products of cyclic groups are isomorphic to lattices in $PSL(2, \mathbb{R})$ with signature of the form $(0; e_1, \dots e_k; t)$, where $t > 0$.

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