Some questions on subgroups of 3-dimensional Poincaré duality groups

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ABSTRACT

We state a number of open questions on 3-dimensional Poincaré duality groups and their subgroups, motivated by considerations from 3-manifold topology.

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The notion of Poincaré duality group of dimension n (or PD_n-group, for short) is essentially an algebraic analogue of the notion of aspherical n-manifold. When $n = 1$ or 2 the analogy is precise; the only such groups are the fundamental groups of the circle and the aspherical closed surfaces, and two such manifolds are homeomorphic if and only if their groups are isomorphic [EM80, EL81]. As is well known, the fundamental group also plays a central role in 3-dimensional topology. Every (orientable) closed 3-manifold admits an essentially unique decomposition with prime factors which either have finite fundamental group, or are $S^1 \times S^2$ or are aspherical. Whether every homotopy equivalence between irreducible aspherical 3-manifolds is homotopic to a homeomorphism is an open question. It is natural to ask also whether every PD_3-group is the fundamental group of some 3-manifold. Affirmative answers to both of these questions would suggest that a large part of the study of 3-manifolds may be reduced to algebra. In fact, each of these questions has a positive answer when the (fundamental) group has sufficiently nice subgroups. It should be noted also that PD_3-groups may occur in other contexts, where the geometry does not immediately provide a corresponding 3-manifold (cf. [H]).

Here we shall present a number of questions on subgroups of PD_3-groups, motivated by results conjectured or already established geometrically for 3-manifold groups. The corresponding questions for subgroups of (finitely generated, indecomposable) groups of cohomological dimension 2 should be considered with these. (See also the lists of problems in [W], [GS] and [NR]). Any group with a finite 2-dimensional Eilenberg - Mac Lane complex is the fundamental group of a compact aspherical 4-manifold with boundary, obtained by attaching 1- and 2-handles to D^4. On applying the orbifold hyperbolization technique of Gromov, Davis and Januszkiewicz to the boundary we see that each such group embeds in a PD_4-group. (See [DJ91] and [Me90]). (Conjecturally such groups are exactly the finitely presentable groups of cohomological dimension 2). Thus the case considered here is critical.

We shall assume throughout that G is an orientable PD_3-group. The normalizer and centralizer of a subgroup H of G shall be denoted by $N_G(H)$ and $C_G(H)$, respectively. We shall also let $\zeta G = C_G(G)$, G' and $G^{(\omega)} = \cap G^{(n)}$ denote the centre, the commutator subgroup and the intersection of the terms of the derived series of G, respectively. A group has a given property *virtually* if it has a subgroup of finite index with that property. Since our interest is in PD_3-groups, we shall use "3-manifold group" to mean fundamental group of an aspherical closed 3-manifold. (This usage is not standard).

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The group

To begin with, the augmentation $\mathbb{Z}[G]$-module Z has a finite projective resolution, and there is a 3-dimensional $K(G,1)$ complex. Hence G is almost finitely presentable (FP_2). The $K(G,1)$-complex is finitely dominated, and hence a Poincaré complex in the sense of [Wl67], if and only if G is finitely presentable. (For every $n \geq 4$ there are PD_n-groups which are not finitely presentable [Da98]). Since every PD_n-complex may be obtained by attaching an n-cell to a complex which is homologically of dimension $< n$, the exotic characteristic class of $K(G,1)$ in $H^3(G;\mathbb{F}_2)$ is trivial, and so $K(G,1)$ has a normal invariant. Bowditch has recently shown that if G is an FP_2 group such that $H^2(G;\mathbb{Z}[G])\cong Z$ then G is virtually a PD_2-group.

If $\beta_1(G)>0$ then G satisfies the Strong Bass Conjecture, as a consequence of [Co95]. If M is an aspherical 3-manifold then M is homeomorphic to a finite complex and so $\pi_1(M)$ is finitely presentable and FF. If moreover M is Haken then $\tilde{K}_0(\mathbb{Z}[\pi_1(M)])=Wh(\pi_1(M))=0$ and $\tilde{M}\cong R^3$, so $\pi_1(M)$ is 1-connected at ∞.

(1) Is G finitely presentable? If so, does it have a presentation of deficiency 0?

(2) Is G FF, i.e., does the augmentation module Z have a finite *free* $\mathbb{Z}[G]$-resolution?

(3) Is $\tilde{K}_0(\mathbb{Z}[G])=0$? Is $Wh(G)=0$? Does G satisfy the (Strong) Bass conjecture?

(4) Is G 1-connected at ∞? Does it have a boundary in the sense of [Be96]?

(5) Is $K(G,1)$ homotopy equivalent to a finite complex?

(6) If G is an FP_3 group such that $H^3(G;\mathbb{Z}[G])\cong Z$ is G virtually a PD_3-group?

Subgroups in general

Since G has cohomological dimension 3 it has no nontrivial finite subgroups. Any nontrivial element g generates an infinite cyclic subgroup $\langle g\rangle$; it is not known whether there need be any other proper subgroups. If a subgroup H of G has finite index then it is also a PD_3-group. The cases when $[G:H]$ is infinite are of more interest, and then either $c.d.H=2$ or H is free, by [St77] and [S]. If there is a finitely generated (respectively, FP_2) subgroup of cohomological dimension 2 there is one such which has one end (i.e., which is indecomposable with respect to free product). An abelian subgroup A of rank ≥ 2 must be finitely generated, since either $[G:A]$ is finite or $c.d.A=2=rankA$.

3-manifold groups are coherent: finitely generated subgroups are finitely presentable. In fact something stronger is true: such a subgroup is the fundamental group of a compact 3-manifold (possibly with boundary) [Sc73]. If $\mathbb{Z}[G]$ is coherent as a *ring* then every finitely generated subgroup of G is FP_2.

In general PD_n-groups with $n \geq 4$ are not coherent. If π is a surface group with $\chi(\pi) < 0$ then it has a noncyclic free subgroup F and so $F \times F$ is a subgroup of $\pi \times \pi$. It is well known that such groups $F \times F$ have finitely generated subgroups which are not finitely related (cf. Section 8.2 of [B]). However $F \times F$ is not a subgroup of any PD_3-group [KR89].

If M is a closed orientable 3-manifold then a transversality argument implies that every element of $H_2(M;\mathbb{Z}) \cong H^1(M;\mathbb{Z}) \cong [M;S^1]$ is represented by an embedded submanifold. As we may assume the submanifold incompressible, if M is aspherical it follows that every element of $H_2(\pi_1(M);\mathbb{Z})$ is represented by a surface subgroup of $\pi_1(M)$. If G/G' is infinite then G is an HNN extension with finitely generated base and associated subgroups [BS78], and so has a finitely generated subgroup of cohomological dimension 2.

(7) Is there a noncyclic proper subgroup? If so, is there one of cohomological dimension 2? and finitely generated?

(8) Is there a subgroup which is a surface group? Is every element of $H_2(G;\mathbb{Z})$ represented by such a subgroup?

(9) Is G coherent? Is $\mathbb{Z}[G]$ coherent as a ring?

(10) Does every (finitely presentable) subgroup of cohomological dimension 2 have a (finite) 2-dimensional Eilenberg-Mac Lane complex?

(11) If H is a finitely generated subgroup with one end, is there a system of monomorphisms σ such that (H,σ) is a PD_3-pair? In particular, does every such subgroup of infinite index have infinite abelianization? contain a surface group?

Subnormal subgroups

If N is an almost finitely presentable subnormal subgroup of G and $c.d.N = 2$ then it is a surface group and G has a subgroup of finite index which is a surface bundle group. If $c.d.N = 1$ then $N \cong Z$ and either G is virtually poly-Z or N is normal in G and $[G : C_G(N)] \leq 2$ [BH91]. In the latter case G is the group of a Seifert fibred 3-manifold [Bo99]. It is easy to find examples among normal subgroups of 3-manifold groups to show that finite generation of N is necessary for these results.

If N is finitely generated, normal and $[G:N]=\infty$ then $H^1(G/N;\mathbb{Z}[G/N])$ is isomorphic to $H^1(G;\mathbb{Z}[G/N])$ and hence to $H_2(G;\mathbb{Z}[G/N])\cong H_2(N;\mathbb{Z})$, by Poincaré duality. In particular, G/N has two ends if and only if $H_2(N;\mathbb{Z})\cong Z$. In the latter case Shapiro's lemma and Poincaré duality together imply that $H^2(N;lim_{\rightarrow}M_i)$ is 0 for any direct system M_i with limit 0. Hence N is FP_2 by Brown's criterion [Br75] and so is a surface group by the above result.

(12) Is there a simple PD_3-group?

(13) Is G virtually representable onto Z?

(14) Must a finitely generated *normal* subgroup N be finitely presentable?

(15) Suppose $N\leq U$ are subgroups of G with U finitely generated, $[G:U]$ infinite, N subnormal in G and N not cyclic. Is $[G:N_G(U)]<\infty$? (Cf. [El84]).

Normalizers

If x is a nontrivial element of G then $\langle x\rangle$ is infinite cyclic and so $[N_G(\langle x\rangle):C_G(x)]\leq 2$. (See below for comments on centralizers). If F is a free subgroup of G of finite rank $\mu>1$ then $c.d.N_G(F)\leq 2$ [Hi 87]. If H is another free subgroup of G in which F is a subnormal subgroup then $[H:F]$ is finite, and $\mu-1=[H:F](rankH-1)$. We may therefore suppose that either $N_G(F)=F$ or $c.d.N_G(F)=2$, and that $N_G(F)$ is an increasing union of finitely generated groups of cohomological dimension 2. By Corollary 8.7 of [B] each such group is the group of a finite graph of free groups. In particular, $N_G(F)/F$ is locally free-by-finite. If G is a 3-manifold group then $N_G(F)/F$ cannot have any nonfinitely generated subgroups [Sc76] and so must be finite or virtually Z.

Let H be an almost finitely presentable subgroup of infinite index in G and with one end, and define an increasing sequence of subgroups $\{H_i|i\geq 0\}$ by $H_0=H$ and $H_i=N_G(H_{i-1})$ for $i>0$. Then $\hat{H}=\cup H_i$ is almost finitely presentable and either $c.d.\hat{H}=2$, $\hat{H}$ has one end and $N_G(\hat{H})=\hat{H}$, or $\hat{H}$ is a PD_3-group and G is virtually the group of a surface bundle [Hi88]. (An easy application of the LHSSS shows that if H is normal in $J\leq G$ and $[J:H]$ is infinite then $H^r(J;W)=0$ for any free $\mathbb{Z}[J]$-module W and for $r\leq 2$. Hence

$c.d.J = 3$ and so $[G : J]$ is finite. If $[H_{i+1} : H_i]$ is finite for all i then $c.d.\hat{H} = 2$ and so $\hat{H}$ is finitely generated, by Theorem 3.3 of [GS81]). In particular, if G has a subgroup H which is a surface group with $\chi(H) = 0$ (respectively, < 0) then either it has such a subgroup which is its own normalizer in G or G is virtually the group of a surface bundle.

The *commensurator* of a subgroup H is the subgroup $Comm_G(H)$ of elements g of G such that $[H : H \cap gHg^{-1}]$ and $[H : H \cap g^{-1}Hg]$ are both finite. It clearly contains $N_G(H)$.

(16) If F is a finitely generated noncyclic free subgroup is $N_G(F)$ finitely generated?

(17) If $c.d.N_G(F) = 2$ must $N_G(F)$ contain a copy of Z^2?

(18) Is $Comm_G(\langle x \rangle) = \cup N_G(\langle x^n \rangle)$? If not, there is an element t in G such that $tx^pt^{-1} = x^q$ for some integers p and q with $|p| \neq |q|$. The subgroup generated by t and x is a quotient of a Baumslag-Solitar group.

Centralizers

Bowditch has recently shown that if G is a PD_3-group with nontrivial centre then G is the fundamental group of an aspherical Seifert fibred 3-manifold [Bo99]. In particular, ζG is finitely generated. Since an elementary amenable group of finite cohomological dimension is virtually solvable [HL 92], it follows also that either G is virtually poly-Z or its maximal elementary amenable normal subgroup is cyclic.

A group has *Max-c* if every strictly increasing sequence of centralizers is finite. Every 3-manifold group has *Max-c* [Kr90a]. On the other hand, as the Baumslag-Solitar groups $\langle x, t \mid tx^pt^{-1} = x^q \rangle$ embed in PD_4-groups, there are PD_4-groups which do not have *Max-c*. The centralizer of any element in the group of a Haken 3-manifold is finitely generated [JS79].

Let $H^\circ = C_G(C_G(H))$. Then $H \subseteq H^\circ$, while $C_G(H^\circ) = C_G(H)^\circ = C_G(H)$ and so $H^{\circ\circ} = H^\circ$. If $C_G(H)$ is abelian then it equals ζH°; if moreover it is not infinite cyclic then either $C_G(H) = H^\circ$ and is a maximal abelian subgroup or $[G : H^\circ]$ is finite. If $C_G(H)$ is nonabelian then either $c.d.C_G(H) = 2$ and $H^\circ = \zeta C_G(H)$ and is infinite cyclic or $[G : C_G(H)]$ is finite. (The possibility that $C_G(H)$ might be a noncyclic free group may be ruled out by [KR89]).

The centralizer $C_G(x)$ clearly contains all the roots of x, i.e., the elements g such that x is in $\langle g\rangle$.

(19) Is $C_G(x)$ always finitely generated?

(20) Does every PD_3-group have Max-c? In particular, suppose that $\{C_n\}$ is an increasing sequence of subgroups of G such that ζC_n is infinite cyclic, $C_n = C_G(\zeta C_n)$ and $c.d.C_n = 2$ for all $n \geq 1$. Must such a sequence stabilize?

The derived series and perfect subgroups

The intersection $P = \cap G^{(\alpha)}$ of the terms of the transfinite derived series for G is a perfect normal subgroup of G and is the maximal perfect subgroup. Either G/P is finite (and is a solvable group with cohomological period dividing 4) or $c.d.P = 2$ or $P = 1$. If $c.d.P = 2$ then P cannot be FP_2, for otherwise it would be a surface group [Hi87]. Note that $P \subseteq G^{(\omega)}$, and if $c.d.P = 2$ then $c.d.G^{(\omega)} = 2$ also. If $[G : P]$ is infinite and $\zeta G \neq 1$ then $P = 1$.

If G is not virtually representable onto Z then either $[G : G^{(\omega)}]$ is finite and $G^{(\omega)}$ is a perfect PD_3-group (and $G/G^{(\omega)}$ is a finite solvable group with cohomological period dividing 4) or $G/G^{(\omega)}$ is a finitely generated infinite residually finite-solvable group with one or infinitely many ends. If ζG is not finitely generated and $[G : G^{(\omega)}]$ is infinite then $G^{(\omega)} \subseteq \zeta G$ and $G/G^{(\omega)}$ has one end.

If H is a nontrivial finitely generated subgroup of a 3-manifold group G such that $H^1(H; Z) = 0$ then $[G : H]$ is finite [Ho82]. However if G is the group of the (aspherical) 3-manifold obtained by 0-framed surgery on a nontrivial knot with $\Delta = 1$ then G' is a perfect normal subgroup which is not finitely generated. Note also that there is a superperfect group with a finite 2-dimensional Eilenberg-Mac Lane complex [DV73].

(21) Can a nontrivial finitely generated normal subgroup of infinite index be perfect?

(22) If a finitely generated, infinite residually solvable group has infinitely many ends must it be virtually representable onto Z?

(23) If $P = 1$ is G residually solvable (i.e., is $G^{(\omega)} = 1$ also)?

The Tits alternative

Let N be the subgroup generated by all the normal subgroups which have no nonabelian free subgroup. Then N is the maximal such subgroup, and clearly it contains the maximal elementary amenable normal subgroup of G. If N is nontrivial then either $N \cong Z$, $c.d.N = 2$ or $N = G$. If N is a rank 1 abelian group then $N \cong Z$ [Bo99]. If $c.d.N = 2$ then N cannot be FP_2, for otherwise it would be a surface group and G would be virtually the group of a surface bundle [Hi87]. Since N has no nonabelian free subgroup this would imply that N and hence G are virtually poly-Z, and so $N = G$. Similarly, if $N = G$ and G/G' has rank at least 2 then there is an epimorphism $\phi : G \to Z$ with finitely generated kernel [BNS87]. Hence $ker\phi$ is a surface group and so G is poly-Z.

Subgroups which are coherent, locally virtually indicable and contain no nonabelian free subgroups are virtually solvable [Hi97]. If G is the fundamental group of a Haken 3-manifold then subgroups which contain no noncyclic free group are virtually poly-Z [EJ73].

(24) Is N the maximal elementary amenable normal subgroup?

(25) If H is a finitely generated subgroup which has no nonabelian free subgroup must it be virtually poly-Z?

Atoroidal groups

We shall say that G is *atoroidal* if all of its finitely generated abelian subgroups are cyclic. Two-generator subgroups of atoroidal 3-manifold groups are either free or of finite index [JS79]. If G is the group of an atoroidal virtually Haken 3-manifold then it can be embedded as a discrete uniform subgroup of $PSL(2, \mathbb{C})$ [Th82, CS83]. Question 18 of [Th82] asks whether every closed hyperbolic 3-manifold has a finite covering space which fibres over the circle.

(26) Is every atoroidal PD_3-group *automatic* [Ep92]? *negatively curved* [AS91]? isomorphic to a discrete uniform subgroup of $PSL(2, \mathbb{C})$?

(27) Does every atoroidal PD_3-group have a nontrivial finitely generated subnormal subgroup of infinite index? (Cf. question 18 of [Th82]).

Splitting

The central role played by incompressible surfaces in the geometric study of Haken 3-manifolds suggests strongly the importance of splitting theorems for PD_3-groups. This issue was raised in [Th84], the first paper on PD_3-groups. Since then the most substantial results on splittings of Poincaré duality groups are to be found in the papers [Du89], [Kr90,93] and [KR88,88a,89,89a].

If G has no noncyclic free subgroup and G/G' is infinite then G is an ascending HNN extension with finitely generated base and associated subgroups. If G is residually finite and has a subgroup isomorphic to Z^2 then either G is virtually poly-Z or it has subgroups of finite index with abelianization of arbitrarily large rank. (By [KR88] we may suppose that G splits over Z^2, and then we may use the argument of [Ko87], which is essentially algebraic).

If G has Max-c and has a subgroup $H \cong Z^2$ then either G splits over a subgroup commensurate with H or it has a nontrivial abelian normal subgroup [Kr90]. In the latter case it is a 3-manifold group [Bo99].

(28) If G is a nontrivial free product with amalgamation or HNN extension does it split over a surface group?

(29) If G is a nontrivial free product with amalgamation is it virtually representable onto Z?

(30) Can G be a properly ascending HNN extension?

(31) If G has an almost finitely presentable, one ended subgroup H such that $c.d.H = 2$ and $N_G(H) = H$ does G have a subgroup of finite index which splits over a subgroup commensurate with H? In particular is this so if $H \cong Z^2$?

(32) Suppose G is not virtually poly-Z and that G/G' is infinite. Does G have subgroups of finite index whose abelianization has rank ≥ 2?

(33) Suppose that G is an HNN extension with stable letter t, base H and associated subgroup $F \subset H$. Is $\mu(G) = \cap t^k F t^{-k}$ finitely generated? (See [Ka 89] for a related result on knot groups, and also [So 91]).

Residual finiteness, hopficity, cohopficity, Baumslag-Solitar groups

Let $K_n = \cap\{H \subset G | [G : H] \ divides \ n!\}$. Then $[G : K_n]$ is finite, for all

$n \geq 1$, and G is residually finite if and only if $\cap K_n = 1$. If G is not virtually representable onto Z then this intersection is also the intersection of the terms in the more rapidly descending series given by $L_n = K_n^{(n)}$, and is contained in $G^{(\omega)}$.

A residually finite PD_3-group which has a subgroup $H \cong Z^2$ is virtually split over a subgroup commensurate with H [KR88]. The groups of virtually Haken 3-manifolds are residually finite [He87]. Finitely generated residually finite groups are hopfian, i.e., onto endomorphisms of such groups are automorphisms. The Baumslag-Solitar groups $\langle x, t \mid tx^pt^{-1} = x^q \rangle$ embed in PD_4-groups. Since these groups are not hopfian, there are PD_4-groups which are not residually finite [Me90].

The dual notion of cohopficity (injective endomorphisms are automorphisms) has been studied for 3-manifold groups in [GW94].

Let X be the class of groups of cohomological dimension 2 which have an infinite cyclic subgroup which is commensurate with all of its conjugates. The finitely generated groups in this class are the fundamental groups of finite graphs of groups in which each vertex group and each edge group is infinite cyclic. This class includes the torus knot groups and the Baumslag-Solitar groups. If G is a PD_3-group with no nontrivial abelian normal subgroup and which contains a subgroup isomorphic to Z^2 then G splits over an $\mathcal{X}$-group [Kr93]. (See also [Kr90b]).

(34) Are all PD_3-groups residually finite?

(35) Which X-groups are subgroups of PD_3-groups?

Other questions

We conclude with some somewhat more topological questions. If G is a PD_3-group which has a subgroup isomorphic to $\pi_1(M)$ where M is an aspherical 3-manifold which is Seifert fibred, Haken or hyperbolic then G is a 3-manifold group (by Section 63 of [Z], [Zi82] and Mostow rigidity, respectively).

Every PD_3-complex is a connected sum of PD_3-complexes whose fundamental groups are indecomposable with respect to free product [Tu89]. The indecomposable summands are either aspherical or have virtually free fundamental group [Cr95]).

If P is a PD_3-complex with fundamental group π then $\pi_2(P) \cong \overline{H^1(\pi;\mathbb{Z}[\pi])}$ as a left $\mathbb{Z}[\pi]$-module. If π is torsion free but not free then $H^2(P;\pi_2(P)) = 0$, $\pi_2(P)$ is a projective $\mathbb{Z}[\pi]$-module and any two of the conditions "π is FF", "P is homotopy equivalent to a finite complex" and "$\pi_2(P)$ is stably free" imply the third. Moreover if π is a nontrivial free group then $\pi_2(P)$ has projective dimension 1 and $H^2(P;\pi_2(P)) \cong Z$. (See [H]). (If π is not torsion free then the projective dimension of $\pi_2(P)$ is infinite). The cohomology group $H^2(P;\pi_2(P))$ arises in studying homotopy classes of self homotopy equivalences of P. Hendriks and Laudenbach show that if N is a P^2-irreducible 3-manifold and $\pi_1(N)$ is virtually free then $H^2(N;\pi_2(N)) \cong Z$, and otherwise $H^2(N;\pi_2(N)) = 0$ [HL74].

(36) Suppose that G is a PD_3-group (*not* necessarily orientable) which is virtually a 3-manifold group. Must G itself be a 3-manifold group?

(37) Let X be an indecomposable, orientable PD_3-complex such that $\pi_1(X)$ is infinite andvirtually free. Must $\pi_1(X)$ be infinite cyclic?

(38) Let X be an orientable PD_3-complex such that $\pi_1(X)$ is not virtually free. Is $H^2(X;\pi_2(X)) = 0$? If $\pi_1(X)$ is torsion free, is $\pi_2(X)$ a free $\pi_1(X)$-module?

(39) What can be said about maps of nonzero degree between PD_3-groups? (See [GW], [GW94] and [Wn91,93]).

(40) Is there a purely algebraic analogue of orbifold hyperbolization which may be used to show that every FP group of cohomological dimension k is a subgroup of a PD_{2k}-group?

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