# IMPLICIT FUNCTION THEOREM FOR LOCALLY BLOW-ANALYTIC FUNCTIONS

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Abstract. In this paper we prove the implicit function theorem for the *locally blow-analytic functions* ( see for instance [12], [11], [1] for locally blow-analytic functions, and [2], [7], [6], [8] for blow-analytic functions), and as an interesting application, of using blow-analytic homeomorphisms, we describe a very easy way to resolve analytic curves on analytic varieties.

## 1. Blow-analytic category

In this work we continue the study started in [3], describing new properties of the blow-analytic maps and finding criteria for blow-analytic homeomorphisms. This paper generalises to arbitrary modifications, some of the results in [3] where we treat mostly the case of toric modifications. Here we enlarge our category to local blow-analytic functions allowing in this way apparently a larger class of modifications.

The main difficulty concerning *blow-analytic category* , as far as calculus is concern, comes from the facts that it is not closed neither to differentiation nor to integration, and also there is no global chain rule.

For the reader's convenience we recall here the basic notions related to the blow-analytic category.

Let $U$ be a neighbourhood of the origin of $\mathbf{R}^n$ and $\pi : M \to U$ be a proper analytic real modification whose complexification (see [4]) is also a proper modification near its real part (we often simply say that "$\pi$ is a modification"). For instance $x \to x^3$ is not a modification in our sense.

Let $f : U \dashrightarrow \mathbf{R}^m$ denote a map defined on $U$ except possibly some thin subset of $U$. In this case, we shall simply say that $f$ is defined almost everywhere. From now on if we do not use dasharrows we understand that the map is defined everywhere. We say that $f$ is *blow-analytic* via $\pi$ if $f \circ \pi$ has an analytic extension on $M$.

We say that $f$ is *blow-analytic* if it does so via some modification. It will follow that *blow-analytic* maps are analytic except a thin set.

Some typical examples of blow-analytic functions are the following:

$$f(x,y) = \frac{x^2+y^2}{x^2+2y^2}$$

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which is defined everywhere except the origin, and

$$f(x,y) = \frac{xy^2}{x^2+y^2}$$

which can be extended continuously everywhere.

We say that $f$ is *blow-meromorphic via* $\pi$ if $f \circ \pi$ can be written as a meromorphic map on $M$.

We say that $f$ is *blow-meromorphic* if it does so via some modification. It will follow that *blow-meromorphic* maps are analytic except a thin set.

Let $P$ be a function defined almost everywhere on $U$.

We say that $P$ is a *blow-analytic unit* via a modification $\pi : M \to U$, if $P \circ \pi$ extends to an analytic function on $M$, which is a unit as an analytic function. It will follow that $P$ is bounded away from zero both from below and above and also it has constant sign.

We say that $f$ is *locally blow-analytic* via a locally finite collection of analytic modifications $\sigma_\alpha : U_\alpha \to \mathbf{R}^n$ if for each $\alpha$ we have

(i) $\sigma_\alpha$ is the composition of finitely many local blowings-up with smooth nowhere dense centres and $f \circ \sigma_\alpha$ has an analytic extension on $U_\alpha$.

(ii) There are subanalytic compact subsets $K_\alpha \subset U_\alpha$ such that $\bigcup \sigma_\alpha(K_\alpha) = \bar{U}$.

Let $U_1$, $U_2$ be two neighbourhoods of the origin of $\mathbf{R}^n$. $h : U_1 \to U_2$ is a *blow-analytic homeomorphism* if $h : U_1 \to U_2$ is a homeomorphism and there is an analytic isomorphism of pairs $H : (M_1, E_1) \to (M_2, E_2)$ so that $h \circ \pi_1 = \pi_2 \circ H$ for some modifications $\pi_i : M_i \to U_i$, $i = 1, 2$, where $E_i$ (as analytic spaces) denote the critical loci of $\pi_i$, $i = 1, 2$. If there is a modification-germ $\pi : (M, \pi^{-1}(0)) \to (\mathbf{R}^n, 0)$ such that $\pi_1$, $\pi_2$ are its representatives, we call $h$ a *blow-analytic homeomorphism(-germ)* via $\pi$.

Let $U_1$, $U_2$ be two neighbourhoods of the origin of $\mathbf{R}^n$ and $\mathbf{R}^p$ respectively, and consider $f : U_1 \dashrightarrow U_2$ a map defined on $U_1$ except possibly some thin subset of $U_1$.

We say that $f$ has the *arc-lifting-property* (**alp** for short) if by definition, for any given analytic arc $\alpha : (\mathbf{R}, 0) \to (U_2, 0)$ there exists an analytic arc $\beta : (\mathbf{R}, 0) \to (U_1, 0)$, such that we have $f \circ \beta = \alpha$. If $f$ has **alp** then clearly $f$ must be surjective.

The importance of this notion in our set up is justified by the fact that a blow-analytic map $h : U_1 \to U_2$, ( $U_1$, $U_2$ neighbourhoods of the origin in $\mathbf{R}^n$), which is also a homeomorphism, is a blow-analytic homeomorphism if and only if it has **alp**.

Note that in the case when $f$ is a finite analytic map we have **alp** if and only if each fibre of $f$ contains at least one point where its jacobian matrix has maximal rank.

**Remark 1.1.** *The notion of (locally) blow-analytic functions (or maps) is very much related to the notion of arc-analytic functions introduced in* [10]. *These are functions $f : U \to \mathbf{R}$ such that $f \circ \alpha$ is analytic for any analytic arc $\alpha : I \to U$ (here $U$ is an open subset of $\mathbf{R}^n$ and $I$ is an open interval. Indeed in* [1] *it is proved in particular that an arc-analytic function has sub-analytic graph if and only if it is locally blow-analytic. It is clear that if $f$ is blow-analytic in our sense then it is automatically arc-analytic with respect to the analytic arcs not contained in the set where $f$ is not defined.*

As we already have mentioned before the main difficulty concerning *blow-analytic category*, as far as calculus is concern, comes from the facts that it is not closed neither to differentiation nor to integration, and also there is no global chain rule. For instance the following blow-analytic homeomorphism has its jacobian matrix with all entries non arc-analytic functions:

**Example 1.2.** *Let $h : (\mathbf{R}^3, C) \to (\mathbf{R}^3, C)$ be the map-germ defined by*

$$(x, y, z) \mapsto (x + f(x + y, z), y + x + f(x + y, z), z + x + f(x + y, z))$$

*where $f(u, v) = \frac{uv^5}{u^4+v^5}\cdot$ and $C$ is the $z$-axis.*

## 2. Blow-Analytic Homeomorphisms

Let $U$ be an open neighbourhood of the origin in $\mathbf{R}^n$ and $f : U \to \mathbf{R}$ be an arc-analytic function. It is easy to see that at each point of $U$ we have well defined partial derivatives. However, in general, they are no longer arc-analytic functions. If moreover $f$ is a blow-analytic function, then it is clear that its partial derivatives are analytic except a thin set. In fact we have even more.

**Lemma 2.1.** *Let $U$ be an open neighbourhood of the origin in $\mathbf{R}^n$ and $f : U \to \mathbf{R}$ be a blow-analytic function via a modification $\pi : (M, E) \to (U, C)$, where $E$ critical locus of $\pi$. Then the partial derivatives of $f : U \setminus C \to \mathbf{R}$ are blow-meromorphic via $\pi$.*

*Proof.* Let $h = f \circ \pi$ denote the function obtained by composing $f$ and $\pi$. By assumption $h$ is analytic. Using the chain rule (outside $C$) we deduce that $\mathrm{grad}(f)(\pi(X))d\pi(X) = \mathrm{grad}(h)(X)$ where $X$ represents some local coordinates systems in $M \setminus E$. From this we can solve

$$\mathrm{grad}(f)(\pi(X)) = \frac{\mathrm{grad}(h)(X)d\pi^*(X)}{\det d\pi(X)}\cdot$$

Actually with more care one can prove that $\mathrm{grad}(f)(x) = \frac{h(x)}{p(x)}$ where $h$ is a blow-analytic map via $\pi$ on $U$ and $p$ is an analytic function on $U$. Indeed we may assume that locally the components of $\pi$ and $\det d\pi(X)$ are monomials (modulo analytic units), so we can find a polynomial $p(x)$ such that $\frac{p(\pi(X))}{\det d\pi(X)}$ is analytic in $X$. This in turn will give us that $h(x) = p(x) \cdot \mathrm{grad} f(x)$ is blow-analytic via $\pi$.

□

**Remark 2.2.** *Note that even if the meromorphic function obtained in this way is defined at some points in $E$ then $\mathrm{grad}(f)(\pi(x))$ may not coincide with the value calculated using the definition at those points. This is true in general for blow-meromorphic maps defined globally on $U$. We do not expect that the preexisting values of our map outside the analyticity set will coincide with those possible extra values coming from blowing-up. The next example will make this clear.*

**Example 2.3.** *Let $h : (\mathbf{R}^3, C) \to (\mathbf{R}^3, h(C))$ be a map-germ defined by*

$$(x, y, z) \mapsto \big(x + z^3, y + z^2, z - 2f(x + z^3, y + z^2)\big)$$

*where $f(u, v) = \frac{uv^5}{u^4+v^5}\cdot$ and $C$ is the $z$-axis.*

This map is a blow-analytic homeomorphism and therefore its differential $dh$ is well defined at every point and so it is its jacobian determinant. An easy direct computation shows that $\frac{\partial h}{\partial z}$ vanishes at the origin. However, the jacobian determinant is equal to 1 outside a thin set, so it is a blow-analytic unit in our sense.

Let $h : (\mathbf{R}^n, C) \to (\mathbf{R}^n, h(C))$ be a germ of a blow-analytic map. It is interesting to look at the eigenvalues and the eigenvectors of its jacobian matrix. It turns out, once again, that the situation is quite complicated. There are blow-analytic homeomorphisms such that the corresponding eigenvalues are not blow-analytic (not even arc-analytic). Here we understand also the case of complex eigenvalues which we treat componentwise. Naturally we have the same bad behaviour for the eigenvectors.

For a given continuous map germ $h : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ we would like to decide when it is a blow-analytic homeomorphism via a modification $\pi$. Here $\pi : M \to \mathbf{R}^n$ is a modification whose critical locus is normal crossing. Let us consider

$f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$, $0 \leq t \leq 1$, a continuous family of maps such that $f_1 = h$ and $f_0$ is a blow analytic homeomorphism via $\pi$. We put $F(x;t) = (F_1(x;t), ..., F_n(x;t)) = f_t(x)$.

In [3] it is introduced a vector field $\xi$, defined by

$$\xi = -(d_x(f_t))^{-1}(\sum_{i=1}^{n} \frac{\partial F_i}{\partial t} \frac{\partial}{\partial x_i}) + \frac{\partial}{\partial t},$$

which trivializes the family $f_t$ whenever defined.

Then $h$ is a blow-analytic homeomorphism via $\pi$, if the following conditions are satisfied.

(i) $\xi$ extends continuously on $U \times [0,1]$.

(ii) $\xi$ admits an analytic lift $\tilde{\xi}$ via $\pi \times \mathrm{id}_{[0,1]}$.

(iii) $\tilde{\xi}$ is tangent to each irreducible component of the critical locus of $\pi$.

For $\xi$ to be well defined we need that the eigenvalues of $d_x(f_t)$ (in particular those of $dh$) are not zero at least on the analyticity set. These are difficult to check for an arbitrary family $F(x,t)$, but in some cases we can use the "segment type" family $F(x,t) = (1-t)I + th$, where we can express everything in terms of the given $h$. For instance in the particular case when we can compose $h$, possibly both sides, with blow-analytic homeomorphisms so that we bring it in a form when $h$ has all the eigenvalues either positive (negative) or non-real, then we can consider the following easy to manipulate family $f_t = (1-t)I + th$ ($f_t = (1-t)I - th$ respectively), which will have always non-zero eigenvalues (here $I$ is the corresponding identity matrix) allowing us to construct the vector field we want $\xi$. Clearly if the composite is a blow-analytic homeomorphism, so will be the initial one.

In the other direction we offer the following constructive proposition.

**Proposition 2.4.** *Let $e_j(x) : \mathbf{R}^n \to \mathbf{C}^n$, $a_j(x) : \mathbf{R}^n \to \mathbf{C}$, $\quad j = 1, \ldots, m \,(\leq n)$, be a family closed under complex conjugation and such that if $e_j = \bar{e_k}$ then $a_j = \bar{a_k}$ as well. Assume that*

$$\|e_j(x)\| = 1, j = 1, \ldots, m,$$

*and $a_j(x)e_j(x)$, $j = 1, \ldots, m$, as vector fields, admit lifts via some modification $\pi$. Assume furthermore that $1 + \langle \mathrm{grad}(a_j(x)), e_j(x) \rangle$, $j = 1, \ldots, m$, are positive blow-analytic units also via the modification $\pi$ (when non real we require that both their real and imaginary parts are blow-analytic and the square of their absolute values are blow-analytic units).*

*Then* $h(x) = x + \sum_{j=1}^m a_j(x)e_j(x)$ *is a blow-analytic homeomorphism via* $\pi$.

*Proof.* It is not hard to see that the above defined $h$ has $e_j(x)$, $j = 1, \ldots, m$, as eigenvectors, and $\lambda_j = 1 + \langle \text{grad}(a_j(x)), e_j(x)\rangle$, $j = 1, \ldots, m$, as the corresponding eigenvalues. Therefore using the segment family, $F(x,t) = (1-t)x + th(x)$, we see that the above vector field $\xi$ is given by $\xi = \sum_{j=1}^m \frac{a_j(x)}{(1-t)+t\lambda_j} e_j(x)$ which clearly satisfies the requirements for $h$ to be a blow-analytic homeomorphism. $\square$

In the case when $e_j(x) = e_j$ the canonical base in $\mathbf{R}^n$ we recover many results from [3]. For instance if $\pi$ is toric and we take $a_1(x) = x_1(P(x) - 1), a_i = 0, i \geq 2$ such that $\frac{\partial(x_1 P(x))}{\partial x_1}$ is a blow-analytic unit via $\pi$ and it extends continuously on $\mathbf{R}^n \setminus \{0\}$ we obtain Lemma 5.1 from [3].

The behaviour of the eigenvectors (eigenvalues) depends on the coordinates we use. Choosing a good coordinate system may give as a better chance in proving that a given map is a blow-analytic homeomorphism. Actually we have the following more general statement.

**Proposition 2.5.** *Let* $h : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ *be a germ of a blow-analytic map. Then* $h$ *is a blow-analytic homeomorphism if and only if there exists a germ of an injective arc-analytic map* $\alpha : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ *such that* $\alpha \circ h$ *is injective and it has* **alp**.

*Proof.* If $\alpha$ is as above then it is easy to check that $h$ is injective and it has **alp** and thus as we already mentioned before, this ensures that $h$ is a blow-analytic homeomorphism. $\square$

## 3. Implicit Function Theorem

In this section we prove an implicit function theorem whith an eye toward a criterion to decide whether or not a given blow-analytic function is blow-analytic equivalent to a coordinate function, say $x_1$.

**Theorem 3.1.** *Let* $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ *be a germ of a locally blow-analytic function. If on a dense subset of a neighbourhood of the origin* $\frac{\partial f}{\partial x_1}$ *is bounded away from zero, both from above and below, then there exists a unique locally blow-analytic map germ,* $\alpha : (\mathbf{R}^{n-1}, 0) \to (\mathbf{R}, 0)$, *such that* $f(\alpha(b), b) = 0$, *for every* $b$ *in a small neighbourhood of the origin in* $\mathbf{R}^{n-1}$.

*Proof.* Using the theorem 1.4 in [1] it is enough to prove the theorem for blow-analytic functions. Because under the hypothesis, $\frac{\partial f}{\partial x_1}$ is a blow-analytic unit (beeing blow-analytic meromorphic), we may assume that it is positive (otherwise we use $-f$ instead). In a neighbourhood $U$ of the origin of $\mathbf{R}^n$ we may write:

$$f(x_1, b) = f(0, b) + x_1 P(x_1, b)$$

with $P$ a positive blow-analytic unit in a small neighbourhood of the origin in $\mathbf{R}^{n-1}$ (Hadamard decomposition [3], where we use decisevely the assumption on $\frac{\partial f}{\partial x_1}$). We can find $c_1, c_2$ positive real numbers such that we have that

$$0 < c_1 \leq P \leq c_2,$$

and $q_1 < 0, q_2 > 0$ such that $[q_1, q_2] \times V \subset U$ where $V$ is a small neighbourhood of the origin in $\mathbf{R}^{n-1}$. Now let us consider $W = f^{-1}(-c_1 q_2, -c_1 q_1) \bigcap 0 \times V$. We claim that for $(0, b) \in W, t \in [q_1, q_2], f(t, b)$ changes the sign.

Indeed for negative $t \in [q_1, q_2]$ we have

$$f(t,b) = f(0,b) + tP(t,b) \leq f(0,b) + c_1 t,$$

and this becomes negative if we put for instance $t = q_1$. For positive $t \in [q_1, q_2]$ we have

$$f(t,b) = f(0,b) + tP(t,b) \geq f(0,b) + c_1 t$$

and this becomes positive if we put for instance $t = q_2$.

These show that for $(0,b) \in W$ there exists at least one $\alpha(b) \in [q_1, q_2]$ such that $f(\alpha(b), b) = 0$. The fact that this $\alpha(b)$ is unique follows once again from our assumption. Indeed $\frac{d}{dt} f(t,b) = \frac{\partial f}{\partial t}(t,b)$, which shows that for each fixed $b$ our function is strictly increasing, giving us the uniqueness of $\alpha(b)$.

The graph of $\alpha$ clearly coincide with the zero-set of $f$ near the origin in $\mathbf{R}^n$ and this is clearly sub-analytic by our assumption. It remains to prove that $\alpha$ is also arc-analytic. First we will prove this for $n = 2$. We may write:

$$0 = f(\alpha(b), b) = f(0,b) + \alpha(b) P(\alpha(b), b).$$

If we put instead of $b$, $\beta(t)$ an analytic arc, we obtain that

$$-f(0, \beta(t)) = \alpha(\beta(t)) P(\alpha(\beta(t)), \beta(t)).$$

The left hand side of this identity is analytic by assumtion vanishing at zero, and in the right hand side $\alpha(\beta(t))$ is a priori a fractional power series (in our set up the composite of two sub-analytic is sub-analytic and we are in the one variable case) and because $P$ is a blow-analytic unit we may conclude that $\alpha(\beta(t))$ must start with integer powers. If $\alpha(\beta(t))$ is not analytic we write $\tau(t) = \alpha(\beta(t)) = \tau_1(t) + \tau_2(t)$, where by definition $\tau_1(t)$ contains all the integer powers in $\tau(t)$. Considering the change of coordinates $\bar{x}_1 = x_1 - \tau(y), \bar{y} = y$ and repeating the decomposition above, we conclude that $\tau_2$ must start also with integer powers which shows that it must be identical zero, thus $\tau(t)$ is analytic. This gives that $\alpha$ is an arc-analytic function which together with sub-analyticity, following [1], gives that actually $\alpha$ is locally blow-analytic.

This finishes the proof in the case $n = 2$. In the general case we use the strong decomposition results, especially theorem 7.5, from [12] to obtain fractional power series for $\alpha$ (on quadrants as in [12] or [11]). By fixing generically $y_1, ..., \hat{y}_i, ..., y_{n-1}$, $i \in \{1, ..., n-1\}$ the theorem reduces to the case $n = 2$. $\square$

**Corollary 3.2.** *Let $f : (\mathbf{R} \times \mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be a germ of a blow-analytic function via a modificatin $\pi$. Assume that $\frac{\partial f}{\partial x}$ is a blow-analytic unit via $\pi$. Then $h(x,y) = (f(x,y), y)$, $(x,y) \in \mathbf{R} \times \mathbf{R}^n$, is a blow-analytic homeomorphism.*

*Proof.* The only thing we have to prove is the direct implication. Consider $g(x,y,z) : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ given by $g(x,y,z) = f(x,y) - z$. Then $g$ satisfies the hypothesis of the implicit function theorem and therefore there exists an arc-analytic function $\alpha(y,z)$ such that $g(\alpha(y,z), y, z) \equiv 0$. We can define the arc-analytic map $h^{-1}(x,y) = (\alpha(y,x), y)$. Also let us notice that in this case, for any analytic arcs $\beta(t) = z$ and $\gamma(t) = y$, there exists an analytic arc $\tau(t)$ such that $f(\tau(t), \gamma(t)) = \beta(t)$. $\square$

We also note the following criterion.

**Theorem 3.3.** *Let $f : (\mathbf{R} \times \mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be a germ of a blow-analytic function and assume that $\frac{\partial f}{\partial x}(x, y)$, $(x, y) \in \mathbf{R} \times \mathbf{R}^n$, is a blow-analytic unit. Then $f$ is blow-analytic equivalent to the coordinate function $x$.*

*Proof.* Indeed under these assumptions it follows that there exists $\alpha(y)$ an arc-analytic function such that $f(\alpha(y), y) \equiv 0$ so $h(x, y) = (x+\alpha(y), y) : \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}\times\mathbf{R}^n$ is a blow-analytic homeomorphism.We have that $g(x, y) = f \circ h(x, y)$ is blow-analytic equivalent to $f$ and $\frac{\partial g}{\partial x}$ is a blow-analytic unit (because $\frac{\partial f}{\partial x}$ is) and moreover $g(0, y) \equiv 0$. This in turn implies that $g(x, y) = xP(x, y)$ with $P$ a blow-analytic unit. The result follows using corollary 3.7 from [3]. □

We also note that the converse of this proposition is no longer true. Indeed there are blow-analytic functions, components of blow-analytic homeomorphismes, whose all partial derivatives are not blow-analytic units (see for instance the Example 1.2). Instead we offer the following criterion.

**Corollary 3.4.** *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be a germ of a blow-analytic function. Then $f$ is blow-analytic equivalent to the coordinate function $x_1$ if and only if there exists a blow-analytic homeomorphism $\tau$ such that for some $i \in \{1, ..., n\}$, $\frac{\partial f\circ\tau}{\partial x_i}$ is a blow-analytic unit.*

*Proof.* Clear. □

Now having a criterion to test a coordinate function in our category, we can easily prove the following.

**Theorem 3.5.** *Let $h = (h_1, h_2) : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ be a blow-analytic map germ. Then $h$ is a blow-analytic homeomorphism if and only if its jacobian determinant is a blow-analytic unit and $h_1$ is blow-analytic equivalent to a coordinate function.*

*Proof.* Assume that $\tau$ is a blow-analytic homeomorphism such that $h_1 \circ \tau(x, y) = x$. Then $h \circ \tau(x, y) = (x, f(x, y))$ and therefore $\frac{\partial f}{\partial x}$ is a blow-analytic unit, so we can apply 3.2 to finish the proof. □

Note that the result above is not true without asking that one component is blow-analytic equivalent to a coordinate as we can see in example 3.4 in [3].

## 4. Resolution of Analytic Curves

In this section we describe a very elementary procedure to resolve analytic curves in $\mathbf{R}^n$ (or $\mathbf{C}^n$). In principal this is the same with Theorem 5.10 [3], but it is more general and much simpler.

Let $U$ be a neighborhood of the origin of $\mathbf{R}^{n-1}$ and $f : U \to \mathbf{R}$ denote a continuous blow-analytic function defined almost everywhere on $U$. Assume that for some analytic arc $\alpha : (\mathbf{R}, 0) \to U$, we have that $\operatorname{ord}_\alpha(f) < \operatorname{ord}(\alpha)$ (here by $\operatorname{ord}(\alpha)$ we understand the minimum of the orders of its components). Then in [13] an "associated" blow-analytic homeomorphism is constructed between two neighborhoods of the origin of $\mathbf{R}^n$ which in particular drops the order along the arc $(\alpha, 0)$. The example [13] shows that via this kind of blow-analytic homeomorphism, a curve like $(t^3, t^2, 0)$ goes to a smooth one, i.e. that blow-analytic homeomorphism makes the cusp smooth (for a related problem see also [5]).

In the sequel we will show that for any analytic curve $\alpha(t)$ in $\mathbf{R}^2$ with a parameterization $\alpha(t) = (ct^p + \cdots + \text{h.o.t.}, t^n)$, $n < p$, we can construct an explicit rational blow-analytic function $f : U \to \mathbf{R}$, $U$ a neighborhood of the origin of $\mathbf{R}^2$, such that $f(\alpha(t))$ is either an analytic arc of order strictly less, or of order $n$ in the case when all the exponents of the initial curve are multiple of $n$. Iterating this procedure will lead to the claimed resolution. This will give us a lot of explicit examples for which we can use the general construction in [13] to provide interesting blow-analytic homeomorphisms.

**Theorem 4.1.** *For any analytic curve $\alpha = (\alpha_1, \alpha_2) : (\mathbf{R}, 0) \to U$, $U$ a neighborhood of the origin of $\mathbf{R}^2$, there is a continuous rational (blow-analytic) function $f : U \to \mathbf{R}$, defined constructively, such that $\operatorname{ord}_\alpha(f) = \gcd(\operatorname{ord}(\alpha_1), \operatorname{ord}(\alpha_2))$.*

*Proof.* After analytic changes of coordinates both in the source and in the target, we may assume that $\alpha$ has the following form.

$$\alpha(t) = (\alpha_1(t), \alpha_2(t)) = (ct^p + \cdots + \text{h.o.t.}, t^n)$$

with

$$q = \gcd(p = \operatorname{ord}(\alpha_1), n = \operatorname{ord}(\alpha_2)) < n < p, c \neq 0 \,\text{or}\, \alpha(t) = (\alpha_1(t), \alpha_2(t)) = (0, t^n).$$

In the case of interest, we can find positive integers $a, b, k$ such that $-ap + bn = q$ and $2kn \geq a$.

Define $f(x, y)$ as:

$$f(x, y) = \frac{x^{2kn-a} y^b}{x^{2kn} + y^{2kp}}.$$

It is clear that $f$ is a blow-analytic function via a convenient modification $\pi$ (either toric or a finite composite of blowing-ups which can be described efectively). $\square$

Now we can proceed as in [13] to lift our given arc to one of a smaller order and therefore inductively this finishes the proof.

Namely $\alpha$ has the following form

$$\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t)) = (c_1 t^{p_1} + \cdots + \text{h.o.t.}, \ldots, c_n t^{p_n} + \cdots + \text{h.o.t.})$$

and we may assume that after some analytic changes of coordinates we have that

$$q_{i,j} = \gcd(p_i = \operatorname{ord}(\alpha_i), p_j = \operatorname{ord}(\alpha_j)) < p_i, p_j; c_i, c_j \neq 0 \,\text{or if}\, c_i = 0, \text{then}\, \alpha_i(t) \equiv 0.$$

We can construct the described function for a pair $i, j$ for which $q_{i,j}$ is minimum, and then proceed as below.

$$\begin{array}{ccc}
(\mathbf{R}^n, 0) \times (\mathbf{R}, 0) & \xrightarrow{\phi} & (\mathbf{R}^n, 0) \times (\mathbf{R}, 0) \\
{\scriptstyle \pi \times id} \big\downarrow & & {\scriptstyle \pi \times id} \big\downarrow \\
(\mathbf{R}^n, 0) \times (\mathbf{R}, 0) & \xrightarrow{h} & (\mathbf{R}^n, 0) \times (\mathbf{R}, 0)
\end{array}$$

Here we define $\phi(u, v.w) = (u, v, w + f \circ \pi(u, v))$ and $h(x, y, z) = (x, y, z + f(x, y))$.

If we denote by $\tilde{\alpha}(t)$ a lift of $h(\alpha(t))$, we note that $\phi^{-1}(\tilde{\alpha}(t))$ is a lift of $\alpha$ whose order is strictly less than the order of the given arc $\alpha$.

To use this procedure to resolve over $\mathbf{C}$ one needs a bit of care in the definition of $f$ to make sure that the denominator does not vanish for the given curve (choose some coefficients). The computation beeing formal, the method will also give the resolution in this case (we note that upstairs everything is analytic).

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