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CONNECTIONS AND GENERALIZED GAUGE TRANSFORMATIONS

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Abstract. A principle based on the geometry of internal symmetry spaces is suggested to explain the types of gauge interactions that arise in the standard model. Elimination of the fibre coordinate from the transformation rule of the connection form implies a Lie group structure is required for vector bosons in gauge theories and application of division algebras to force unification must begin with fermions comprising the elementary particle multiplets. Constraints on the spinor spaces and the corresponding gauge groups are obtained, and consistency with higher-dimensional superstring theory is discussed.

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1. Introduction

It has recently been shown that superstring theory possesses improved finiteness properties at each order [1][2][3] and at large orders in the series expansion for scattering amplitudes [4], suggesting that a consistent quantum theory containing general relativity in the low-energy limit has been obtained. Phenomenologically relevant models of elementary particle interactions might also be included for a suitable choice of superstring vacuum [5]. A principle for selecting a specific superstring vacuum would be needed as a theoretical basis for any choice of ground state. One possibility, motivated by topology change arising in the path integral for quantum gravity [6], would involve an integration over different background configurations, dominated by a sum over solutions to the string equations of motion, with the geometries weighted by a string field theory action [7][8]. Alternatively, one might wish to consider a principle directly constraining geometries associated with a Kaluza-Klein unification of general relativity with the elementary interactions.

The attempt to unify the elementary particle interactions with gravity has gradually led to the development of theories which may be regarded as generalizations of standard gauge theories [7][9][10][11]. In establishing a theoretical basis for the known gauge groups in the standard model, refinements of various mathematical structures such as fibre bundles may be introduced. A formulation of generalized gauged theories, for example, can be achieved by considering a larger category of bundles than the principal bundles. In a previous investigation [12], a physical requirement imposed on the transformation rule of the connection form for a general bundle led to constraints on the types of geometries corresponding to Yang-Mills theories, and the category of principal bundles was selected uniquely by these conditions. The diffeomorphic sphere theorem, implying the existence of a diffeomorphism from any n-dimensional complete simply connected manifold with appropriate constraints on the sectional curvature to S^n, the derivation of n-dimensional regular algebras from smooth fibrations of S^{2n-1} by a great $(n-1)$-sphere [14], and the topological equivalence of any smooth fibration of S^7 by great three-spheres and S^{15} by great seven-spheres to the classical Hopf fibrations [14][15] indicate that division algebras also represent a special category of mathematical structures relevant for a unified theory of elementary particle interactions. This is particularly relevant for an approach to force unification utilizing a link between the internal symmetries and the different division algebras [16][17][18][19], relating the number of types of elementary particle interactions with the number of division algebras [20][21][22][23].

2. Gauge Transformation Constraints for an SO(4) Action on an S^3 Bundle

Gauge transformations may be viewed as active transformations on the standard fibre. Let $\sigma(x) = y \in G$, $\sigma'(x) = \sigma(x) \cdot g(x) = y \cdot g \in G$. If $\xi \in T_x(M)$,

$$\begin{aligned} \sigma'_* \cdot \xi &= R_{g*}\sigma_* \cdot \xi + L_{y*}g_* \cdot \xi \\ &= R_{g*}\sigma_* \cdot \xi + L_{(y\cdot g)*}L_{g^{-1}*}(g_* \cdot \xi) \end{aligned} \tag{1}$$

Given a connection form

$$\begin{aligned} \omega(\sigma'_* \cdot \xi) = L^{-1}_{(y\cdot g)*}\mathcal{V}(\sigma'_* \cdot \xi) &= L^{-1}_{(y\cdot g)*}[\mathcal{V}(R_{g*}\sigma_* \cdot \xi) + \mathcal{V}(L_{(y\cdot g)*}L_{g^{-1}*}g_* \cdot \xi)] \\ &= L^{-1}_{(y\cdot g)*}R_{g*}\mathcal{V}(\sigma_* \cdot \xi) + L_{g^{-1}*}g_* \cdot \xi \end{aligned} \tag{2}$$

where $\mathcal{V}$ represents the projection onto the vertical subspace $V_{(x,\sigma(x))}(E) \sim V_{\sigma(x)}(E)$. Since $\omega(\sigma_* \cdot \xi) = L^{-1}_{y*}\mathcal{V}(\sigma_* \cdot \xi)$

$$\begin{aligned} \omega(\sigma'_* \cdot \xi) &= L^{-1}_{(y\cdot g)*}R_{g*}L_{y*}\omega(\sigma_* \cdot \xi) + L_{g^{-1}*}g_* \cdot \xi \\ &= ad(g^{-1})\omega(\sigma_* \cdot \xi) + L_{g^{-1}*}g_* \cdot \xi \end{aligned} \tag{3}$$

Setting ξ equal to ∂_μ, one recovers the standard gauge transformation law

$$A'_\mu = ad(g^{-1})A_\mu + g^{-1}\partial_\mu g \tag{4}$$

The dependence on y in equation (16) has disappeared, which is necessary if the theory is to be formulated in the four-dimensional space-time M.

The transformation group could be made larger than the standard fibre in the principal bundle. Consider, for example, a bundle which has standard fibre S^3 and suppose that the group acting on this fibre is enlarged from SU(2) to SO(4), with SO(4) acting on S^3 by right multiplication. Two different homomorphisms $\iota_R,\ \iota_L : S^3 \to SO(4)$ are defined by $y = (y_0, y_1, y_2, y_3) \in S^3$

$$\iota_R(y) = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ -y_1 & y_0 & -y_3 & y_2 \\ -y_2 & y_3 & y_0 & -y_1 \\ -y_3 & -y_2 & y_1 & y_0 \end{pmatrix} \qquad \iota_L(y) = \begin{pmatrix} y_0 & -y_1 & -y_2 & -y_3 \\ y_1 & y_0 & -y_3 & y_2 \\ y_2 & y_3 & y_0 & -y_1 \\ y_3 & -y_2 & y_1 & y_0 \end{pmatrix} \tag{5}$$

so that $y{\cdot}y' = \iota_L(y)\, y'$, with y' as a column vector, and $y'{\cdot}y = y'^T\, \iota_R(y)$. Recalling further that $R_g y' \equiv y'{\cdot}g \leftrightarrow y'^T{\cdot}R_g^T$, after identifying the row vector y'^T with the unit quaternion $y' \in S^3$, $L^{-1}_{(y\cdot g)*}R_{g*}L_{y*}$ is the tangent mapping induced by the SO(4) transformation

$\iota_L(y\cdot g)^{-1}R_g\iota_L(y)$, or equivalently, $\iota_L(y)^T R_g^T[\iota_L(y\cdot g)^{-1}]^T$ acting on four-component row vectors by right multiplication. Since SO(4) is locally isomorphic to $SU(2)\times SU(2)$,

$$R_g^T \;=\; \iota_R(g_0)\iota_L(g_0')^T \; for\ some\ g_0,\ g_0'\in S^3 \tag{6}$$

It then follows that

$$\begin{aligned}\iota_L(y)^T R_g^T[\iota_L(y\cdot g)^{-1}]^T &= \iota_L(y)^T\iota_R(g_0)\iota_L(g_0')^T[\iota_L((g_0'\cdot y)\cdot g_0)^{-1}]^T\\ &= \iota_R(g_0)\iota_L(y)^T\iota_L(g_0')^T[\iota_L(g_0'\cdot y)^{-1}]^T\iota_L(g_0^{-1})^T\\ &= \iota_R(g_0)\iota_L(g_0^{-1})^T\end{aligned} \tag{7}$$

which is independent of y.

The second term in equation (17), $L_{g^{-1}*}g_*\cdot\xi$ can be an arbitrary element of $T_e(SO(4))$.

$$L_{g^{-1}*}g_*\cdot\xi \;=\; \begin{pmatrix} 0 & c_1 & c_2 & c_3\\ -c_1 & 0 & -c_6 & c_5\\ -c_2 & c_6 & 0 & -c_4\\ -c_3 & -c_5 & c_4 & 0\end{pmatrix} \tag{8}$$

Let $y\cdot g\equiv y'=(y_0',y_1',y_2',y_3')$. Then

$$\begin{aligned}\lambda_{y'_*}\cdot L_{g^{-1}*}g_*\cdot\xi \;=\; (&-c_1y_1'-c_2y_2'-c_3y_3',\; c_1y_0'+c_6y_2'-c_5y_3',\; c_2y_0'-c_6y_1'+c_4y_3',\\ &c_3y_0'+c_5y_1'-c_4y_2')\end{aligned} \tag{9}$$

where $\lambda_{y'*}$ is a map from tangent vectors in $T_e(SO(4))$ to tangent vectors in $T_{y'}(S^3)$ induced by the mapping $\lambda_{y'}:\ g\ =\ exp(c_IX_I)\ \to\ y'\cdot g\ =\ y'\cdot exp(c_IX_I)$ with $\{X_I\}$ being the generators of SO(4). $\lambda_{y'_*}\cdot L_{g^{-1}*}(g_*\cdot\xi)\cdot\iota_L(y'^{-1})^T$ is only independent of y' if $c_1\ =\ c_4,\ c_2\ =\ c_5,\ c_3\ =\ c_6$. So $L_{g^{-1}*}\cdot(g_*\cdot\xi)$ must be an element of an SU(2) subalgebra, and $g(x)$ must be an element of an SU(2) subgroup to maintain y-independence of the gauge transformation.

Let $\{X_i\},\ i=1,\ 2,\ 3$ be the left-invariant basis of vector fields on S^3. A section $\sigma(x)$ can also be regarded as a mapping from $g\in SO(4)$ to $\sigma(x)\cdot g\in E$, so that it induces a tangent mapping from $T_e(SO(4))$ to $V_{\sigma(x)}(E)$, the vertical subspace of the tangent space of the bundle. The connection form is a rule for comparing vertical components of tangent vectors at different points on the fibre and is therefore an isomorphism from $V_{\sigma(x)}(E)$ to $T_e(SO(4))\big|_{\{X_i\}}$. While one requires $\omega(\sigma_*\cdot\xi_\mu)\in T_e(SO(4))\big|_{\{X_i\}}$, for an arbitrary section $\sigma'(x)\ =\ \sigma(x)\cdot g(x)$, $ad(g^{-1})[\omega(\sigma_*\cdot\xi_\mu)]\ +\ L_{g^{-1}*}(g_*\cdot\xi_\mu)\notin T_e(SO(4))\big|_{\{X_i\}}$.

To maintain $\omega(\sigma'_* \cdot \xi_\mu) \in T_e(SO(4))\big|_{\{X_i\}}$, a compensating gauge transformation is needed. Since

$$\begin{aligned}
\omega[(\sigma(x)\cdot g(x)h'(x))_* \cdot \xi_\mu] &= ad(h'^{-1})[ad(g^{-1})\omega(\sigma_* \cdot \xi_\mu)] \\
&+ ad(h'^{-1})\omega[\sigma'(x)_* \cdot (L_{g^{-1}(x)*}(g_* \cdot \xi_\mu))] + \omega[\sigma'(x)_* \cdot (h'_* \cdot \xi_\mu)] \\
&= ad(h'^{-1})[\omega((\sigma(x)\cdot g)_* \cdot \xi_\mu)] + \omega[\sigma'(x)\cdot (h'_* \cdot \xi_\mu)]
\end{aligned} \tag{10}$$

one may define

$$\omega(\sigma'_*\cdot\xi_\mu) = ad(h'^{-1}(\sigma,g))\{ad(g^{-1})[\omega(\sigma_*\cdot\xi_\mu)] + L_{g^{-1}*}(g_*\cdot\xi_\mu)\} + L_{h'^{-1}(\sigma,g)*}(h'(\sigma,g)_*\cdot\xi_\mu) \tag{11}$$

where $h'(\sigma,g)$ is an element of the stability group of $\sigma'(x)$.

Theorem. There exists a unique $h'(\sigma,g)$ such that $\omega(\sigma'_* \cdot \xi_\mu) \in T_e(SO(4))\big|_{\{X_i\}}$.
Proof.
Existence. Let $g_0 = exp(t_i X_i)$.

$$L_{g_0^{-1}*}(g_{0*} \cdot \xi_\mu) = exp(-t_i X_i)\ \partial_\mu t_j\ X_j\ exp(t_i X_i) \in T_e(SO(4))\big|_{\{X_i\}} \tag{12}$$

Now $\sigma(x)\cdot exp(t_i X_i)$ covers all of S^3. Thus, if $\sigma'(x) = \sigma(x)\cdot g(x)$, in general, one can write $g(x) = exp(t_i X_i)\cdot h'^{-1}$. Therefore, there exists an h' such that $ad((gh')^{-1})[\omega(\sigma_* \cdot \xi_\mu)] + L_{(gh')^{-1}}((gh')_* \cdot \xi_\mu) \in T_e(SO(4))\big|_{\{X_i\}}$. The result then follows from

$$\begin{aligned}
ad((gh')^{-1})[\omega(\sigma_*\cdot\xi_\mu)] + L_{(gh')^{-1}*}((gh')_*\cdot\xi_\mu) &= ad(h'^{-1}(\sigma,g))\{ad(g^{-1})[\omega(\sigma_*\cdot\xi_\mu)] \\
&+ L_{g^{-1}*}(g_*\cdot\xi_\mu)\} \\
&+ L_{h'^{-1}(\sigma,g)*}(h'(\sigma,g)_*\cdot\xi_\mu)
\end{aligned} \tag{13}$$

Uniqueness. Suppose $ad(h''^{-1})\{ad(g^{-1})[\omega(\sigma_*\cdot\xi_\mu)] + L_{g^{-1}*}(g_*\cdot\xi_\mu)\} + L_{h''^{-1}*}(h''\cdot\xi_\mu) \in T_e(SO(4))\big|_{\{X_i\}}$. Let $h'' = h'h$. Then this implies $L_{h^{-1}*}(h_*\cdot\xi_\mu) \in T_e(SO(4))\big|_{\{X_i\}}$. Take $h = exp(t_i X_i + \tilde{t}_i \tilde{X}_i)$.

$$\begin{aligned}
L_{h^{-1}*}(h_*\cdot\xi_\mu) &= exp(-(t_i X_i + \tilde{t}_i\tilde{X}_i))(\partial_\mu t_j X_j + \partial_\mu \tilde{t}_j \tilde{X}_j) exp(t_i X_i + \tilde{t}_i \tilde{X}_i) \\
&= (\partial_\mu t_j) exp(-(t_i X_i + \tilde{t}_i\tilde{X}_i)) X_j exp(t_i X_i + \tilde{t}_i\tilde{X}_i) \\
&+ (\partial_\mu \tilde{t}_j) exp(-(t_i X_i + \tilde{t}_i\tilde{X}_i)) \tilde{X}_j exp(t_i X_i + \tilde{t}_i\tilde{X}_i)
\end{aligned} \tag{14}$$

As the generators of the two SU(2) subgroups commute, $[X_i, \tilde{X}_j] = 0$,

$$L_{h^{-1}*}(h_*\cdot\xi_\mu) = \partial_\mu t_j exp(-t_i X_i) X_j exp(t_i X_i) + \partial_\mu \tilde{t}_j exp(-\tilde{t}_i\tilde{X}_i)\tilde{X}_j exp(\tilde{t}_i\tilde{X}_i) \tag{15}$$

and $L_{h^{-1}*}(h_* \cdot \xi_\mu) \in T_e(SO(4))\big|_{\{X_i\}}$ if and only if $\partial_\mu \tilde{t}_j \ = \ 0$. We require that the relation

$$\begin{aligned}\omega(\sigma'_* \cdot \xi_\mu) \ &= \ ad(h''^{-1})\{ad(g^{-1})[\omega(\sigma_* \cdot \xi_\mu)] \\ &\qquad + \ L_{g^{-1}*}(g_* \cdot \xi_\mu)\} \ + \ L_{h''^{-1}*}(h''_* \cdot \xi_\mu) \in T_e(SO(4))\big|_{\{X_i\}}\end{aligned} \tag{16}$$

holds not just at a single point $x_0 \in M$ but for all x in some neighbourhood U of x_0. Therefore $\partial_\mu \tilde{t}_j \ = \ 0$ must hold in all of U, implying that $\tilde{t}_j(x) \ = \ \tilde{t}_j(x_0) \ = \ constant$ in U and $h(x) \ = \ exp(t_i(x)X_i \ + \ \tilde{t}_i(x_0)\tilde{X}_i)$. Both $h'(x)$ and $h''(x)$ must stabilize $\sigma'(x) \ = \ \sigma(x) \cdot g(x) \ \forall x \in U$ so that $\sigma'(x) \cdot h''(x) \ = \ \sigma'(x) exp(t_i(x)X_i + \tilde{t}_i(x_0)\tilde{X}_i)$. Defining $\sigma'(x) \ = \ \sigma(x_0) exp(s_j(x)X_j)$, $s_j(x_0) \ = \ 0$ where $\sigma(x_0) exp(t_i(x_0)X_i + \tilde{t}_i(x_0)\tilde{X}_i) \ = \ \sigma(x_0)$, it follows that

$$\sigma'(x) \ exp(t_i(x)X_i + \tilde{t}_i(x_0)\tilde{X}_i) \ = \ \sigma(x_0) \ exp(-t_i(x_0)X_i) \ exp(s_j(x)X_j) \ exp(t_i(x)X_i) \tag{17}$$

Thus,

$$exp(-t_i(x_0)X_i) \ exp(s_j(x)X_j) \ exp(t_i(x)X_i) \ = \ exp(s_j(x)X_j) \tag{18}$$

is required for the path $\sigma'(x) \ = \ \sigma(x_0)exp(s_j(x)X_j)$ to be stabilized by $h''(x)$. Since

$$\begin{aligned} e^{s_j(x)X_j} e^{t_i(x)X_i} \ &= \ exp[(s_i(x) \ + \ t_i(x))X_i \ + \ \frac{1}{2} s_i(x) t_j(x) c_{ijk} X_k \\ &\qquad + \ \frac{1}{12} s_i(x) s_j(x) t_k(x) c_{jkl} c_{ilm} X_m \\ &\qquad + \ \frac{1}{12} s_i(x) t_j(x) t_l(x) c_{ijk} c_{klm} X_m \ + \ ... \ ] \\ e^{t_j(x_0)X_j} e^{s_i(x)X_i} \ &= \ exp[(s_i(x) \ + \ t_i(x_0))X_i \ + \ \frac{1}{2} t_i(x_0) s_j(x) c_{ijk} X_k \\ &\qquad + \ \frac{1}{12} t_i(x) t_j(x_0) s_k(x) c_{jkl} c_{ilm} X_m \\ &\qquad + \ \frac{1}{12} t_i(x_0) s_j(x) s_l(x) c_{ijk} c_{klm} X_m \ + \ ... \ ] \end{aligned} \tag{19}$$

Equating the powers gives

$$\begin{aligned} t_m(x) \ + \ \frac{1}{2} s_i(x) t_j(x) c_{ijm} \ + \ \frac{1}{12}(s_i(x) t_j(x) t_l(x) \ + \ t_i(x) s_j(x) s_l(x)) c_{ijk} c_{klm} \ + \ ... \\ = \ t_m(x_0) \ - \ \frac{1}{2} s_i(x) t_j(x_0) c_{ijm} \ + \ \frac{1}{12} s_i(x) t_j(x_0) t_l(x_0) \\ + \ t_i(x_0) s_j(x) s_l(x)) c_{ijk} c_{klm} \ + \ ... \end{aligned} \tag{20}$$

Assuming that $s_i(x)$ is small in the neighbourhood U of x_0 and equating terms with the same power of $s_i(x)$ gives

$$\begin{aligned} t_m(x) &= t_m(x_0) \\ \frac{1}{2}s_i(x)t_j(x)c_{ijm} + \frac{1}{12}s_i(x)t_j(x)t_l(x)c_{ijk}c_{klm} + ... &= -\frac{1}{2}s_i(x)t_j(x_0)c_{ijm} \\ &\quad + \frac{1}{12}s_i(x)t_j(x_0)t_l(x_0)c_{ijk}c_{klm} \\ &\quad + ... \\ &\vdots \end{aligned} \tag{21}$$

Substituting the first equation into the second implies that $s_i(x)t_j(x_0)c_{ijm} + ... = 0$ or equivalently that $M_{ij}(x)t_j(x_0) = 0$, where $M_{ij}(x)$ is a matrix of rank 3, and consequently, $t_j(x_0) = 0$. Moreover, $exp(\tilde{t}_i(x_0)\tilde{X}_i)$ does not stabilize $\sigma'(x)$ since it does not stabilize any point on S^3, particularly $\sigma(x_0)$. Thus, $\tilde{t}_i(x_0) = 0$ and $h(x) = Id$. The last result also follows from the property of $h(x) = exp(t_i(x_0)X_i + \tilde{t}_i(x_0)\tilde{X}_i)$ being a fixed element in SO(4), which cannot stabilize more than one point on the path $\sigma'(x) = \sigma(x_0)exp(s_j(x)X_j)$, unless it is the identity element, which is the only common element in each of the stability subgroups of SO(4). Therefore, $h''(x) = h'(x)$ is unique.

Thus, one can make a comparison between the vector fields in the space spanned by $\{X_i\}$ even under the action of an SO(4) transformation for the S^3 bundle. From equation (9), however, it can be seen that independence of the term $L_{g^{-1}*}g_* \cdot \xi$ with respect to the fibre coordinate requires that $g(x)$ must be an element of an SU(2) subgroup.

3. Gauge Transformation Constraints for an SO(8) and SU(4) Actions on an S^7 Bundle

If the fibre is S^7, it admits a parallelism associated with the existence of octonions as an 8-dimensional division algebra over the real numbers. Consider the action of SO(8) by right multiplication on a trivial S^7 bundle. Suppose $L_{g^{-1}*}g_* \cdot \xi_\mu$ is an arbitrary element of the 28-dimensional Lie algebra of SO(8). Let ι_L an embedding of octonions in SO(8), so that left. multiplication by y is represented by right multiplication by the matrix $\iota_L(y)$. Then, independence with respect to the fibre coordinate of the inhomogeneous term in the gauge transformation implies that $\lambda_{y*} \cdot L_{g^{-1}*} \cdot (g_* \cdot \xi) \cdot (\iota_L(y^{-1})^T)_*$, or equivalently that the row vector $y \cdot (d_{AB}J_{AB}) \cdot (\iota_L(y^{-1})^T)$, where J_{AB} are generators of SO(8), is independent of y. Since

$$\iota_L(y^{-1})^T \;=\; \begin{pmatrix} y_0 & -y_1 & -y_2 & -y_3 & -y_4 & -y_5 & -y_6 & -y_7 \\ y_1 & y_0 & -y_3 & y_2 & -y_5 & y_4 & -y_7 & y_6 \\ y_2 & y_3 & y_0 & -y_1 & -y_6 & y_7 & y_4 & -y_5 \\ y_3 & -y_2 & y_1 & y_0 & y_7 & y_6 & -y_5 & -y_4 \\ y_4 & y_5 & y_6 & -y_7 & y_0 & -y_1 & -y_2 & y_3 \\ y_5 & -y_4 & -y_7 & -y_6 & y_1 & y_0 & y_3 & y_2 \\ y_6 & y_7 & -y_4 & y_5 & y_2 & -y_3 & y_0 & -y_1 \\ y_7 & -y_6 & y_5 & y_4 & -y_3 & -y_2 & y_1 & y_0 \end{pmatrix} \tag{22}$$

the vector $y \cdot (d_{AB}J_{AB}) \cdot \iota_L(y^{-1})^T \;=\; (0\ c_1\ c_2\ c_3\ c_4\ c_5\ c_6\ c_7)$ where

$$\begin{aligned} c_1 \;=\;& d_{01}(y_0^2+y_1^2)+(d_{02}-d_{13})(y_1y_2+y_0y_3)+(d_{03}+d_{12})(y_1y_3-y_0y_2) \\ &+\;(d_{04}-d_{15})(y_1y_4+y_0y_5)+(d_{05}+d_{14})(y_1y_5-y_0y_4)+(d_{06}-d_{17})(y_1y_6+y_0y_7) \\ &+\;(d_{07}+d_{16})(y_1y_7-y_0y_6)-d_{23}(y_2^2+y_3^2)+(-d_{24}-d_{35})(y_3y_4-y_2y_5) \\ &+\;(-d_{25}+d_{34})(y_3y_5+y_2y_4)+(-d_{26}-d_{37})(y_3y_6-y_2y_7)+(-d_{27}+d_{36})(y_3y_7+y_2y_6) \\ &-\;d_{45}(y_4^2+y_5^2)+(-d_{46}-d_{57})(y_5y_6-y_4y_7)+(-d_{47}+d_{56})(y_5y_7+y_4y_6) \\ &-\;d_{67}(y_6^2+y_7^2) \end{aligned} \tag{23}$$

$$\begin{aligned} c_2 \;=\;& (d_{01}+d_{23})(y_1y_2-y_0y_3)+d_{02}(y_0^2+y_2^2)+(d_{03}+d_{12})(y_2y_3+y_0y_1) \\ &+\;(d_{04}-d_{26})(y_2y_4+y_0y_6)+(d_{05}+d_{27})(y_2y_5-y_0y_7)+(d_{06}+d_{24})(y_2y_6-y_0y_4) \\ &+\;(d_{07}-d_{25})(y_2y_7+y_0y_5)+d_{13}(y_1^2+y_3^2)+(d_{14}-d_{36})(y_3y_4+y_1y_6) \\ &+\;(d_{15}+d_{37})(y_3y_5-y_1y_7)+(d_{16}+d_{34})(y_3y_6-y_1y_4)+(d_{17}-d_{35})(y_3y_7+y_1y_5) \\ &+\;(d_{45}-d_{67})(-y_4y_7-y_5y_6)-d_{46}(y_4^2+y_6^2)+(d_{47}-d_{56})(-y_6y_7+y_4y_5) \\ &+d_{57}(y_5^2+y_7^2) \end{aligned} \tag{24}$$

$$\begin{aligned} c_3 \;=\;& (d_{01}+d_{23})(y_1y_3+y_0y_2)+(d_{02}-d_{13})(y_2y_3-y_0y_1)+d_{03}(y_0^2+y_3^2) \\ &+\;(d_{04}+d_{37})(y_3y_4-y_0y_7)+(d_{05}+d_{36})(y_3y_5-y_0y_6)+(d_{06}-d_{35})(y_3y_6+y_0y_5) \\ &+\;(d_{07}-d_{34})(y_3y_7+y_0y_4)-d_{12}(y_1^2+y_2^2)+(d_{14}-d_{27})(-y_2y_4-y_1y_7) \\ &+\;(d_{15}-d_{26})(-y_2y_5-y_1y_6)+(d_{16}+d_{25})(-y_2y_6+y_1y_5)+(d_{17}+d_{24})(-y_2y_7+y_1y_4) \\ &+\;(d_{45}-d_{67})(y_5y_7-y_4y_6)+(d_{46}+d_{57})(y_6y_7+y_4y_5)+d_{47}(y_4^2+y_7^2) \\ &+d_{56}(y_5^2+y_6^2) \end{aligned} \tag{25}$$

$$\begin{aligned} c_4 \;=\;& (d_{01}+d_{45})(y_1y_4-y_0y_5)\;+\;(d_{02}+d_{46})(y_2y_4-y_0y_6)+(d_{03}-d_{47})(y_3y_4+y_0y_7) \\ &+\;d_{04}(y_0^2+y_4^2)+(d_{05}+d_{14})(y_4y_5+y_0y_1)+(d_{06}+d_{24})(y_4y_6+y_0y_2) \\ &+\;(d_{07}-d_{34})(y_4y_7-y_0y_3)+(d_{12}+d_{56})(y_2y_5-y_1y_6)+(d_{13}-d_{57})(y_3y_5+y_1y_7) \\ &+\;d_{15}(y_1^2+y_5^2)+(d_{16}+d_{25})(y_5y_6+y_1y_2)+(d_{17}-d_{35})(d_5d_7-y_1y_3) \end{aligned}$$

$$
\begin{aligned}
&+\ (d_{23}-d_{67})(y_3y_6+y_2y_7)+d_{26}(y_2^2+y_6^2)+(d_{27}-d_{36})(y_6y_7-y_2y_3)\\
&-\ d_{37}(y_3^2+y_7^2) \qquad (26)\\
c_5 &= (d_{01}+d_{45})(y_1y_5+y_0y_4)+(d_{02}-d_{57})(y_2y_5+y_0y_7)+(d_{03}-d_{56})(y_3y_5+y_0y_6)\\
&+\ (d_{04}-d_{15})(y_4y_5-y_0y_1)\ +\ d_{05}(y_0^2+y_5^2)+(d_{06}-d_{35})(y_5y_6-y_0y_3)\\
&+\ (d_{07}-d_{25})(y_5y_7-y_0y_2)+(d_{12}+d_{47})(-y_2y_4+y_1y_7)+(d_{13}+d_{46})(-y_3y_4+y_1y_6)\\
&-\ d_{14}(y_1^2+y_4^2)+(d_{16}+d_{34})(-y_4y_6-y_1y_3)+(d_{17}+d_{24})(-y_4y_7-y_1y_2)\\
&+\ (d_{23}-d_{67})(-y_3y_7+y_2y_6)+(d_{26}+d_{37})(-y_6y_7-y_2y_3)-d_{27}(y_2^2+y_7^2)\\
&-\ d_{36}(y_3^2+y_6^2) \qquad (27)\\
c_6 &= (d_{01}+d_{67})(y_1y_6-y_0y_7)+(d_{02}+d_{46})(y_2y_6+y_0y_4)+(d_{03}-d_{56})(y_3y_6-y_0y_5)\\
&+\ (d_{04}-d_{26})(y_4y_6-y_0y_2)\ +\ (d_{05}+d_{36})(y_5y_6+y_0y_3)+d_{06}(y_0^2+y_6^2)\\
&+\ (d_{07}+d_{16})(y_6y_7+y_0y_1)+(d_{12}+d_{47})(y_2y_7+y_1y_4)+(d_{13}-d_{57})(y_3y_7-y_1y_5)\\
&+\ (d_{14}-d_{27})(y_4y_7-y_1y_2)+(d_{15}+d_{37})(y_5y_7+y_1y_3)+d_{17}(y_1^2+y_7^2)\\
&+\ (d_{23}-d_{45})(-y_3y_4-y_2y_5)-d_{24}(y_2^2+y_4^2)+(d_{25}-d_{34})(-y_4y_5+y_2y_3)\\
&+\ d_{35}(y_3^2+y_5^2) \qquad (28)\\
c_7 &= (d_{01}+d_{67})(y_1y_7+y_0y_6)+(d_{02}-d_{57})(y_2y_7-y_0y_5)+(d_{03}-d_{47})(y_3y_7-y_0y_4)\\
&+\ (d_{04}+d_{37})(y_4y_7+y_0y_3)\ +\ (d_{05}+d_{27})(y_5y_7+y_0y_2)+(d_{06}-d_{17})(y_6y_7-y_0y_1)\\
&+\ d_{07}(y_0^2+y_7^2)+(d_{12}+d_{56})(-y_2y_6-y_1y_5)\ +\ (d_{13}+d_{46})(-y_3y_6-y_1y_4)\\
&+\ (d_{14}-d_{36})(-y_4y_6+y_1y_3)+(d_{15}-d_{26})(-y_5y_6+y_1y_2)-d_{16}(y_1^2+y_6^2)\\
&+\ (d_{23}-d_{45})(-y_2y_4+y_3y_5)+(d_{24}+d_{35})(y_4y_5+y_2y_3)+d_{25}(y_2^2+y_5^2)\\
&+\ d_{34}(y_3^2+y_4^2) \qquad (29)
\end{aligned}
$$

Altogether, independence of the c_i with respect to y leads to 21 independent constraints

$$
\begin{array}{lllllll}
d_{01} &=& -d_{23} &=& -d_{45} &=& -d_{67}\\
d_{02} &=& d_{13} &=& -d_{46} &=& d_{57}\\
d_{03} &=& -d_{12} &=& d_{47} &=& d_{56}\\
d_{04} &=& d_{15} &=& d_{26} &=& -d_{37}\\
d_{05} &=& -d_{14} &=& -d_{27} &=& -d_{36}\\
d_{06} &=& d_{17} &=& -d_{24} &=& d_{35}\\
d_{07} &=& -d_{16} &=& d_{25} &=& d_{34}
\end{array}
\qquad (30)
$$

and $L_{g^{-1}*}g_*\cdot\xi$ is required to be in the seven-dimensional subspace spanned by the SO(8) generators $\{X_i|\ i=1,...,7\}$. This result can be traced to the fact that although $\iota_R(\tilde{g})$ and

$\iota_L(y^{-1})^T$ do not commute as elements of SO(8),

$$y \cdot \iota_R(\tilde{g}) \cdot \iota_L(y^{-1})^T \;=\; y \cdot \iota_L(y^{-1})^T \cdot \iota_R(\tilde{g}) \tag{31}$$

because any *two* elements of the octonions form an associative algebra:
$y^{-1}(y \cdot \tilde{g}) \;=\; (y^{-1}y) \cdot \tilde{g}$ for $y,\ \tilde{g} \in \mathbb{O}$.

Let $v_\mu \;=\; \sigma_* \cdot \xi_\mu \in T_{\sigma(x)}(E)$. Then $\omega(v_\mu) \;=\; L^{-1}_{y*} \cdot \mathcal{V} v_\mu$ and

$$\omega(R_{g*}v_\mu) \;=\; L^{-1}_{(y\cdot g)*}\mathcal{V}(R_{g*}v_\mu) \tag{32}$$

If v_μ lies in the vertical subspace of $T_{\sigma(x)}(E)$, $\mathcal{V}(R_{g*}v_\mu) \;=\; R_{g*}L_{y*}\omega(v_\mu)$. Representing left multiplication by a unit octonion L_y as right multiplication by an SO(8) transformation, denoted by R_{a_y}, one finds that

$$\omega(R_{g*}v_\mu) \;=\; R_{a^{-1}_{(y\cdot g)}*}R_{g*}R_{a_y*}\omega(v_\mu) \;=\; R_{(a_y g a^{-1}_{y\cdot g})*}\omega(v_\mu) \tag{33}$$

Defining $a_y g a^{-1}_{y\cdot g}$ to be $h(y,g) \in H$, the stability subgroup of SO(8) for the origin o in S^7, the connection form satisfies $\omega(R_{g*}v_\mu) \;=\; R_{h(y,g)*}\omega(v_\mu)$. Independence of the homogeneous part of the gauge transformation with respect to the fibre coordinate requires that $h(y,g)$ does not depend on y. If $g \;\leftrightarrow\; R_g^T \;=\; (c_{ij}), i,\ j \;=\; 0,1,...,7$, y-independence of the homogeneous term is confirmed by a calculation of $\iota_L(y)^T R_g^T[\iota_L(y\cdot g)^{-1}]^T$. However,

$$\begin{aligned}
\tilde{g}'\iota_L(y)^T[\iota_L(y\cdot\tilde{g})^{-1}] \;&=\; \tilde{g}'^T\iota_L(y)^T\tilde{g}\iota_L(y^{-1})^T\iota_L(\tilde{g}^{-1})^T \\
&\leftrightarrow\; g^{-1}\ y^{-1}\ [(y\cdot\tilde{g}')\cdot\tilde{g}] \\
&\neq\; g^{-1}(y^{-1}(y\cdot\tilde{g}')\cdot\tilde{g})) \\
&=\; g^{-1}\ (\tilde{g}'\ \cdot\ \tilde{g})
\end{aligned} \tag{34}$$

Thus, a y-dependent matrix is obtained when R_g^T is given by $e^{t_i X_i}$. A y-independent gauge transformation rule from the SO(8) action on an S^7 fibre, by analogy with the tranformation of the connection form resulting from an SO(4) action on S^3, therefore cannot be constructed.

The same procedure can also be applied to a bundle with S^7 fibre and an SU(4) structure group. This allows for the possibility of obtaining an action with the appropriate SU(3) gauge symmetry, based on an action with a larger symmetry group SU(4) more directly connected with the Clifford algebra $R_{1,9}$.

The action of SU(4) on the seven-sphere follows from the invariance of the bilinear form $\bar{z}'_0 z_0 + \bar{z}'_1 z_1 + \bar{z}'_2 z_2 + \bar{z}'_3 z_3$, where $z = (z_0\ z_1\ z_2\ z_3) \in \mathbb{C}^4$. To define the equivalent of the inhomogeneous term in the transformation rule of the connection form, one needs the embedding of S^7 into SU(4). Left multiplication by a unit octonion maps any point $y' \in S^7$ to a point $y'' = yy'$, and since a transitive group action on the sphere would take any pair of points into each other, left multiplication by the octonion y can be represented as $\iota_L(y) \in SU(4)$, where $\iota_L : S^7 \to SU(4)$.

Quaternions can be included in the group SO(4), as noted earlier, through

$$(y_0\ y_1\ y_2\ y_3) \rightarrow \begin{pmatrix} y_0 & -y_1 & -y_2 & -y_3 \\ y_1 & y_0 & -y_3 & y_2 \\ y_2 & y_3 & y_0 & -y_1 \\ y_3 & -y_2 & y_1 & y_0 \end{pmatrix} \tag{35}$$

and they can also be represented as SU(2) matrices

$$\begin{pmatrix} y_0 + iy_2 & y_1 + iy_3 \\ -y_1 + iy_3 & y_0 - iy_2 \end{pmatrix} \tag{36}$$

acting on the column vectors

$$\begin{pmatrix} y_0 + iy_2 \\ y_1 + iy_3 \end{pmatrix} \tag{37}$$

Similar considerations apply to the embedding of $S^7 \to SU(4)$. The problem of representing left multiplication by octonions therefore reduces to the problem of determining whether there is a correct choice of coordinate axes.

Proposition. There are no choices of octonion multiplication rules and coordinate axes in $\mathbb{C}^4$ such that left multiplication by unit octonions can be represented as a 4×4 matrix with entries of the form $\pm y_{r_i} \pm iy_{r_j}$.

Proof. Consider the transformation

$$y'' = Z\, y' \tag{38}$$

where $Z = (z_{ij})$ and y' and y'' are column vectors with the coordinate axes $(0\ r_1)$, $(r_2\ r_3)$, $(r_4\ r_5)$ and $(r_6\ r_7)$. Since

$$\begin{aligned} y''_0 + iy''_{r_1} = &[Re\, z_{00}\, y'_0 - Im\, z_{00}\, y'_{r_1} + Re\, z_{01}\, y'_{r_2} - Im\, z_{01}\, y'_{r_3}] \\ &- [Re\, z_{02}\, y'_{r_4} - Im\, z_{02}\, y'_{r_5} + Re\, z_{03}\, y'_{r_6} - Im\, z_{03}\, y'_{r_7}] \\ &+ i\,[Im\, z_{00}\, y'_0 + Re\, z_{00}\, y'_{r_1} + Im\, z_{01}\, y'_{r_3} + Re\, z_{01}\, y'_{r_3}] \\ &+ i\,[Im\, z_{02}\, y'_{r_4} + Re\, z_{02}\, y'_{r_5} + Im\, z_{03}\, y'_{r_6} + Re\, z_{03}\, y'_{r_7}] \end{aligned} \tag{39}$$

$$
\begin{aligned}
Re\ z_{00} &= y_0 & Re\ z_{01} &= -y_{r_2} \\
Re\ z_{02} &= -\ y_{r_4} & Re\ z_{03} &= -y_{r_6} \\
Im\ z_{00} &= y_{r_1} & Im\ z_{01} &= y_{r_3} \\
Im\ z_{02} &= y_{r_5} & Im\ z_{03} &= y_{r_7}
\end{aligned} \tag{40}
$$

so that

$$
\begin{aligned}
y''_{r_1} \;=\; & y_{r_1} y'_0 \;+\; y_0 y'_{r_1} \;-\; y_{r_2} y'_{r_3} \;+\; y_{r_3} y'_{r_2} \\
& -\; y_{r_4} y'_{r_5} \;+\; y_{r_5} y'_{r_4} \;-\; y_{r_6} y'_{r_7} \;+\; y_{r_7} y'_{r_6}
\end{aligned} \tag{41}
$$

which is consistent with the triples $(r_1 r_3 r_2)$, $(r_1 r_5 r_4)$ and $(r_1 r_7 r_6)$. The relation

$$
\begin{aligned}
y''_{r_2} \;+\; iy''_{r_3} \;=\; & [Re\ z_{10}\ y'_0 \;-\; Im\ z_{10}\ y'_{r_1} \;+\; Re\ z_{11}\ y'_{r_2} \;-\; Im\ z_{11}\ y'_{r_3}] \\
& +\; [Re\ z_{12}\ y'_{r_4} \;-\; Im\ z_{12}\ y'_{r_5} \;+\; Re\ z_{13}\ y'_{r_6} \;-\; Im\ z_{13}\ y'_{r_7}] \\
& +\; i\,[Im\ z_{10}\ y'_0 \;+\; Re\ z_{10}\ y'_{r_1} \;+\; Im\ z_{11}\ y'_{r_2} \;+\; Re\ z_{11}\ y'_{r_3}] \\
& +\; i\,[Im\ z_{12}\ y'_{r_4} \;+\; Re\ z_{12}\ y'_{r_5} \;+\; Im\ z_{13}\ y'_{r_6} \;+\; Re\ z_{13}\ y'_{r_7}]
\end{aligned} \tag{42}
$$

implies that

$$
\begin{aligned}
Re\ z_{10} &= y_{r_2} & Re\ z_{11} &= y_0 \\
Re\ z_{12} &= -\ y_{r_{T(r_2,r_4)}} & Re\ z_{13} &= -y_{r_{T(r_2,r_6)}} \\
Im\ z_{10} &= y_{r_3} & Im\ z_{11} &= -y_{r_1} \\
Im\ z_{12} &= y_{r_{T(r_2,r_5)}} & Im\ z_{13} &= y_{r_{T(r_2,r_7)}}
\end{aligned} \tag{43}
$$

where $e_{r_{T(r_i,r_j)}} \;\equiv\; e_{r_i} e_{r_j}$. Since

$$
\begin{aligned}
y''_{r_3} \;=\; & y_{r_3} y'_0 \;+\; y_{r_2} y'_{r_1} \;+\; y_0 y'_{r_3} \;-\; y_{r_1} y'_{r_2} \\
& +\; y_{r_{T(r_2,r_5)}} y'_{r_1} \;-\; y_{r_{T(r_2,r_4)}} y'_{r_5} \;+\; y_{r_{T(r_2,r_7)}} y'_{r_6} \;-\; y_{r_{T(r_2,r_6)}} y'_{r_7}
\end{aligned} \tag{44}
$$

the following triples, $(r_3\ r_{T(r_2,r_5)}\ r_4)$, $(r_3\ r_5\ r_{T(r_2,r_4)})$, $(r_3\ r_{T(r_2,r_7)}\ r_6)$ and $(r_3\ r_7\ r_{T(r_2,r_6)})$. The relation

$$
\begin{aligned}
y''_{r_4} \;+\; iy''_{r_5} \;=\; & [Re\ z_{20}\ y'_0 \;-\; Im\ z_{20}\ y'_{r_1} \;-\; Re\ z_{21}\ y'_{r_2} \;-\; Im\ z_{21}\ y'_{r_3}] \\
& +\; [Re\ z_{22}\ y'_{r_4} \;-\; Im\ z_{22}\ y'_{r_5} \;+\; Re\ z_{23}\ y'_{r_6} \;-\; Im\ z_{23}\ y'_{r_7}] \\
& +\; i\,[Im\ z_{20}\ y'_0 \;+\; Re\ z_{20}\ y'_{r_1} \;+\; Im\ z_{21}\ y'_{r_2} \;+\; Re\ z_{21}\ y'_{r_3}] \\
& +\; i\,[Im\ z_{22}\ y'_{r_4} \;+\; Re\ z_{22}\ y'_{r_5} \;+\; Imz_{23}\ y'_{r_6} \;+\; Re\ z_{23}\ y'_{r_7}]
\end{aligned} \tag{45}
$$

implies

$$
\begin{aligned}
Re\ z_{20} &= y_{r_4} & Re\ z_{21} &= y_{r_{T(r_2,r_4)}} \\
Re\ z_{22} &= y_0 & Re\ z_{23} &= -y_{r_{T(r_4,r_6)}} \\
Im\ z_{20} &= y_{r_5} & Im\ z_{21} &= -y_{r_{T(r_3,r_4)}} \\
Im\ z_{22} &= -\ y_{r_1} & Im\ z_{23} &= y_{r_{T(r_4,r_7)}}
\end{aligned} \tag{46}
$$

and

$$\begin{aligned} y''_{r_5} &= y_{r_5}y'_0 \ + \ y_{r_4}y'_{r_1} \ + \ y_{r_{T(r_2,r_4)}}y'_{r_3} \ - \ y_{r_{T(r_3,r_4)}}y'_{r_2} \\ &\quad - \ y_{r_1}y'_{r_4} \ + \ y_0y'_{r_5} \ + \ y_{r_{T(r_4,r_7)}}y'_{r_6} \ - \ y_{r_{T(r_4,r_6)}}y'_{r_7} \\ &= \ y_{r_5}y'_0 \ + \ y_{r_4}y'_{r_1} \ + \ y_{r_{T(r_3,r_5)}}y'_{r_3} \ + \ y_{r_{T(r_2,r_5)}}y'_{r_2} \\ &\quad - \ y_{r_1}y'_{r_4} \ + \ y_0y'_{r_5} \ + \ y_{r_{T(r_4,r_7)}}y'_{r_6} \ - \ y_{r_{T(r_4,r_6)}}y'_{r_7} \end{aligned} \tag{47}$$

consistent with the triples $(r_{T(r_4,r_7)}\ r_6\ r_5)$ and $(r_{T(r_4\ r_6)}\ r_5\ r_7)$. Consequently, as

$$\begin{aligned} y''_{r_6} \ + \ iy''_{r_7} &= [Re\ z_{30}\ y'_0 \ - \ Im\ z_{30}\ y'_{r_1} \ + \ Re\ z_{31}\ y'_{r_2} \ - \ Im\ z_{31}\ y'_{r_3}] \\ &\quad + \ [Re\ z_{32}\ y'_{r_4} \ - \ Im\ z_{32}\ y'_{r_5} \ + \ Re z_{33}\ y'_{r_6} \ - \ Im\ z_{33}\ y'_{r_7}] \\ &\quad + \ i\ [Im\ z_{30}\ y'_0 \ + \ Re\ z_{30}\ y'_{r_1} \ + \ Im\ z_{31}\ y'_{r_2} + \ Re\ z_{31}\ y'_{r_2}] \\ &\quad + \ i[Re\ z_{32}\ y'_{r_4} \ + \ Im\ z_{32}\ y_{r_4} \ + \ Re\ z_{33}\ y'_{r_7} \ + \ Im\ z_{33}\ y'_{r_6}] \end{aligned} \tag{48}$$

the matrix elements are

$$\begin{aligned} Re\ z_{30} &= y_{r_6} & Re\ z_{31} &= y_{r_{T(r_2,r_6)}} \\ Re\ z_{32} &= y_{r_{T(r_4,r_6)}} & Re\ z_{33} &= y_0 \\ Im\ z_{30} &= y_{r_7} & Im\ z_{31} &= -y_{r_{T(r_3,r_6)}} \\ Im\ z_{32} &= -\ y_{r_{T(r_5,r_6)}} & Im\ z_{33} &= -y_{r_1} \end{aligned} \tag{49}$$

and

$$\begin{aligned} y''_{r_7} &= \ y_{r_7}y'_0 \ + \ y_{r_{T(r_2,r_6)}}y'_{r_3} \ - \ y_{r_{T(r_3,r_6)}}y'_{r_2} \\ &\quad + \ y_{r_{T(r_4,r_6)}}y'_{r_5} \ - \ y_{r_{T(r_5,r_6)}}y'_{r_4} \ + \ y_0y'_{r_7} \ - \ y_{r_1}y'_{r_6} \\ &= \ y_{r_7}y'_0 \ + \ y_{r_6}y'_{r_1} \ + \ y_{r_{T(r_3,r_7)}}y'_{r_3} \ + \ y_{r_{T(r_2,r_7)}}y'_{r_2} \\ &\quad + \ y_{r_{T(r_5,r_7)}}y'_{r_5} \ + \ y_{r_{T(r_4,r_7)}}y'_{r_4} \ + \ y_0y'_{r_7} \ - \ y_{r_1}y'_{r_6} \end{aligned} \tag{50}$$

The four possible choices for the pair $\{r_{T(r_2,r_4)}, r_{T(r_2,r_5)}\}$ are $\{r_6, r_7\}$, $\{r_7, r_6\}$ and $\{r_6, -r_7\}$ and $\{-r_7, r_6\}$, where $e_{-r_i} \equiv -e_{r_i}$, there are no sets of triples consistent with any of the octonion multiplication tables. The same conclusion could be reached for right multiplication using the transpose, dropping the primes in the row vector and placing primes on the elements of the 4×4 matrix.

$$\begin{aligned} &(y''_0 \ + \ iy''_{r_7} \ \ y''_{r_2} \ + \ iy''_{r_3} \ \ y''_{r_4} \ + \ iy''_{r_5} \ \ y''_{r_6} \ + \ iy''_{r_7}) \\ &\quad = (y_0 \ + \ iy_{r_1} \ \ y_{r_2} \ + \ iy_{r_3} \ \ y_{r_4} \ + \ iy_{r_5} \ \ y_{r_6} \ + \ iy_{r_7}) \begin{pmatrix} z'_{00} & z'_{10} & z'_{20} & z'_{30} \\ z'_{01} & z'_{11} & z'_{21} & z'_{31} \\ z'_{02} & z'_{12} & z'_{22} & z'_{32} \\ z'_{03} & z'_{13} & z'_{23} & z'_{33} \end{pmatrix} \end{aligned} \tag{51}$$

This result also can be verified by noting that the first column of the 4×4 matrix mapping the origin o to y must have entries $z_{00} \ = \ y_0 \ + \ iy_1$, $z_{10} \ = \ y_2 \ + \ iy_3$, $z_{20} \ = \ y_4 \ + \ iy_5$

and $z_{30} = y_6 + iy_7$ and the unitarity relations for this matrix cannot be satisfied when the other entries are of the form $\sigma_{r_i r_{i+1}}(y_{r_i} \pm iy_{r_{i+1}})$, where $\sigma_{r_i r_{i+1}} = \pm 1$.

Another way of representing the embedding of S^7 into SU(4) might be achieved through the identification of the unit octonion $y = y_0 + y_1 e_1 + y_2 e_2 + y_3 e_3 + y_4 e_4 + y_5 e_5 + y_6 e_6 + y_7 e_7 = (y_0 + iy_1) + (y_2 + iy_3)e_2 + (y_4 + iy_5)e_4 + (y_6 + iy_7)e_6$ with a 4×4 matrix based on the quaternionic subalgebra $(1, e_2, e_4, e_6)$

$$\iota_L(y) = \begin{pmatrix} y_0 + iy_1 & (y_2 + iy_3)e_2 & (y_4 + iy_5)e_4 & (y_6 + iy_7)e_6 \\ (y_2 + iy_3)e_2 & y_0 + iy_1 & (y_6 + iy_7)e_6 & (y_4 + iy_5)e_4 \\ (y_4 + iy_5)e_4 & (y_6 + iy_7)e_6 & y_0 + iy_1 & (y_2 + iy_3)e_2 \\ (y_6 + iy_7)e_6 & (y_4 + iy_5)e_4 & (y_2 + iy_3)e_2 & y_0 + iy_1 \end{pmatrix} \tag{52}$$

so that

$$y'' = yy' = \iota_L(y) \begin{pmatrix} y'_0 + iy'_1 \\ (y'_2 + iy'_3)e_2 \\ (y'_4 + iy'_5)e_4 \\ (y'_6 + iy'_7)e_6 \end{pmatrix} \tag{53}$$

It is also of interest to note that since $\bar{y}'' y'' = \bar{y}' y' = 1$, $\iota_L(y)$ satisfies the identity $\iota_L(y)^\dagger \iota_L(y) = 1$, the defining relation for unitary matrices. However, the determinant of this matrix contains non-trivial expressions involving quaternions and non-associativity properties of the octonions. It is nevertheless possible to use the matrix $\iota_L(y)$ in the computation of the inhomogeneous term in the gauge transformation rule.

The generators J_{AB} of SO(8) in the expression $y \cdot (d_{AB} J_{AB}) \iota_L(y^{-1})^T$ may be replaced by the 4×4 traceless anti-hermitian generators of SU(4). Since the generators J_A are anti-hermitian and traceless,

$$d_A J_A = \begin{pmatrix} id_7 & d_1 + id_4 & d_2 + id_5 & d_3 + id_6 \\ -d_1 + id_4 & -id_7 + id_{14} & d_8 + id_9 & d_{10} + id_{11} \\ -d_2 + id_5 & -d_8 + id_9 & -id_{14} + id_{15} & d_{10} + id_{11} \\ -d_3 + id_6 & -d_{10} + id_{11} & -d_{12} + id_{13} & -id_{15} \end{pmatrix} \tag{54}$$

Using the row vector $y = (y_0 + iy_1 \; (y_2 + iy_3)e_2 \; (y_4 + iy_5)e_4 \; (y_6 + iy_7)e_6)$, it follows that $y(d_A J_A)\iota_L(y^{-1})^T = (c_0 \; c_1 \; c_2 \; c_3)$ where

$$\begin{aligned} c_0 = & \left[\, id_7 \, (y_0^2 + y_1^2) + i(d_7 - d_{14})(y_2^2 + y_3^2) + i(d_{14} - d_{15})(y_4^2 + y_5^2) + id_{15}(y_6^2 + y_7^2) \, \right] \\ & + \left[\, -2(y_0 + iy_1)(y_2 + iy_3)(d_1 + id_4) + 2d_{12}(y_4 y_6 + y_5 y_7) - 2d_{13}(y_4 y_7 - y_5 y_6) \, \right] e_2 \\ & + \left[\, -2(y_0 + iy_1)(y_4 + iy_5)(d_2 + id_5) + 2d_{10}(y_2 y_6 + y_3 y_7) - 2d_{11}(y_2 y_7 - y_3 y_6) \, \right] e_4 \\ & + \left[\, -2(y_0 + iy_1)(y_6 + iy_7)(d_3 + id_6) + 2d_8(y_2 y_4 + y_3 y_5) - 2d_9(y_2 y_5 + y_3 y_4) \, \right] e_6 \end{aligned}$$

$$
\begin{aligned}
c_1 \ &= [\ (y_0+iy_1)id_7 \ + \ (y_2+iy_3)e_2(-d_1+id_4) \ + \ (y_4+iy_5)e_4(-d_2+id_5) \\
&\qquad + \ (y_6+iy_7)e_6(-d_3+id_6)\ ]\ (-y_2-iy_3)e_2 \\
&+ [\ (y_0+iy_1)(d_1+id_4) \ + \ (y_2+iy_3)e_2(-id_7+id_{14}) \ + \ (y_4+iy_5)e_4(-d_8+id_9) \\
&\qquad + \ (y_6+iy_7)e_6(-d_{10}+id_{11})\ ]\ (y_0-iy_1) \\
&+ [\ (y_0+iy_1)(d_2+id_5) \ + \ (y_2+iy_3)e_2(d_8+id_9) \ + \ (y_4+iy_5)e_4(-id_{14}+id_{15}) \\
&\qquad + \ (y_6+iy_7)e_6(-d_{12}+id_{13})\ ]\ (-y_6-iy_7)e_6 \\
&+ [\ (y_0+iy_1)(d_3+id_6) \ + \ (y_2+iy_3)e_2(d_{10}+id_{11}) \ + \ (y_4+iy_5)e_4(d_{12}+id_{13}) \\
&\qquad + \ (y_6+iy_7)e_6(-id_{15})\ ]\ (-y_4-iy_5)e_4 \\
&\ \vdots
\end{aligned}
\tag{55}
$$

c_0 can only be independent of y if

$$
\begin{aligned}
&d_7 \ = \ d_7 \ - \ d_{14} \ = \ d_{14} \ - \ d_{15} \ = \ d_{15} \\
&d_1 \ + \ id_4 \ = \ 0 \qquad d_8 \ = \ d_9 \ = \ 0 \\
&d_2 \ + \ id_5 \ = \ 0 \qquad d_{10} \ = \ d_{11} \ = \ 0 \\
&d_3 \ + \ id_6 \ = \ 0 \qquad d_{12} \ = \ d_{13} \ = \ 0
\end{aligned}
\tag{56}
$$

and then

$$
c_1 \ = \ -2\ [\ id_4(y_2+iy_3)e_2 \ + \ id_5(y_4+iy_5)e_4 \ + \ id_6(y_6+iy_7)e_6\ ]\ (y_2+iy_3)e_2 \tag{57}
$$

which is only independent of y if $d_4 \ = \ d_5 \ = \ d_6 \ = \ 0$. Independence of the transformation rule of the connection form with respect to the fibre coordinate y leads to enough constraints on the gauge matrix so that the coefficients d_A must all vanish. Thus, there appear to be no non-trivial combinations of the generators such that the inhomogeneous term is independent of the fibre coordinate.

Another embedding of S^7 into the space of matrices satisfying the unitary relation and its effect on the terms in the connection form transformation rule will therefore be determined. The general 4×4 matrix is given by $g \ = \ D(\delta_1, \delta_2, \delta_3, -\delta_1 - \delta_2 - \delta_3) \cdot U_{23}(\phi_3, \sigma_6)U_{12}(\theta_3, \sigma_5)U_{13}(\phi_2, \sigma_4) \cdot \ U_{01}(\theta_2, \sigma_3)U_{02}(\theta_1, \sigma_2)U_{03}(\phi_1, \sigma_1)$, where $D(\delta_1, \delta_2, \delta_3, -\delta_1 - \delta_2 - \delta_3)$ is the diagonal matrix with elements $e^{i\delta_1}$, $e^{i\delta_2}$, $e^{i\delta_3}$, $e^{-i\delta_1-\delta_2-\delta_3}$ and $U_{pq}(\phi, \sigma)$, which has all diagonal elements equal to 1, except for u_{pp} and u_{qq}, which should be $cos\ \phi$, and non-zero off-diagonal entries $u_{pq} \ = \ -sin\ \phi\ e^{-i\sigma}$, $u_{qp} \ = \ sin\ \phi\ e^{i\sigma}$, represents a unitary transformation in the (p,q)-plane. Consequently, the elements of g are

$$
\begin{aligned}
g_{00} &= cos\ \theta_2\ cos\ \theta_1\ cos\ \phi_1\ e^{i\delta_1} \\
g_{01} &= -sin\ \theta_2\ e^{i(\delta_1-\sigma_3)} \\
g_{02} &= -cos\ \theta_2\ sin\ \theta_1\ e^{(\delta_1-\sigma_2)} \\
g_{03} &= -cos\ \theta_2\ cos\ \theta_1\ sin\ \phi_1\ e^{i(\delta_1-\sigma_1)} \\
g_{10} &= cos\ \theta_3\ sin\ \theta_2\ cos\ \theta_1\ cos\ \phi_1 e^{i(\delta_2+\sigma_3)} \\
&\quad -\ sin\ \theta_3\ sin\ \theta_1\ cos\ \phi_1 e^{i(\sigma_2+\delta_2-\sigma_5)}\ -cos\ \theta_3\ \sin\ \phi_2\ sin\ \phi_1 e^{i(\sigma_1+\delta_2-\sigma_4)} \\
g_{11} &= cos\ \theta_2\ cos\ \theta_3\ cos\ \phi_2\ e^{i\delta_2} \\
g_{12} &= -cos\ \theta_3\ cos\ \phi_2\ sin\ \theta_2\ sin\ \theta_1\ e^{i(\delta_2+\sigma_3-\sigma_2)}\ -\ sin\ \theta_3\ cos\ \theta_1\ e^{i(\delta_2-\sigma_5)} \\
g_{13} &= -cos\ \theta_3\ cos\ \phi_2\ sin\ \theta_2\ sin\ \phi_1\ cos\ \theta_1 e^{i(\delta_2+\sigma_3-\sigma_1)} \\
&\quad +\ sin\ \theta_3\ sin\ \theta_1\ sin\ \phi_1\ e^{i(\delta_2+\sigma_2-\sigma_1-\sigma_5)} \\
&\quad -cos\ \theta_3\ sin\ \phi_2\ cos\ \phi_1\ e^{i(\delta_2-\sigma_4)} \\
g_{20} &= cos\ \phi_3\ sin\ \theta_3\ cos\ \phi_2\ sin\ \theta_2\ cos\ \theta_1\ cos\ \phi_1\ e^{i(\delta_3+\sigma_3+\sigma_5)} \\
&\quad +\ cos\ \theta_3\ cos\ \phi_3\ sin\ \theta_1\ cos\ \phi_1\ e^{i(\delta_3+\sigma_2)} \\
&\quad -cos\ \phi_3\ sin\ \theta_3\ sin\ \phi_2\ sin\ \phi_1\ e^{i(\sigma_1+\delta_3+\sigma_5-\sigma_4)} \\
g_{21} &= cos\ \phi_3\ sin\ \theta_3\ cos\ \phi_2\ cos\ \theta_2\ e^{i(\delta_3+\sigma_5)} \\
g_{22} &= -cos\ \phi_3\ sin\ \theta_3\ cos\ \phi_2\ sin\ \theta_2\ sin\ \theta_1\ e^{i(\delta_3+\sigma_3+\sigma_5-\sigma_2)} \\
&\quad +\ cos\ \phi_3\ cos\ \theta_3\ cos\ \theta_1\ e^{i\delta_3} \\
g_{23} &= -cos\ \phi_3\ sin\ \theta_3\ cos\ \phi_2\ sin\ \theta_2\ cos\ \theta_1\ sin\ \phi_1\ e^{i(\delta_3+\sigma_3-\sigma_1)} \\
&\quad -\ cos\ \phi_3\ cos\ \theta_3\ sin\ \theta_1\ sin\ \phi_1\ e^{i(\delta_3+\sigma_2-\sigma_1)} \\
&\quad -cos\ \phi_3\ sin\ \theta_3\ sin\ \phi_2\ cos\ \phi_1\ e^{i(\delta_3+\sigma_5-\sigma_4)} \\
g_{30} &= sin\ \phi_3\ sin\ \theta_3\ cos\ \phi_2\ sin\ \theta_2\ cos\ \theta_1\ cos\ \phi_1\ e^{i(\sigma_3+\sigma_5+\sigma_6-\delta_1-\delta_2-\delta_3)} \\
&\quad +\ cos\ \phi_3\ sin\ \phi_2\ sin\ \theta_2\ cos\ \theta_1\ cos\ \phi_1\ e^{i(\sigma_3+\sigma_4-\delta_1-\delta_2-\delta_3)} \\
&\quad +sin\ \phi_3\ cos\ \theta_3\ sin\ \theta_1\ cos\ \phi_1\ e^{i(\sigma_2+\sigma_6-\delta_1-\delta_2-\delta_3)} \\
&\quad +\ cos\ \phi_3\ cos\ \phi_2\ sin\ \phi_1\ e^{i(\sigma_1-\delta_1-\delta_2-\delta_3)} \\
&\quad -\ sin\ \phi_3\ sin\ \theta_3\ sin\ \theta_2\ sin\ \phi_1\ e^{i(\sigma_1+\sigma_5+\sigma_6-\sigma_4-\delta_1-\delta_2-\delta_3)} \\
g_{31} &= sin\ \phi_3\ sin\ \theta_3\ cos\ \phi_2\ cos\ \theta_2 e^{i(\sigma_5+\sigma_6-\delta_1-\delta_2-\delta_3)} \\
&\quad +\ cos\ \phi_3\ sin\ \phi_2\ cos\ \theta_2\ e^{i(\sigma_4-\delta_1-\delta_2-\delta_3)} \\
g_{32} &= -sin\ \phi_3\ sin\ \theta_3\ cos\ \phi_2\ sin\ \theta_2\ sin\ \theta_1 e^{i(\sigma_3+\sigma_5-\sigma_2-\sigma_6-\delta_1-\delta_2-\delta_3)} \\
&\quad -\ cos\ \phi_3\ sin\ \theta_2\ sin\ \phi_2\ e^{i(\sigma_3+\sigma_4-\sigma_2-\delta_1-\delta_2-\delta_3)} \\
&\quad -\ sin\ \phi_3\ cos\ \theta_3\ cos\ \theta_1\ e^{i(\sigma_6-\delta_1-\delta_2-\delta_3)}
\end{aligned}
$$

$$
\begin{aligned}
g_{33} \;=\; & -sin\ \phi_3\ cos\ \theta_3\ sin\ \theta_1\ sin\ \phi_1\ e^{i(\sigma_2+\sigma_6-\sigma_1-\delta_1-\delta_2-\delta_3)} \\
& -\ sin\ \phi_3\ sin\ \theta_3\ cos\ \phi_2\ sin\ \theta_2\ cos\ \theta_1\ sin\ \phi_1 e^{(\sigma_3+\sigma_5+\sigma_6-\sigma_1-\delta_1-\delta_2-\delta_3)} \\
& -cos\ \phi_3\ sin\ \phi_2\ sin\ \theta_2\ cos\ \theta_1\ cos\ \phi_1\ e^{i(\sigma_3+\sigma_4-\sigma_1-\delta_1-\delta_2-\delta_3)} \\
& -\ sin\ \phi_3\ sin\ \theta_3\ sin\ \phi_2\ cos\ \phi_1\ e^{i(\sigma_5+\sigma_6-\sigma_4-\delta_1-\delta_2-\delta_3)} \\
& +cos\ \phi_3\ cos\ \phi_2\ cos\ \phi_1\ e^{-i(\delta_1+\delta_2+\delta_3)}
\end{aligned} \tag{58}
$$

The embedding of $\iota_L : S^7 \;\rightarrow\; SU(4)$ will be defined so that

$$
\begin{aligned}
\iota_L(y) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} y_0+iy_1 \\ y_2+iy_3 \\ y_4+iy_5 \\ y_6+iy_7 \end{pmatrix} \\
(1\ 0\ 0\ 0)\ \iota_L(y)^T &= (y_0+iy_1\ y_2+iy_3\ y_4+iy_5\ y_6+iy_7)
\end{aligned} \tag{59}
$$

For simplicity, one may choose g to be $\iota_L(y)^T$ so that

$$
\begin{aligned}
y_0+iy_1 &= g_{00} = cos\ \theta_2\ cos\ \theta_1\ cos\ \phi_1\ e^{i\delta_1} \\
y_2+iy_3 &= g_{01} = -sin\ \theta_2\ e^{i(\delta_1-\sigma_3)} \\
y_4+iy_5 &= g_{02} = -cos\ \theta_2\ sin\ \theta_1\ e^{i(\delta_1-\sigma_2)} \\
y_6+iy_7 &= g_{03} = -cos\ \theta_2\ cos\ \theta_1\ sin\ \phi_1\ e^{i(\delta_1-\sigma_1)}
\end{aligned} \tag{60}
$$

which clearly satisfies $\bar{y}y \;=\; 1$. The proposition concerning the existence of SU(4) matrices mapping o to y, or y' to $y'' \;=\; yy'$ is circumvented because the other entries of $g \;=\; \iota_L(y)^T$ are related nonlinearly to $g_{00},\ g_{01},\ g_{02}$ and g_{03}. Any SU(4) matrix with the first row given by (52) will map o to y. To simplify the calculations involving the entire matrix, the other eight parameters, $\delta_2,\ \delta_3,\ \phi_2,\ \phi_3,\ \theta_3,\ \sigma_4,\ \sigma_5,\ \sigma_6$ shall be set to zero. As $\iota_L(y^{-1})^T \ = \ g^\dagger$,

$$
\begin{aligned}
& y \cdot (d_A J_A)(\iota_L(y^{-1})^T) \ \equiv \ (c_0\ c_1\ c_2\ c_3) \\
& = \ (cos\ \theta_2\ cos\ \theta_1\ cos\ \phi_1\ e^{i\delta_1} \ \ - sin\ \theta_2\ e^{i(\delta_1-\sigma_3)} \ - \ cos\ \theta_2\ sin\ \theta_1\ e^{i(\delta_1-\sigma_2)} \\
& \qquad\qquad\qquad\qquad - cos\ \theta_2\ cos\ \theta_1\ sin\ \phi_1\ e^{i(\delta_1-\sigma_1)}) \\
& \begin{pmatrix} id_7 & d_1+id_4 & d_2+id_5 & d_3+id_6 \\ -d_1+id_4 & -id_7+id_{14} & d_8+id_9 & d_{10}+id_{11} \\ -d_2+id_5 & -d_8+id_9 & -id_{14}+id_{15} & d_{12}+id_{13} \\ -d_3+id_6 & -d_{10}+id_{11} & -d_{12}+id_{13} & -id_{15} \end{pmatrix} \\
& \begin{pmatrix} cos\ \theta_2\ cos\ \theta_1\ cos\ \phi_1\ e^{-i\delta_1} & sin\ \theta_2\ cos\ \theta_1\ cos\ \phi_1\ e^{-i\sigma_3} & sin\ \theta_1\ cos\ \phi_1\ e^{-i\sigma_2} & sin\ \phi_1\ e^{i(\delta_1-\sigma_1)} \\ - sin\ \theta_2\ e^{-i(\delta_1-\sigma_3)} & cos\ \theta_2 & 0 & 0 \\ - cos\ \theta_2\ sin\ \theta_1\ e^{-i(\delta_1-\sigma_2)} & -sin\ \theta_2\ sin\ \theta_1\ e^{i(\sigma_2-\sigma_3)} & cos\ \theta_1 & 0 \\ - cos\ \theta_2\ cos\ \theta_1\ sin\ \phi_1\ e^{-i(\delta_1-\sigma_1)} & -sin\ \theta_2\ cos\ \theta_1\ sin\ \phi_1\ e^{i(\sigma_1-\sigma_3)} & -sin\ \theta_1 sin\ \phi_1\ e^{i(\sigma_1-\sigma_2)} & cos\ \phi_1\ e^{i\delta_1} \end{pmatrix}
\end{aligned}
\tag{61}
$$

so that

$$
\begin{aligned}
c_0 \ = \ & (cos^2\theta_2\ cos^2\theta_1\ cos^2\phi_1)(id_7) \ + \ sin^2\theta_2(-id_7+id_{14}) \\
& + \ (cos^2\theta_2\ sin^2\theta_1)(-id_{14}+id_{15}) \ + \ (cos^2\theta_2\ cos^2\theta_1\ sin^2\phi_1)(-id_{15}) \\
& + \ 2i(-sin\ \theta_2\ cos\ \theta_2\ cos\ \theta_1\ cos\ \phi_1)\ [d_1\ sin\ \sigma_3\ + \ d_4\ cos\ \sigma_3] \\
& + \ 2i(-cos^2\theta_2\ sin\ \theta_1\ \cos\ \theta_1\ cos\ \phi_1)\ [d_2\ sin\ \sigma_2\ + \ d_5\ cos\ \sigma_2] \\
& + \ 2i(-cos^2\theta_2\ cos^2\theta_1\ sin\ \phi_1\ sin\ \phi_1)\ [d_3\ sin\ \sigma_1\ + \ d_6\ cos\ \sigma_1] \\
& + \ 2i(sin\ \theta_2\ cos\ \theta_2\ cos\ \theta_1\ sin\ \phi_1)\ [d_8\ sin\ (\sigma_2-\sigma_3)\ + \ d_9\ cos\ (\sigma_2-\sigma_3)] \\
& + \ 2i(sin\ \theta_2\ cos\ \theta_2\ cos\ \theta_1\ sin\ \phi_1)\ [d_{10}\ sin\ (\sigma_1-\sigma_3)\ + \ d_{11}\ cos\ (\sigma_1-\sigma_3)] \\
& + \ 2i(cos^2\theta_2\ sin\ \theta_1\ cos\ \theta_1\ sin\ \phi_1)\ [d_{12}\ sin\ (\sigma_1-\sigma_2)\ + \ d_{13}\ cos\ (\sigma_1-\sigma_2)]
\end{aligned}
\tag{62}
$$

Independence of c_0 with respect to the angular coordinates requires

$$
\begin{aligned}
& id_7 \ = \ -id_7 \ + \ id_{14} \ = \ -id_{14} \ + \ id_{15} \ = \ -id_{15} \\
& \sigma_1 \ = \ \sigma_2 \ = \ \sigma_3 \ = \ 0 \\
& d_5 \ = \ d_6 \ = \ d_9 \ = \ d_{11} \ = \ d_{13} \ = \ 0
\end{aligned}
\tag{63}
$$

The remaining coefficients are $d_1,\ d_2,\ d_3,\ d_8,\ d_{10}$ and d_{12} and the fibre is restricted to a four-dimensional submanifold of S^7, $SU(2)\times U(1)$. Similarly,

$$
c_1 \ = \ d_1\ cos\ \theta_1\ cos\ \phi_1\ e^{i\delta_1} \ + \ d_8\ sin\ \theta_1\ e^{i\delta_1} \ + \ d_{10}\ cos\ \theta_1\ sin\ \phi_1\ e^{i\delta_1} \tag{64}
$$

which implies the vanishing of $d_1,\ d_8$ and d_{10}, and

$$c_2 \;=\; d_2\ cos\ \theta_2\ cos\ \phi_1\ e^{i\delta_1}\ +\ d_{12}\ cos\ \theta_2\ sin\ \phi_1\ e^{i\delta_1} \tag{65}$$

which is independent of the angles if $d_2 \;=\; d_{12} \;=\; 0$. Finally,

$$c_3 \;=\; d_3\ cos\ \theta_2\ cos\ \theta_1\ e^{i\delta_1} \tag{66}$$

so that independence with respect to the fibre coordinates can be achieved either by setting $d_3 \;=\; 0$ or $\theta_1 \;=\; \theta_2 \;=\; \delta_1 \;=\; 0$. The latter choice is obviously preferable as it still leaves a non-trivial action on the submanifold of the fibre parametrized by ϕ_1, namely the action of U(1) on S^1.

Independence of the homogeneous part of the gauge transformation with respect to the fibre coordinates follows from the relation

$$L_y^T\ R_g^T\ L_{(y\cdot g)^{-1}}^T \;=\; \iota_L(y)^T\ exp(d_3\ J_3)\ [\iota_L(y\cdot g)^{-1})]^T \;=\; Id_4 \tag{67}$$

The remaining gauge symmetry is therefore associated with a U(1) gauge potential transforming as

$$\begin{aligned} A_\mu\ &\to\ A_\mu\ +\ (\partial_\mu g)g^{-1} \\ A_\mu^3\ &\to\ A_\mu^3\ +\ \partial_\mu\ d_3 \end{aligned} \tag{68}$$

4. Constraints on Gauge Transformations acting on a Submanifold in an S^7 Bundle

Constraints on the connection form transformation rule can also be obtained for bundles with fibres that are submanifolds of S^7. The structure group initially can be chosen to be any subgroup of SO(8) which preserves the submanifold. However, there also should be less conditions on the allowed transformations, because there are fewer fibre coordinates to be eliminated. Therefore, this leaves open the possibility of a residual gauge symmetry associated with the action of a subgroup of the original structure group on a submanifold of the standard fibre.

For the S^7 bundle, one choice for the submanifold is $S^3 \;=\; \{y \in S^7\ |\ y_0^2+y_1^2+y_2^2+y_3^2 = 1,\ y_4 = y_5 = y_6 = y_7 = 0\}$. Independence of the inhomogeneous term, or equivalently

$c_1, \ ..., \ c_7$, with respect to $y_1, \ ..., \ y_3$ leads to the 15 conditions

$$\begin{aligned}
d_{01} &= -d_{23} \qquad d_{02} = d_{13} \qquad d_{03} = -d_{12} \\
d_{04} &= d_{15} = d_{26} = -d_{37} \\
d_{05} &= -d_{14} = -d_{27} = -d_{36} \\
d_{06} &= d_{17} = -d_{24} = d_{35} \\
d_{07} &= -d_{16} = d_{25} = d_{34}
\end{aligned} \tag{69}$$

and the generators of the remaining 13 transformations are $J_{01} - J_{23}, \ J_{02} + J_{13}, \ J_{03} - J_{12}, \ J_{04} + J_{15} + J_{26} - J_{37}, \ J_{05} - J_{14} - J_{25} - J_{36}, \ J_{06} + J_{17} - J_{24} + J_{35}, \ J_{07} - J_{16} + J_{25} + J_{34}, \ J_{45}, \ J_{46}, \ J_{47}, \ J_{56}, \ J_{57}$ and J_{67}. Independence of the homogeneous terms can be checked for these generators by establishing the y-independence of $L_y^T R_g^T L_{(y \cdot g)^{-1}}^T$, with L_y^T given by (22) after replacing y_i by $-y_i$ when $i = 1, 2, 3$ and setting $y_j \ = \ 0, \ j \geq 4$, and R_g^T equal to the exponential of the generator. This property can be verified for the first seven generators and holds trivially for the last six generators.

It is now necessary to note that the transformations generated by $J_{04} + J_{15} + J_{26} - J_{37}, \ J_{05} - J_{14} - J_{25} - J_{36}, \ J_{06} + J_{17} - J_{24} + J_{35}$ and $J_{07} - J_{16} + J_{25} + J_{34}$ do not leave S^3 invariant and instead map it into four-dimensional submanifolds of S^7, consisting of a one-parameter family of three-spheres. The coordinates of these four-dimensional submanifolds are, respectively,

$$\begin{aligned}
&(y_0 \ cos \ d_{04}, \ y_1 \ cos \ d_{04}, \ y_2 \ cos \ d_{04}, \ y_3 \ cos \ d_{04}, \ y_0 \ sin \ d_{04}, \ y_1 \ sin \ d_{04} \\
&\qquad\qquad\qquad\qquad y_2 \ sin \ d_{04}, \ - y_3 \ sin \ d_{04}) \\
&(y_0 \ cos \ d_{05}, \ y_1 \ cos \ d_{05}, \ y_2 \ cos \ d_{05}, \ y_3 \ cos \ d_{05}, \ - y_1 \ sin \ d_{05}, \ y_0 \ sin \ d_{05}, \\
&\qquad\qquad\qquad\qquad - y_3 \ sin \ d_{05}, \ - y_2 \ sin \ d_{05}) \\
&(y_0 \ cos \ d_{06}, \ y_1 \ cos \ d_{06}, \ y_2 \ cos \ d_{06}, \ y_3 \ cos \ d_{06}, -y_2 \ sin \ d_{06}, \ y_3 \ sin \ d_{06}, \\
&\qquad\qquad\qquad\qquad y_0 \ sin \ d_{06}, \ y_1 \ sin \ d_{06}) \\
&(y_0 \ cos \ d_{07}, \ y_1 \ cos \ d_{07}, \ y_2 \ cos \ d_{07}, \ y_3 \ cos \ d_{07}, \ y_3 \ sin \ d_{07}, \ y_2 \ sin \ d_{07}, \\
&\qquad\qquad\qquad\qquad - y_1 \ sin \ d_{07}, \ y_0 \ sin \ d_{07})
\end{aligned} \tag{70}$$

and since there is a bijective, continuous map to the coordinates $\{((y_0, y_1, y_2, y_3), \ \theta) \mid y_0^2 + y_1^2 + y_2^2 + y_3^2 = 1\}$, each of these submanifolds is topologically $S^3 \times S^1$. Moreover, since the maps between the coordinates are also diffeomorphisms, one might consider constructing an $S^3 \times S^1$ bundle.

However, the enlargement of the fibre from S^3 to $S^3 \times S^1$ with non-zero entries in the last four components of the row vectors (70) implies that the quantities c_i in (23) - (29) will not necessarily be independent of y. Three extra conditions

$$\begin{aligned} d_{01} \ + \ d_{45} \ &= \ d_{23} \ - \ d_{67} \\ d_{02} \ + \ d_{46} \ &= \ -d_{13} \ + \ d_{57} \\ d_{03} \ - \ d_{47} \ &= \ d_{12} \ + \ d_{56} \end{aligned} \tag{71}$$

must be satisfied before c_4, ..., c_7 are independent of the coordinates (50). There would then be 10 remaining generators $J_{01} - J_{23} + \alpha J_{45} - (\alpha+2)J_{67}$, $J_{02} + J_{13} + \beta J_{46} + (\beta+2)J_{57}$, $J_{03} - J_{12} + \gamma J_{47} + (2-\gamma)J_{56}$, $J_{04} + J_{15} + J_{26} - J_{37}$, $J_{05} - J_{14} - J_{25} - J_{36}$, $J_{05} - J_{14} - J_{25} - J_{36}$, $J_{06} + J_{17} - J_{24} + J_{35}$, $J_{07} - J_{16} + J_{25} + J_{34}$, $J_{45} - J_{67}$, $J_{46} + J_{57}$ and $J_{47} - J_{56}$.

The computation of $L_y^T R_g^T L_{(y\cdot g)^{-1}}^T$ must now be repeated for these 10 generators with the new coordinates (70) to determine whether the dependence on y_i and θ can be eliminated in the homogeneous term in the transformation rule of the connection form. Denoting the coordinates of $S^3 \times S^1$ by $y_\theta \ = \ y \cdot g_{0l}(\theta), l \ = \ 4,5,6,7$ so that

$$\begin{aligned} g_{04}(\theta) \ &= \ (cos \ \theta, \ 0, \ 0, \ 0, \ sin \ \theta, \ 0, \ 0, \ 0) \\ g_{05}(\theta) \ &= \ (cos \ \theta, \ 0, \ 0, \ 0, \ 0, \ sin \ \theta, \ 0, \ 0) \\ g_{06}(\theta) \ &= \ (cos \ \theta, \ 0, \ 0, \ 0, \ 0, \ 0, \ sin \ \theta, \ 0) \\ g_{07}(\theta) \ &= \ (cos \ \theta, \ 0, \ 0, \ 0, \ 0, \ 0, \ 0, \ sin \ \theta) \end{aligned} \tag{72}$$

the following identity for alternative algebras [24]

$$L_{y\cdot y'} \ = \ L_y L_{y'} \ + \ [L_y, R_{y'}] \tag{73}$$

implies that

$$\begin{aligned} L^T_{y\cdot g_{0l}(\theta)} \ &= \ (L_y L_{g_{0l}(\theta)} \ + \ L_y R_{g_{0l}(\theta)} \ - \ R_{g_{0l}(\theta)} L_y)^T \\ &= \ L^T_{g_{0l}(\theta)} L^T_y \ + \ R^T_{g_{0l}(\theta)} L^T_y \ - \ L^T_y R^T_{g_{0l}(\theta)} \end{aligned} \tag{74}$$

Since

$$\begin{aligned} L_{((y\cdot g_{0l}(\theta))\cdot R_g^T)^{-1}} y' \ &= \ ((y \cdot g_{0l}(\theta)) \cdot R_g^T)^{-1} y' \ = \ (R_g^T)^{-1}(y \cdot g_{0l}(\theta))^{-1} y' \\ &= \ (R_g^T)^{-1}(g_{0l}(\theta)^{-1} \cdot y^{-1}) \cdot y' \ = \ (R_g^T)^{-1} L_{g_{0l}(\theta)^{-1}\cdot y^{-1}} \cdot y' \end{aligned} \tag{75}$$

and

$$L_{g_{0l}(\theta)^{-1}\cdot y^{-1}} \ = \ L_{g_{0l}(\theta)^{-1}} L_{y^{-1}} \ + \ L_{g_{0l}(\theta)^{-1}} R_{y^{-1}} \ - \ R_{y^{-1}} L_{g_{0l}(\theta)^{-1}} \tag{76}$$

it can be shown that

$$
\begin{aligned}
L^T_{y_\theta}\ R^T_g\ L^T_{(y_\theta\cdot g)^{-1}} &= L^T_{y\cdot g_{0l}(\theta)}\ R^T_g\ L^T_{((y\cdot g_{0l}(\theta))\cdot g)^{-1}} \\
&= (L^T_{g_{0l}(\theta)} L^T_y \ +\ R^T_{g_{0l}(\theta)} L^T_y \ -\ L^T_y R^T_{g_{0l}(\theta)}) R^T_g \\
&\qquad\qquad (L^T_{y^{-1}} L^T_{g_{0l}(\theta)^{-1}} \ +\ R^T_{y^{-1}} L^T_{g_{0l}(\theta)^{-1}} \ -\ L^T_{g_{0l}(\theta)^{-1}} R^T_{y^{-1}}) R_g^{-1}
\end{aligned}
\tag{77}
$$

The next step in determining whether the fibre-coordinate dependence can be eliminated from this expression would involve moving all of the θ-dependent matrices to the center. Although it can be verified that $L_{g_{0l}(\theta)} L_y = L_y L_{g_{0l}(\theta)}$, the commutators $[L_y, R_{g_{0l}(\theta)}]$ and $[R_y, L_{g_{0l}(\theta)}]$ do not similarly vanish. Thus, the product (77) will be dependent on y and θ for the general group element obtained by exponentiating the 10 generators listed above.

Consequently, to find any residual gauge symmetry, the coordinates must be restricted to the S^3 submanifold considered initially. From the above considerations, it follows that only 9 generators $J_{01} - J_{23},\ J_{02} + J_{13},\ J_{03} - J_{12},\ J_{45},\ J_{46},\ J_{47},\ J_{56},\ J_{57},\ J_{67}$ leave this submanifold invariant, and the last 6 generators act trivially on this three-sphere. As the remaining generators form an su(2) Lie algebra, there remains an SU(2) group of symmetry transformations acting on an S^3 fibre, which represents the gauge invariance of an SU(2) Yang-Mills theory corresponding to the SU(2) principal bundle.

When the submanifold is $S^4 = \{y \in S^7 \,|\, y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1,\ y_5 = y_6 = y_7 = 0\}$, independence of the inhomogenous term with respect to y leads to 18 conditions

$$
\begin{aligned}
d_{01} &= -d_{23} = -d_{45} \\
d_{02} &= d_{13} = -d_{46} \\
d_{03} &= -d_{12} = d_{47} \\
d_{04} &= d_{15} = d_{26} = -d_{37} \\
d_{05} &= -d_{14} = -d_{27} = -d_{36} \\
d_{06} &= d_{17} = -d_{24} = d_{35} \\
d_{07} &= -d_{16} = d_{25} = d_{34}
\end{aligned}
\tag{78}
$$

and the generators of the remaining ten generators are $J_{01} - J_{23} - J_{45},\ J_{02} + J_{13} - J_{46},\ J_{03} - J_{12} + J_{47},\ J_{04} + J_{15} + J_{26} - J_{37},\ J_{05} - J_{14} - J_{27} - J_{36},\ J_{06} + J_{17} - J_{24} + J_{35},\ J_{07} - J_{16} + J_{25} + J_{34},\ J_{56},\ J_{57},\ J_{67}$.

The exponentials of the generators $J_{01} - J_{23} - J_{45}$, $J_{02} + J_{13} - J_{46}$ and $J_{03} - J_{12} + J_{47}$

map S^4 into a five-dimensional submanifold diffeomorphic to $S^4 \times S^1$, with coordinates

$$
\begin{aligned}
&(y_0 \, cos \, d_{01} + y_1 \, sin \, d_{01}, \; -y_0 \, sin \, d_{01} + y_1 \, cos \, d_{01}, \; y_2 \, cos \, d_{01} - y_3 \, sin \, d_{01}, \\
&\qquad y_2 \, sin \, d_{01} + y_3 \, cos \, d_{01}, \; y_4 \, cos \, d_{01}, \; y_4 \, sin \, d_{01}, \; 0, \; 0) \\
&(y_0 \, cos \, d_{02} + y_2 \, sin \, d_{02}, y_1 \, cos \, d_{02} + y_3 \, sin \, d_{02}, -y_0 \, sin \, d_{02} + y_3 \, cos \, d_{02}, \\
&\qquad -y_1 \, sin \, d_{02} + y_3 \, cos \, d_{02}, \; y_4 \, cos \, d_{02}, \; 0, \; y_4 \, sin \, d_{02}, \; 0) \\
&(y_0 \, cos \, d_{03} + y_3 \, sin \, d_{03}, y_1 \, cos \, d_{03} - y_2 \, sin \, d_{03}, \; y_1 \, sin \, d_{03} + y_3 \, cos \, d_{03}, \\
&\qquad -y_0 \, sin \, d_{03} + y_3 \, cos \, d_{03}, \; y_4 \, cos \, d_{03}, \; 0, \; 0, \; -y_4 \, sin \, d_{03})
\end{aligned}
\tag{79}
$$

The transformations obtained by exponentiating the four generators $J_{04} + J_{15} + J_{26} - J_{37}, \; J_{05} - J_{14} - J_{27} - J_{36}, \; J_{06} + J_{17} - J_{24} + J_{35}, \; J_{07} - J_{16} + J_{25} + J_{34}$ map S^4 into a five-dimensional submanifold diffeomorphic to $S^4 \times S^1$, with coordinates

$$
\begin{aligned}
&(y_0 \, cos \, d_{04} - y_4 \, sin \, d_{04}, y_1 \, cos \, d_{04}, \; y_2 \, cos \, d_{04}, \; y_3 \, cos \, d_{04}, \\
&\qquad y_0 \, sin \, d_{04} + y_4 \, cos \, d_{04}, \; y_1 \, sin \, d_{04}, \; y_2 \, sin \, d_{04}, \; -y_3 \, sin \, d_{04}) \\
&(y_0 \, cos \, d_{05}, \; y_1 \, cos \, d_{05} - y_4 \, sin \, d_{05}, \; y_2 \, cos \, d_{05}, \; y_3 \, cos \, d_{05}, \\
&\qquad y_1 \, sin \, d_{05} + y_4 \, cos \, d_{05}, \; y_0 \, sin \, d_{05}, \; -y_3 \, sin \, d_{05}, \; -y_2 \, sin \, d_{05}) \\
&(y_0 \, cos \, d_{06}, \; y_1 \, cos \, d_{06}, \; y_2 \, cos \, d_{06} - y_4 \, sin \, d_{06}, \; y_3 \, cos \, d_{06}, \\
&\qquad y_2 \, sin \, d_{06} + y_4 \, cos \, d_{06}, \; -y_3 \, sin \, d_{06}, \; -y_0 \, sin \, d_{06}, \; -y_1 \, sin \, d_{06}) \\
&(y_0 \, cos \, d_{07}, \; y_1 \, cos \, d_{07}, \; y_2 \, cos \, d_{07}, \; y_3 \, cos \, d_{07} + y_4 \, sin \, d_{07}, \\
&\qquad -y_3 \, sin \, d_{07} + y_4 \, cos \, d_{07}, \; -y_2 \, sin \, d_{07}, \; y_1 \, sin \, d_{07}, \; -y_0 \, sin \, d_{07})
\end{aligned}
\tag{80}
$$

It can be confirmed that $L_y^T \; R_g^T \; L_{(y \cdot g)^{-1}}^T$ depends on y for each set of coordinates. Consequently, there is no residual symmetry group when the submanifold is chosen to be S^4.

When the submanifold is S^5, independence of the inhomogeneous term with respect to the fibre coordinate gives rise to 20 conditions leaving 8 generators. The action of these generators on $(y_0, y_1, y_2, y_3, y_4, y_5, 0, 0, 0)$ produces a new set of coordinates and the homogeneous term depends on y for each of these coordinates. This follows from the results for S^4, as the restriction of the homogeneous term to $y_5 = 0$ already reveals a dependence on y.

The embedding of S^5 into $SU(3)$ can be deduced from the earlier embedding of S^7 into $SU(4)$ by setting $\delta_1 + \delta_2 + \delta_3 = \phi_3 = \sigma_6 = \phi_2 = \sigma_1 = \phi_1 = \sigma_1 = 0$. The remaining parameters are θ_1, θ_2, θ_3, σ_2, σ_3, σ_5, δ_1, δ_2 and the entries of the general SU(3) matrix in Euler angles, $D(\delta_1, \delta_2 - \delta_1 - \delta_2)\ U_{23}(\theta_3, \sigma_5) U_{12}(\theta_2, \sigma_3)\ U_{13}(\theta_1, \sigma_2)$ is

$$\begin{aligned}
g_{00} &= cos\ \theta_2\ cos\ \theta_1\ e^{i\delta_1} \\
g_{01} &= -sin\ \theta_2\ e^{i(\delta_1-\delta_3)} \\
g_{02} &= -cos\ \theta_2\ sin\ \theta_1\ e^{i(\delta_1-\sigma_2)} \\
g_{10} &= cos\ \theta_3\ cos\ \theta_2\ sin\ \theta_2\ cos\ \theta_1\ e^{i(\delta_2+\sigma_3)}\ -\ sin\ \theta_3\ sin\ \theta_1\ e^{i(\sigma_2+\delta_2-\sigma_3)} \\
g_{11} &= cos\ \theta_2\ cos\ \theta_3\ e^{i\delta_2} \\
g_{12} &= -cos\ \theta_3\ sin\ \theta_2\ sin\ \theta_1\ e^{i(\delta_2+\sigma_3-\sigma_2)}\ -\ sin\ \theta_3\ cos\ \theta_1\ e^{i(\delta_2-\sigma_5)} \\
g_{20} &= sin\ \theta_3\ sin\ \theta_2\ cos\ \theta_1\ e^{i(-\delta_1-\delta_2+\sigma_3+\sigma_5)}\ +\ cos\ \theta_3\ sin\ \theta_1\ e^{i(\sigma_2\ -\ \delta_1\ -\ \delta_2)} \\
g_{21} &= sin\ \theta_3\ cos\ \theta_2\ e^{i(\sigma_5-\delta_1-\delta_2)} \\
g_{22} &= -sin\ \theta_3\ sin\ \theta_2\ sin\ \theta_1\ e^{i(-\delta_1-\delta_2+\sigma_3+\sigma_5-\sigma_2)}
\end{aligned} \tag{81}$$

Restriction to S^5 is obtained by setting $\theta_3 = \sigma_5 = \delta_2 = 0$ and using the map $\iota_L : S^5 \to SU(3)$

$$\begin{aligned}
\iota_L(y) &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} y_0 + iy_1 \\ y_2 + iy_3 \\ y_4 + iy_5 \end{pmatrix} \\
&= \begin{pmatrix} cos\ \theta_2\ cos\ \theta_1\ e^{i\delta_1} \\ -sin\ \theta_2\ e^{i(\delta_1-\sigma_3)} \\ -cos\ \theta_2\ sin\ \theta_1\ e^{i(\delta_1-\sigma_2)} \end{pmatrix}
\end{aligned} \tag{82}$$

If $(c_0,\ c_1,\ c_2) \equiv y \cdot (d_A J_A)(\iota_L(y^{-1})^T)$, independence of c_0 with respect to the fibre coordinates implies that $d_5 = d_7 = d_9 = d_{14} = 0$ and $\sigma_2 = \sigma_3 = 0$. The remaining coefficients are d_1, d_2, d_8. Since $c_1 = d_1\ cos\ \theta_1\ e^{i\delta_1} + d_8\ sin\ \theta_1\ e^{i\delta_1}$, independence with respect to the fibre coordinate y implies that $d_1 = d_8 = 0$ and $c_2 = d_2\ cos\ \theta_2\ e^{i\delta_1}$. If $\theta_2 = \delta_1 = 0$, there is still a non-trivial U(1) action parametrized by θ_1. This is consistent with the existence of a U(1) affinor structure on S^5 [25].

Similarly, independence of the inhomogenous term for an S^6 submanifold implies 21 conditions and leaves 7 remaining generators. Dependence of the homogeneous term for the coordinates generated by the action of these generators on $(y_0, y_1, y_2, y_3, y_4, y_5, y_6, 0)$ again can be verified.

When G/H is a reductive homogeneous space, the G-invariant affine connections are in one-to-one correspondence with the set of algebra multiplications on $(\mathbf{m}, *)$ with $(\mathbf{g}, \mathbf{h})$ a reductive pair, $\mathbf{g} = \mathbf{m} \oplus \mathbf{h}$ and $AdH \subseteq Aut(\mathbf{m}, *)$ where $\mathbf{g}$, $\mathbf{h}$ are Lie algebras of G and

H, where G/H is a reductive homogeneous space [26], and this property has been used to characterize G_2-invariant affine connections on $S^6 = G_2/SU(3)$ [27] and Spin(7)-invariant connections on $S^7 = Spin(7)/G_2$ [28]. The affine connection make take values in the vector space representing the vertical tangent space, but the torsion and curvature forms are acting on vectors in this vector space. A connection form may be introduced for the total bundle by extending evaluation of the form to the horizontal lifts of vectors in the base space and the projection of tangent vectors to the sections of the total space onto the the vertical subspace. While the transformation of these connection forms would be of the standard type, the connections defined in this manner would generally depend on the fibre coordinates. Invariants for the groups G_2 and $Spin(7)$ also have been defined [29][30], but they consist of trace polynomials rather than the gauge potentials and the transformation rule does not contain the inhmomogenous terms required for connections.

Since this study began with an SO(8) structure group, it appeared possible that an allowed symmetry group larger than the fibre could be obtained if the fibre was chosen to be an appropriate submanifold of S^7. The computation of the fibre coordinate dependence of the connection form transformation rule for the S^3 submanifold demonstrates that the constraints reduce the structure group to SU(2), producing the standard principal bundle structure. This property is verified for general bundles in Appendix A.

5. Division Algebras and the Standard Model

A connection between the gauge groups in the standard model and the division algebras has been established recently using Clifford algebras. In this approach, the division algebras are the spinor spaces, while the Clifford algebras, which may be known as adjoint division algebras, may be used to derive the gauge symmetries of the action [31]. Denoting this Clifford algebra of $R^{p,q}$ by $R_{p,q}$, generated by the elements Υ_α satsifying

$$\begin{aligned} \Upsilon_\alpha \Upsilon_\beta + \Upsilon_\beta \Upsilon_\alpha &= 2\eta_{\alpha\beta}\epsilon \\ \eta_{\alpha\beta} &= diag(1, ..., 1, -1, ..., -1), \ \epsilon = Id_{p+q} \end{aligned} \tag{83}$$

the Pauli algebra is $P \sim R_{3,0} \sim \mathbb{C}\otimes\mathbb{H}$. Given elements $u_1, ..., u_n,\ x$ in an algebra A, the left adjoint map $x \to u_n(...(u_2(u_1 x))...)$ defines an associative algebra of left actions. Using the basis $\{e_a\}$ of A, A_L consists of elements of the form $1_L,\ e_{La},\ e_{Lab} = e_{La} \cdot e_{Lb}$, ... For the division algebras $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, it can be shown that $\mathbb{C}_L = \mathbb{C}_R = \mathbb{C}$, $\mathbb{H}_L \sim \mathbb{H}_R \sim \mathbb{H}$, and $\mathbb{O}_L = \mathbb{O}_R \sim R(8)$. While the adjoint left algebra P_L acts on the space of 2×1 complex Pauli spinors, $P_L(2) = R(2) \otimes P_L \sim C(4) \sim \mathbb{C} \otimes R_{1,3}$ is the Dirac algebra, which is the complexification of the Clifford algebra $R_{1,3} \sim \mathbb{H}(2)$. The algebras of left actions and right actions of $\mathbb{H}(2)$ on the space of 2×1 matrices over $\mathbb{H}$ commute, so that the algebra of left actions represent the Clifford algebra of 4-dimensional Minkowski space-time and the algebra of right actions generates a $SU(2) \times U(1)$ internal symmetry. Now considering the tensor product $T = \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, the adjoint $T_L \sim R_{0,9}$ corresponds to the Pauli algebra, whereas $T_L(2) \sim \mathbb{C}(32)$, the complexification of $R_{1,9}$, the equivalent of the Dirac algebra in the 10-dimensional Minkowski space-time. As the spinor space T is 64-dimensional, it is just large enough to describe a family consisting of a lepton doublet and a quark doublet with three distinct colours and the corresponding anti-family.

The subspace of 2-vectors of $R_{p,q}$ closes under commutation and it is isomorphic to the Lie algebra so(p,q). Thus, the two-vector basis $\{e_{Lpq},\ p, q = 1, ..., 6, p \neq q\}$ of $\mathbb{O}_L \sim R_{0,6}$ as 15-dimensional and isomorphic to $so(6) \sim su(4)$. The intersection of su(4) with $LG_2 = \{e_{Lab} - e_{Lcd} : e_a e_b = e_c e_d\}$ is $su(3) = \{e_{Lpq} - e_{Lrs} : e_p e_q = e_r e_s, p, q, r, s \neq 7\}$. The SU(3) gauge symmetry of the strong interactions therefore arises as part of the SO(1,9) Lorentz transformations and not as an internal symmetry in 10 dimensions, although it can be regarded as an internal symmetry in 4 dimensions. This is consistent with the use of string theory to describe the strong interactions and gravity.

These considerations suggest that one may begin with a ten-dimensional Lagrangian

$$
\begin{aligned}
\mathcal{L}_{1,9} &= \mathcal{L}_{gauge} + \mathcal{L}_{\phi} + \mathcal{L}_{1,9}^{ferm} \\
\mathcal{L}_{1,9}^{ferm} &= \langle \Psi | \partial\!\!\!/_{1,9} \Psi \rangle \\
\mathcal{L}_{\phi} &= \sum_{a=0}^{9} \langle \partial_a \phi | \partial^a \phi \rangle - \mu^2 \langle \phi | \phi \rangle - \lambda \langle \phi | \phi \rangle^2 \qquad (84) \\
\sum_{p=4}^{9} (\partial_p \partial^p) \phi_i &= 0
\end{aligned}
$$

and $\mathcal{L}_{gauge}$ is a ten-dimensional action for the spin-one gauge field. After using the projector distinguishing between matter and anti-matter multiplets, $R_{1,9}$ is projected to $R_{1,3} \otimes SO(6)$, the bosonic part of the action is based on the covariant derivative appropriate for $R_{1,3} \otimes SU(4)$. The SU(4) symmetry must then be broken to SU(3) to reproduce the QCD action, while the $SU(2) \times U(1)$ symmetry, arising from $R_{1,3}$ leads to the Weinberg-Salam model. While the scalar Lagrangian is used for spontaneous symmetry breaking, the fermion Lagrangian may be reduced to the standard lepton-quark Lagrangian in four dimensions. Using the projector $\rho_{\pm} \; = \; \frac{(1 \; \pm i \; e_{L7})}{2}$, one may re-express the fermion term as

$$\begin{aligned} \langle \rho_+ \Psi | \not{\partial}_{1,3}(\rho_+ \Psi) \rangle \; &+ \; \langle \rho_- \Psi | \not{\partial}_{1,3}(\rho_- \Psi) \rangle \\ &+ \; \langle \rho_+ \Psi | \not{\partial}_{0,6}(\rho_- \Psi) \rangle \; + \; \langle \rho_- \Psi | \not{\partial}_{0,6}(\rho_+ \Psi) \rangle \end{aligned} \tag{85}$$

The last two terms represent matter/anti-matter transitions that are not observed, and they vanish upon imposing the conditions

$$\not{\partial}_{0,6}(\rho_{\pm} \Psi) \; = \; 0 \tag{86}$$

The solutions to these constraints have a dependency on the coordinates in the extra six dimensions which gives rise to the SU(2) and SU(3) symmetries of the standard model.

Since ten-dimensional superstring theory contains a supersymmetric extension of the standard model, it will be necessary to enlarge the spinor space T to include the additional fields.

6. Dimensional Reduction over Coset Manifolds and Residual Gauge Symmetry

The action of generalized gauge transformations on bundles with an S^7 fibre, and the residual gauge symmetries, have been studied in §3. These results can be compared with the dimensional reduction of 11-dimensional supergravity over $M_4 \times S^7$ [32] and dimensional reduction of superstring theory from ten to four dimensions [33].

Given a coset manifold S/R, dimensional reduction of the theory can be achieved automatically when all of the fields are required to be invariant under the group S. For a tensor field , with $X^{\mu} \; = \; (x, y)$, this implies that

$$T^{\mu_1 \cdot\cdot \mu_n}(g(x, y)) \; = \; \frac{\partial g(x, y)^{\mu_1}}{\partial X^{\rho_1}} \cdots \frac{\partial g(x, y)^{\mu_n}}{\partial X^{\rho_n}} T^{\rho_1 \cdots \rho_n}(x, y) \tag{87}$$

It has been shown that S-invariance of a field on $M_4 \times S/R$ follows from R-invariance of the field at a designated base point [32] of the homogeneous coset manifold. When the coset manifold is S^7, and S is the group SU(4), then SU(4) invariance of a vector field $V^{\alpha}(x, y)$,

$$V^{\alpha}(x, g(y)) \; = \; \frac{\partial (g(y))^{\alpha}}{\partial y^{\gamma}} V^{\gamma}(x, y) \tag{88}$$

is equivalent to SU(3) invariance at the base point

$$V^\alpha(x,o) \ = \ \frac{\partial(h(y))^\alpha}{\partial y^\gamma}|_{y=o} V^\gamma(x,o) \tag{89}$$

where $h \in R$, the stabilizing group of the origin o. Consequently, the only unconstrained fields over the base space M_4 are the SU(3) singlets.

In this case, the decomposition of the adjoint representation of S into irreducible representations of R provides the gauge groups of the dimensionally reduced field theory. Denoting the isotropy representation of R by I_R, defined by an isomorphism of R into SO(N), where N is the dimension of S/R, the adjoint representation of S decomposes as $ad\ S \ \rightarrow \ ad\ R\ \oplus I_R$. When $S \ = \ SO(8)$, $\underline{28} \rightarrow \ \underline{21} \ + \ \underline{7}$ and when $S \ = \ SU(4)$, $\underline{15} \ \rightarrow \ \underline{8} \ + \ \underline{3} \ + \ \bar{\underline{3}} \ + \ \underline{1}$. The number of unconstrained gauge potentials is given by the number of R-singlets in I_R, or equivalently, the dimension of C(R), the centralizer of R in S. Consequently, dimensional reduction of an S-invariant gauge theory in $M_4 \times S/R$ should give a theory with unconstrained gauge potentials on M_4 transforming under the symmetry group $C(R)$. When $S \ = \ SO(8)$, this residual symmetry group only consists of the identity element, whereas, when $S \ = \ SU(4)$, the symmetry group is U(1). These results are consistent with those obtained through the calculation of the transformation of the connection form in §3.

In a modification of this technique, S-invariance can be extended to symmetric gauge fields which satisfy a generalized gauge invariance law [32]

$$\begin{aligned} g(s,x)\ A_\rho\ g^{-1}(s,x) \ + \ \partial_\rho g(s,x) g^{-1}(s,x) \ &= \ A_\mu(s(x))\ J^\mu_\rho(s,x) \\ s \ \in \ S,\ g(s,x) \ &\in \ G \end{aligned} \tag{90}$$

indicating that the potential is invariant up to a gauge transformation. If the initial gauge group is G, and K(R) is the homomorphic image of R in G, then after dimensional reduction, the gauge group is the centralizer in G of K(R) [32]. For different choices of G, the use of symmetric gauge fields leads to a wider variety of dimensional reduction schemes [32][33] and therefore might be used to obtain larger residual gauge groups beginning with an invariance under $S \ = \ SU(4)$. The choice of G would be somewhat arbitrary in general [34].

Since it has been demonstrated that the strong-coupling limits of ten-dimensional Type IIA theory and the $E_8 \times E_8$ heterotic string are related to 11-dimensional supergravity compactified over S_1 [35] and $\frac{S_1}{\mathbb{Z}_2}$ [36] respectively, and that it is necessary to dimensionally reduce the ten-dimensional SU(4)-invariant theory to four-dimensional SU(3)-invariant theory in the unification scheme in §5, it is of interest to consider solutions to the $d = 11$ supergravity equations of motion with SU(4) symmetry, including, for example, the solution with an S^7 metric obtained by stretching U(1) fibres over $\mathbb{CP}^3$ [37][38]. The higher-dimensional action may consist of terms containing fields transforming under different irreducible representations of SU(3), and integration over the fibre coordinates can be performed giving rise to an action on M_4. Specifically, when $S = SU(4)$ and $R = SU(3)$, amongst the non-trivial representations in the decomposition of $ad\ SU(4)$ are those corresponding to the SU(3) gauge field, which could also be included in the dimensionally reduced action. This would therefore providing a method for obtaining the QCD gauge theory for strong interactions in four dimensions from a higher-dimensional theory through the geometrical procedure of reduction over a coset space.

In addition to this method for obtaining an SU(3) gauge theory, the results of a systematic study of Lagrangians containing fields transforming linearly under a group R but nonlinearly under a larger group S [39] could be used in this case. Since it is known that a theory with fields transforming linearly under R can be shown to be equivalent to a theory with fields transforming nonlinearly under S, pure gauge fields on S/R may be added to gauge fields on R to obtain a Lagrangian with local gauge invariance under S [40]. Given an element of the coset space $\phi_0(x) \in S/R$, and gauge fields in the Lie algebra of R, $A_\mu \in \mathcal{R}$, the pure gauge fields [41] are defined to be

$$B_\mu \ = \ \phi_0(x)(\partial_\mu \ + \ A_\mu)\phi_0^{-1}(x) \qquad (91)$$

and the set $\{A_\mu,\ B_\mu\}$ forms a nonlinear representation of S and the field content of an S-invariant Lagrangian. It may be noted that the extra fields are derived from scalar quantities and therefore resemble the coordinate fields of a higher-dimensional theory. Within the context of the nonlinear realization approach, the physical equivalence of the R-invariant and S-invariant theories follows from the elimination of the pure gauge fields by gauge transformations, whereas the dimensional reduction of a higher-dimensional theory produces a closely related but nevertheless distinct theory. The usefulness of pure gauge fields on coset spaces depends on whether the spin-one gauge field in the ten-dimensional action (84) possesses an SU(4) symmetry or an SU(3) symmetry that is being viewed as SU(4) through the method of induced representations.

In §5, the standard model was derived through an algebraic procedure for obtaining the gauge symmetry from the left action algebra acting on the tensor product $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, which, in turn, is equivalent to the Clifford algebra associated with the Lorentz metric in $R^{1,9}$. It was demonstrated in that section that this formulation of the model naturally led to the introduction of the group SU(4) as that part of the Lorentz group SO(1,9), the exponential of two-vector generators e_{Lab} in $R_{1,9}$, corresponding to $\mathbb{O}_L$. The restriction to SU(3), however, required that extra conditions be imposed on the Clifford algebra, and these arose as a consequence of certain properties that necessarily had to satisfied by the fermion terms. While derivation of the fermionic part of the standard model was presented in [31], inclusion of the spin-one gauge field action appears to begin with an SU(4) symmetry rather than an SU(3) symmetry. The geometrical methods studied in §3 and in this section suggest a way of breaking the SU(4) gauge symmetry. While additional input is required to obtain an SU(3) residual symmetry, the techniques reviewed in this section could this is feasible without introducing a scalar potential. These results therefore complement the algebraic procedure of §5 and provide further evidence for the derivation of the SU(3) symmetry from a higher-dimensional Lagrangian. Having established the origin of the local SU(3) symmetry, all of the gauge groups of the standard model can then be naturally included in a higher-dimensional theory, as $SU(2)\times U(1)$ directly corresponds to left action algebra of $\mathbb{R}\otimes\mathbb{C}\otimes\mathbb{H}$, the remaining part of the tensor product T.

Although reduction over $S^7 = SU(4)/SU(3)$ has been considered exclusively thus far in this section, it may be noted that compactification of ten-dimensional superstring theories on six-dimensional compact spaces is also known to lead to a breaking of an SU(4) symmetry through an SU(3) subgroup. For example, one may consider the reduction of ten-dimensional supergravity to four dimensions when all of the fields are independent of the extra six coordinates y^α. The resulting N=4 supersymmetry is generated by 4 spinors Q^A transforming under the fundamental representation of SU(4). As they also transform as $1 \oplus 3$ under an SU(3) subgroup of SU(4), this invariance under this subgroup breaks the $N = 4$ supersymmetry to $N = 1$, with the surviving supersymmetry being an SU(3) singlet [41]. Similar considerations apply to the breaking of E_8 to E_6, useful for grand unification, in the compactification of ten-dimensional $E_8 \times E_8$ supergravity theories on Calabi-Yau manifolds with SU(3) holonomy.

A grand unified theory with an SO(10) gauge symmetry may be provided with an intermediate scale at which the symmetry breaks down to a group such as $SU(4)_C \times SU(2)_L \times \times SU(2)_R$ or $SU(4)_C \times SU(2)_L \times U(1)_R$. For the first intermediate gauge group, one-loop calculations based on symmetry breaking resulting from the $\underline{126}\oplus\underline{\overline{126}}$ representation of SO(10) an energy scale of 10^{14} GeV is obtained [42][43].

However, computations based on symmetry breaking using the Higgs representation $\underline{54}$ and the representation $\underline{210}$ of SO(10) and inclusion of two-loop and threshold effects give an intermediate energy scale of $10^{10} - 10^{13}$ GeV, preferable for resolution of the solar neutrino problem [44]. For the latter symmetry group, the intermediate enrgy scales of $10^{11} - 10^{13}$ GeV can be achieved at one loop [42][43].

While the octonions and SU(4) can be immediately associated with the dimensional reduction of eleven-dimensional supergravity to four dimensions, it has already been mentioned that they also arise naturally in the ten-dimensional setting used for superstring theory. Seven-dimensional Yang-Mills instantons based on the gauge group G_2 [45] and eight-dimensional octonionic instantons [46][47] have been extended to string solitons. Equivalence of compactifications of D=11 supergravity and heteretotic string theory to four dimensions using compact manifolds of holonomy SU(3) and G_2 respectively has been suggested [48]. The bosonic sector of N=1, D=10 supergravity has soliton solutions interpolating between ten-dimensional Minkowski space and $(AdS)_3 \times S^7$ [49].

The free Green-Schwarz superstring can be formulated in three, four and six dimensions and a Lorentz covariant and unitary interacting Green-Schwarz superstring exists in ten dimensions [50]. These dimensions are necessary for local supersymmetry, which depends on Γ-matrix identities derived from the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ [51], corresponding to the transverse directions. Classical solutions of the equations of motion of the Green-Schwarz Lagrangian in $D = 10$ have been found by expressing ten-dimensional vectors as 2×2 octonionic matrices and 32-real-component Majorana spinors as spinors with four octonionic components [52]. These equivalences follow from the isomorphism $\tilde{SO}(1,9) \simeq SL(2, \mathbb{O})$ [53][54], which is the last in a sequence of isomorphisms involving space-time and division algebras, $\tilde{SO}(1,\nu+1) \simeq SL(2;\mathbb{K}_\nu),\ \nu = 1,2,4,8$ [55].

These isomorphisms can be used in the representation of space-time vectors as 2×2 hermitian matrices and null vectors as fermion bilinears [55]. The null vector P^μ is equivalent to

$$P = \lambda\lambda^\dagger = \begin{pmatrix} \xi\xi^\dagger & \xi\eta^\dagger \\ \eta\xi^\dagger & \eta\eta^\dagger \end{pmatrix} \tag{92}$$

where $\lambda = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$. Since the determinant of this matrix, which equals $P^\mu P_\mu$, vanishes, P^μ must lie on the forward light cone. Lorentz transformations on the space-time vector P^μ can be regarded as $SL(2,\mathbb{K}_\nu)$ transformations on $\lambda\lambda^\dagger$ derived from the multiplication of $SL(2,\mathbb{K}_\nu)$ matrices and the spinor λ [31].

The space of light-like lines at a point in ten dimensions is S^8 and the above equivalence implies that it can be represented as the set of spinors λ modulo transformations which leave $\lambda\lambda^\dagger$ invariant. These transformations form the algebra S^7 and the action on the space of light-like lines is given by the Hopf fibration $S^{15} \to S^8$ [55]. This construction has also been extended to the action of S^7 on the physical twistor space $\mathcal{N} \subset \mathbb{O}P^3$ [56] and the S^7 Kac-Moody algebra $\hat{S}^7$ [54], which arises as a symmetry algebra of the twistor-string theory [58] and the light-cone superstring [59].

A solution in to the problem of the fibre coordinate dependence of the transformation rule of the connection form for a general bundle and an enlargement of the symmetry of the Yang-Mills theory based on octonions, given in Appendices A and B, imply that a Lie group structure is necessary for a pure gauge theory. While supersymmetrical light-like lines have been used in ten-dimensional super-Yang-Mills theories to integrate the constraint equations [60] and an S^7 symmetry on the space of light-like lines exists, the fact that the vector bosons in the standard model must be components of a Lie-algebra-valued potential indicates that, initially, division algebras can only be used in the organization of the fermion multiplets in a unified description of elementary particle interactions. The work of [31] briefly described in §5 shows that this novel, and perhaps preferable, formulation of the standard model can be achieved, with an immediate connection to the Clifford algebra $R_{1,9}$ and the Lorentz group SO(1,9).

The problem of force unification can be refined by the classification of matter multiplets, since the components of the spinors must take values in the division algebras. Basing the fermionic part of the standard model on the spinor space $T = \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, it may be noted that the amplitudes for elementary particle interactions typically involve the product of two fermions $\psi_1,\ \psi_2$ and a vector boson A^μ at the vertices of the perturbative diagrams. If the fermions took values in an algebra that is not a division algebra, this vertex factor could vanish because there would then exist $\psi_1,\ \psi_2 \neq 0$ such that $\psi_1 \cdot \psi_2 = 0$. When fermions take values in $\mathbb{R},\ \mathbb{C},\ \mathbb{H}$ or $\mathbb{O}$,

$$\psi_1 \cdot \psi_2 \neq 0 \tag{93}$$

if $\psi_1 \neq 0$ and $\psi_2 \neq 0$.

General composition algebras, characterized by the composition law $(xy,\ xy) = (x,x)(y,y)$ have been classified, and it has been established that the finite-dimensional composition algebras over fields of characteristic not equal to 2 should have dimensions 1, 2, 4 or 8 [61]. Also, if the composition algebra has the unit element, the dimension 4 and 8 composition algebras are $\mathbb{H}$ and $\mathbb{O}$. If a composition algebra has zero divisors, then there will exist two elements e_l and e_m such that their product vanishes. It follows that since $(e_k e_l)e_m = e_k(e_l e_m) + \{e_k, e_l, e_m\}$, this expression vanishes if the associator $\{e_k, e_l, e_m\}$ is zero. For Cayley-Dickson algebras of dimension 2^n, there always will be triples of basis elements for which $\{e_k, e_l, e_m\} = 0$. In fact, the dimension of the kernel of the operator L_a, representing left multiplication by a zero divisor a, satisfies the condition $2^n - 4 \geq dim\ ker\ L_a \equiv 0 \ (\mathrm{mod}\ 4)$ [62]. The same elements of a Cayley-Dickson algebra will represent $ker\ R_a$. Since it can be verified that the zero-divisors in the Cayley-Dickson algebras are combinations of the basis elements, one may apply a transformation to the standard basis $\{e_0,\ e_1,\ e_2,\ ...,\ e_{2^n-1}\}$ such that the new basis include the zero divisors. The multiplication table will then have block form so that the fermion terms in the Lagrangian and the corresponding vertex factors will vanish. Since the blocks will be 4×4, it should possible to express the particle content again in terms of sums of tensor products of spinor spaces which are isomorphic to $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. The overall symmetry of the action would also be reduced according to the requirement that block form be maintained.

In addition to the finite-dimensional Clifford algebras, $\mathbb{C}_L \times \mathbb{H}_L \times \mathbb{O}_L = R_{1,9}$ and $R_{1,9}$, infinite-dimensional Clifford algebras, in principle, might be used to describe elementary particle interactions. Consider a set of unitary operators $U_i : R^n \to R^n$ such that $U_i^2 = -I$ and $U_i U_j = -U_j U_i$, Let $n = (2a(n)+1)2^{b(n)}$, $b(n) = c(n) + 4d(n)$ $0 \leq c(n) \leq 3$. The maximal number of unitary operators on R^n satisfying the Clifford algebra relations, is equal to $\rho(n) - 1$, where
$\rho(n) = 2^{c(n)} + 8d(n)$ by the Hurwitz-Radon-Eckmann theorem [63]. When n is finite, it is known that $\rho(n) \leq n$ and that $\rho(n) = n$ only for $n = 1,\ 2,\ 4,\ 8$. Moreover, $\rho(\infty) = \infty$, so that the infinite-dimensional Clifford algebra necessarily acts on an infinite-dimensional vector space. It may be noted that $\rho(n) - 1$ also represents the maximal number of linearly independent vector fields on S^{n-1}, replacing the products of basis operators U_i by Lie brackets of unit vector fields on the sphere, X_i. An infinite-dimensional division algebra can be viewed as a module for the Clifford algebra, and therefore no finite-dimensional representation can be used for a spinor space. Since there are only a finite number of fermion degrees of freedom in a realistic unified field theory, infinite-dimensional division algebras would not be relevant in the formulation of this theory.

The vector boson, the carrier of the force, is initially massless when the gauge symmetry is unbroken and its momentum vector can be expressed as a spinor bilinear as above. This also suggests that it might be possible to relate the gauge potential itself to a spinor bilinear as one would expect from the diagram representing the interaction. Since the potential belongs to the adjoint representation of the gauge group, the symmetry groups of the theories describing the elementary particle interactions would then be determined by the fermions. The restriction of the fermions to the division algebras should then provide a theoretical principle for explaining the types of gauge groups that appear in the standard model. The groups which act most naturally on $T = \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ are those which are subgroups of the adjoint left algebra $T_L(2) \sim \mathbb{C}(32)$ or the Clifford algebra $R_{1,9}$. These considerations lead to the subspace of two-vectors of $R_{1,9}$, so(1,9), $\mathbb{O}_L$, $so(6) \sim su(4)$, and $\mathbb{C}_L \otimes \mathbb{H}_L$, su(2). This approach might therefore provide a deeper insight into the different types of matter presently known in elementary particle physics and their interactions.

7. Conclusion

It has been shown that elimination of the fibre coordinate in the transformation rule of the connection form in bundles with a structure group larger than the standard fibre, and in particular for bundles with the structure group given the isometry group of the standard fibre, leads to restrictions on the bundle. For the S^3 bundle, the allowed group of gauge transformations is reduced from the isometry group SO(4) to SU(2). For the S^7 bundle, not all of the conditions deriving from fibre-coordinate independence can be satisfied, so that the transformation rules retain a dependence on the fibre coordinate. The requirement of independence with respect to the coordinates of the entire S^7 fibre leaves no residual gauge symmetry beginning with an SO(8) structure group and only a U(1) symmetry starting with an SU(4) structure group. Similar conditions can be placed on the structure group and gauge transformations in quantum principal bundles. It is established that a Lie group structure is required for the pure gauge theory and that any application of the division algebras to force unification must initially be restricted to the organization of the fermions multiplets in the standard model. Nevertheless, this suggests a theoretical principle which distinguishes the specific gauge groups that do arise in theories of elementary particle interactions. It has been noted that the fermion part of the standard model can be based on the spinor space $T = \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, and the necessity of the division algebras in the organization of the fermion multiplets is explained. Amplitudes for elementary particle interactions typically involve the product of two fermions ψ_1, ψ_2 and a vector boson A^μ at the vertices of perturbative diagrams, and their non-vanishing follows directly from the fermions taking values in the division algebras. Since the unified theory is initially formulated in ten dimensions, the masslessness of the vector boson when the gauge symmetry is unbroken implies that the momentum vector can be expressed as a spinor bilinear as above. Relating the gauge potential itself to a spinor bilinear, the symmetry groups of the theories describing the elementary particle interactions would then be determined by the restriction of the fermions to the division algebras. As the exchange of intermediate string states describes the exchange of vector bosons in the field theory limit, this approach points toward a connection between the geometry of the internal symmetry spaces arising in the standard model and superstring theory.

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Appendix

A. Independence of the Transformation of the Connection Form with respect to the Fibre Coordinate

Let us recall that for a general fibre bundle with atlas $\{U_\alpha, \psi_\alpha\}$, the trivializations $\psi_\alpha,\ \psi_\beta$ determine two local sections $\sigma_\alpha(x) = \psi_\alpha^{-1}(x, y_0), \sigma_\beta(x) = \psi_\beta^{-1}(x, y_0)$ which may be mapped by a diffeomorphism to $(x, y(x))$ and $(x, y'(x))$. If $\psi_{\beta\alpha} : U \times F \to F$, where $U \subset U_\alpha \cap U_\beta$, is defined by $\psi_{\beta\alpha}(x, y) \ = \ y'$, then the tangent spaces to the two sections are given by (ξ_x, V^α) and $(\xi_x, T\psi_{\beta\alpha}(\xi_x, V^\alpha))$. A connection provides a splitting of the tangent bundle into vertical and horizontal sub-bundles, and the image of the horizontal subspaces associated with the two sections in $TU \times TF$ are spanned by vectors of the form $(\xi_x, C^\alpha(y))$ and $(\xi_x, C^\beta(y'))$. Defining the connection forms Γ^α by $V^\alpha - C^\alpha$, one finds that they transform under a change of section as

$$\Gamma^\beta(\xi_x, y') \ = \ T\psi^y_{\beta\alpha}(\xi_x) \ + \ T\psi^x_{\beta\alpha} \cdot \Gamma^\alpha(\xi_x, y) \tag{A1}$$

where $\psi^x_{\beta\alpha}(y) \equiv \psi^y_{\beta\alpha}(x) = y'$.

The tangent mapping $T\psi^x_{\beta\alpha} : TF \to TF$ is an isomorphism if the tangent bundle of the fibre is trivializable. Defining ϕ_y to be a mapping from a vector space V to the tangent space $T_y(F)$, so that $\Gamma^\alpha(\xi_x, y) = \phi_y \Gamma^\alpha(\xi_x),\ \Gamma^\alpha(\xi_x) \in V$, equation (A1) becomes

$$\phi_{y'}\Gamma^\beta(\xi_x) \ = \ T\psi^y_{\beta\alpha}(\xi_x) \ + \ T\psi^x_{\beta\alpha} \cdot \phi_y \Gamma^\alpha(\xi_x) \tag{A2}$$

To interpret the relation between $\Gamma^\alpha(\xi_x)$ and $\Gamma^\beta(\xi_x)$ as a gauge transformation of potentials taking values on the base space, it is necessary to eliminate the fibre coordinate dependence. This can be achieved if the right-hand side of the equation can be expressed as $\phi_{y'}X,\ X \in V$ and ϕ_y is an injective mapping. Moreover, V can be extended to have the same dimension as F, so that $\Gamma^\alpha(\xi_x, y)$ ranges over all of $T_y(F)$ and ϕ_y is a surjective mapping. Thus ϕ_y should be a bijection. Writing $T\psi^y_{\beta\alpha}(\xi_x)$ as $\phi_{y'}X_1,\ X_1 \in V$ and $T\psi^x_{\beta\alpha} \cdot \phi_y \Gamma^\alpha(\xi_x)$ as $\phi_{y'}X_2,\ X_2 \in V$, a fibre coordinate-independent gauge transformation rule may be obtained if

$$\phi_{y'}(X_1) \ + \ \phi_{y'}(X_2) \ = \ \phi_{y'}(X_1 \ + \ X_2) \tag{A3}$$

expressing linearity of ϕ_y. It may be noted that a bijective mapping between the vector spaces that is nonlinear, analogous to the mapping $x \to x^3$ on $(\mathbb{R}, +)$, could also have been considered. As $\Gamma^\alpha(\xi_x, \cdot)$ is a $C^r,\ r > 1$ vector field on F, the mapping $\phi : V \times F \to TF,\ \phi(\cdot, y) = \phi_y$ must be a differentiable function of the fibre coordinate. Moreover, ϕ_y

should also be a vector space isomorphism, since one takes $\Gamma^\alpha(\xi_x, y) + \Gamma^\alpha(\eta_x, y)$ to be $\Gamma^\alpha(\xi_x + \eta_x, y)$ as a result of the trivializations $\{\psi_\alpha\}$, $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ being diffeomorphisms. The requirement that ϕ_y be a vector space isomorphism would exclude nonlinear mappings and ensure that equation (A2) can be used in reducing the transformation rule of the connection form to a gauge transformation involving a dependence on the base space coordinates only.

It follows that ϕ_y is a parallelism on the fibre F. There are in fact several types of fibre parallelisms [64]:

i) fibre parallelism
There exists a map $C : E \times_M E \to \underset{(y,y')\in E\times_M E}{\cup} Isom(T_yE, T_{y'}E)$ such that $C(y, y') = \omega_{y'} \circ \omega_y^{-1} : T_yE \to T_{y'}E$ where $y, y' \in E_x$ and $\omega_y : (T_{red}E)_x \to T_yE$. The last map gives rise to an isomorphism $\Omega : T_{red}E \times_M E \to TE$.

ii) vertical fibre parallelism
There exists a map $\tilde{C} : E \times_M E \to \underset{(y,y')\in E\times_M E}{\cup} Isom(T_yE_x, T_{y'}E_x)$ such that $\tilde{C}(y, y') : T_yE_x = V_yE \to T_{y'}E_x = V_{y'}E$. This parallelism is given by the isomorphism $\tilde{\Omega} : VT_{red}E \times_M E \to VTE$.

iii) integrable fibre parallelism
A fibre parallelism is integrable if, for any $(y, y') \in E \times_M E$, there exists a translation $\tau_{y'y} : E \to E$, $\tau_{y'y}(y) = y'$ such that $T\tau_{y'y} \in C(E \times_M E)$. For a vertical fibre parallelism, there exists a translation $\tau_{y'y}^{vert} : E_x \to E_x$ such that $T\tau_{y'y}^{vert} \in \tilde{C}(E \times_M E)$. It may be noted that a surmersion (E, M, π) that admits a vertical fibre parallelism and a connection is a projectable fibre parallelism. [If (E, M, π) is a fibration, then a vertical fibre parallelism implies the existence of a connection.] A projectable fibre parallelism with an integrable parallelism on each fibre, or equivalently such that the commutator of invariant vertical vector fields is invariant, can also be regarded as an integrable vertical fibre parallelism.

iv) globally integrable fibre parallelism
It can be shown that a fibration which admits a globally integrable fibre parallelism is a principal bundle. The translation $\tau_{y'y} : E \to E$ can be extended to a global diffeomorphism on E.

For a principal bundle,

$$\begin{aligned} VTP \;&=\; T_e(G) \times P \;=\; VTP/G \times_M P \;=\; VT_{red}E \times_M P \\ VTP/G \;&=\; \{right-invariant\ vector\ fields\ tangent\ to\ the\ fibres\ of\ E\} \\ VT_{red}E \;&\sim\; T_e(G) \times M \end{aligned} \qquad (A4)$$

VTP/G is clearly a trivial bundle, because $(VTP/G)_x \sim T_e(G)\ \forall\ x \in M$, and the transition functions are elements of G, leaving invariant any vector field in VTP/G.

The parallelism in a principal bundle can be taken to be the one induced by left multiplication and the diffeomorphism $\psi^x_{\beta\alpha}$ to be right multiplication by a group element. Since

$$\begin{aligned} R_{g*}\ L_{y*} \;&=\; L_{(y\cdot g)*}\ Ad(g^{-1}) \\ T\psi^y_{\beta\alpha}(\xi_x) \;&=\; L_{y*}\cdot(g_*\cdot\xi_x) \;=\; L_{y\cdot g*}[L^{-1}_{g*}(g_*\cdot\xi_x)] \;=\; \phi_{y'}[L^{-1}_{g*}(g_*\cdot\xi_x)] \end{aligned} \qquad (A5)$$

the standard gauge transformation

$$\Gamma^\beta(\xi_x) \;=\; Ad(g^{-1})\Gamma^\alpha(\xi_x) \;+\; L^{-1}_{g*}(g_*\cdot\xi_x) \qquad (A6)$$

is a consequence of equation (A2).

The problem of determining which bundles allow the dependence on the fibre coordinate to be eliminated has been considered previously [12]. A necessary condition is

$$T\psi^x_{\beta\alpha}\phi_y\Gamma^\alpha(\xi_x) \;=\; \phi_{y'}A(x)\Gamma^\alpha(\xi_x) \qquad (A7)$$

or equivalently, that there exists a global diffeomorphism which is a solution to the differential system

$$T\psi^x_{\beta\alpha} \;=\; \phi_{y'}\cdot A\cdot\phi^{-1}_y \qquad (A8)$$

By Frobenius' theorem, the integrability conditions for the differential system require that the commutator of vector fields on the fibre are invariant with respect to the map $\phi_{y'}\cdot A\cdot\phi^{-1}_y$. Thus,

$$[\phi_{y'}\cdot A\cdot\phi^{-1}_y X, \phi_{y'}\cdot A\cdot\phi^{-1}_y Y]_{y'_0} \;=\; \phi_{y'}\cdot A\phi^{-1}_y[X,Y]_{y_0} \qquad (A9)$$

for any vector fields $X, Y \in TF$, implying

$$c_{ijk}(y_0)A_{mk} \;=\; c_{klm}(y'_0)A_{ki}A_{lj} \qquad (A10)$$

with the coefficients $c_{ijk}(y)$ given by $[\xi_i(y), \xi_j(y)] \;=\; c_{ijk}(y)\xi_k(y)$, with $\xi_i(y) \;=\; \phi_y\cdot e_i, e_i \in V$ representing an orthonormal basis for $T_y(F)$.

The most general gauge matrix A that can be allowed is $A = A_1(y')A_2(x)A_1(y)^{-1}$ as this gives

$$T\psi^x_{\beta\alpha} = \phi_{y'} \cdot A_1(y')A_2(x)A_1(y)^{-1} \cdot \phi_y^{-1} = \phi'_{y'} \cdot A_2(x) \cdot \phi'^{-1}_y \qquad (A11)$$

if the new parallelism ϕ'_y is given by $\phi_y \cdot A_1(y)$. The integrability condition for the differential system given by equation (A8) is

$$\begin{aligned}[\phi_{y'} \cdot A_1(y')A_2(x)A_1(y)^{-1} \cdot \phi_y^{-1} \cdot X,\ \phi_{y'} \cdot A_1(y')A_2(x)A_1(y)^{-1} \cdot \phi_y^{-1} \cdot Y]_{y'_0} \\ = \phi_{y'_0} \cdot A_1(y')A_2(x)A_1(y)^{-1} \cdot \phi_{y_0}^{-1} \cdot [X, Y]_{y_0}\end{aligned} \qquad (A12)$$

Letting $X = \xi_i(y) = \phi_y \cdot e_i$, one finds that

$$c_{ijk}[A_1(y'_0) \cdot A_2(x) \cdot A_1(y_0)^{-1}]_{mk} = c_{klm}[A_1(y'_0)A_2(x)A_1(y_0)^{-1}]_{ki}[A_1(y'_0)A_2(x) \cdot A_1(y_0)^{-1}]_{lj} \qquad (A13)$$

Defining the basis $\xi'_i(y)$ using the new parallelism $\phi'_y \cdot e_i = \phi_y \cdot A_1(y)e_i$, and the structure constants $c'_{ijk}(y)$ by the commutation relations $[\xi'_i, \xi'_j]_y = c'_{ijk}(y)\xi'_k(y)$, it follows that

$$c'_{ijk}(y)(A_1(y))_{nk} = c_{lmn}(A_1(y))_{li}(A_1(y))_{mj} \qquad (A14)$$

Using this relation and multiplying equation (70) by $(A_1(y_0))_{ir}(A_1(y_0))_{js}(A^{-1}(y'_0))_{pm}$ gives

$$c'_{rsn}(y_0)A_{2pn} = c'_{lmp}(y'_0)A_{2lr}A_{2ms} \qquad (A15)$$

which is equivalent to the previous condition (A8).

It may be noted that the proof of the existence of globally integrable parallelism on the standard fibre F depends on the following lemma, with a more detailed proof than that provided in reference [12] given below:

Lemma. The set $\mathcal{C}_{y_1} = \{y \in F | c_{ijk}(y) = c_{ijk}(y_1)\}$ either contains all of F or is a set of dimensionality less than one in F.

Proof. It is shown in [12] that an identity satisfied by the vector fields $\{\xi_i\}$ induced by the parallelism ϕ_y [65] implies that

$$c_{ijl}(y')c_{klm}(y') - \xi_k(c_{ijm})(y') = c_{ijl}(y)c_{klm}(y) - \xi_k(c_{ijm})(y) \qquad (A16)$$

where $\xi_k(c_{ijm})$ is defined by the action of the vector field on a function on F. Assuming initially that $\mathcal{C}_{y_1}$ is a continuous curve through y_1 with tangent vector $X_{y_1} = X_k\xi_k(y_1)$ and contracting equation (A16) by X_k,

$$c_{ijl}(y)c_{klm}(y)X_k - X_k\xi_k(c_{ijm})(y) = c_{ijl}(y_1)c_{klm}(y_1)X_k - X_k\xi_k(c_{ijm})(y_1) \qquad (A17)$$

which implies, on $\mathcal{C}_{y_1}$, that $X_k\xi_k(c_{ijm})(y) = X_k\xi_k(c_{ijm})(y_1) = 0$ when X_k is constant. Similarly, the vanishing of the derivative of $c_{ijl}(y)c_{klm}(y) - \xi_k(c_{ijm})(y)$ by equation (A16) gives

$$\begin{aligned}(X_n\xi_n(c_{ijl})(y_1))X_kc_{klm}(y_1) + X_kc_{ijl}(y_1)(X_n\xi_n(c_{klm})(y_1)) - X_n\xi_nX_k\xi_k(c_{ijm})(y_1)\\ = -X_n\xi_nX_k\xi_k(c_{ijm})(y_1) = 0\end{aligned} \qquad (A18)$$

and repeated differentiation leads to $X_{n_1}\xi_{n_1}...X_{n_r}\xi_{n_r}(c_{ijm})(y_1) = 0$. Consequently, c_{ijm} is constant along the integral curve of X, and using the assumption that the set $\mathcal{C}_{y_1}$ is at least one-dimensional, one may conclude that it contains the integral curve of a parallel vector field X passing through y_1.

A two-dimensional surface S_Y is spanned by the integral curves of another parallel vector field Y intersecting $\mathcal{C}_{y_1}$ and an n-dimensional neighbourhood of y_1 is covered by the surfaces $S_{\sum_i a_iY_i}$ corresponding to arbitrary linear combinations of the independent vector fields $Y_1, ..., Y_{n-1}$. The surfaces S_Y and $S_{Y'}$ will initially coincide at $\mathcal{C}_{y_1}$ if $Y' = aX + bY$. However, translation of the vector field X a distance t along the integral curve of Y using the one-parameter family of diffeomorphisms $\{\chi_t\}$ gives a vector field

$$\chi_{t*}X = X - t[X,Y] + \frac{t^2}{2}[[X,Y],Y] - ... \qquad (A19)$$

Parallel transport of the vector field X along the integral curve of Y' using the one-parameter family of diffeomorphisms $\{\chi'_{v*}\}$ gives the vector field

$$\chi'_{v*}X = X - bv[X,Y] + b^2\frac{v^2}{2}[[X,Y],Y] - ... + ab\frac{v^2}{2}[[X,Y],X] - ... \qquad (A20)$$

Even after setting t equal to $\sum_j b_j v^j$, with $b_1 = b$, higher-order terms such as that containing $ab\frac{v^2}{2}$ will vitiate the possibility of equating $\chi_{t*}X$ with $\chi'_{v*}X$. Thus, the tangent spaces to S_Y and $S_{Y'}$, spanned by the bases $\{\chi_{t*}X, Y\}$ and $\{\chi'_{v*}X, Y'\} = \{aX + bY, \chi'_{v*}X\}$ respectively, are equivalent at $\mathcal{C}_{y_1}$ but differ when $t,\ v \neq 0$. Since $S_{Y'}$ does not always coincide with S_Y, it must intersect with a surface $S_{Y''}$, where Y'' is another linear combination of the vector fields $Y_1,\ ...,\ Y_{n-1}$, $\sum_i a''_iY_i$.

Translation of $\mathcal{C}_{y_1}$ a distance v along the integral curves of Y' results in a curve of constant c_{ijk}, $\mathcal{C}_{\chi'_v(y_1)}$, because

$$(Y'_n\xi_n c_{ijl}(y))Y'_k c_{klm}(y) \ + \ (Y'_k c_{ijl}(y))(Y'_n\xi_n c_{klm}(y)) \ - \ Y'_n\xi_n Y'_k\xi_k(c_{ijm})(y) \ = \ 0 \quad (A21)$$

together with equation (A16) implies that

$$Y'_n\xi_n Y'_k\xi_k(c_{ijm})(y) \ = \ Y'_n\xi_n Y'_k\xi_k(c_{ijm})(y_1) \qquad (A22)$$

and repeated differentiation gives

$$Y'_{n_1}\xi_{n_1}...Y'_{n_r}\xi_{n_r}(c_{ijm})(y) \ = \ Y'_{n_1}\xi_{n_1}...Y'_{n_r}\xi_{n_r}(c_{ijm})(y_1) \qquad \forall y \in \mathcal{C}_{y_1} \qquad (A23)$$

Similarly, translation of $\mathcal{C}_{y_1}$ a distance w along the integral curves of Y'' produces another curve of constant c_{ijk}, $\mathcal{C}_{\chi''_w(y_1)}$.

If a point y_3 lies in the intersection of $S_{Y'}$ and $S_{Y''}$, then it must be located on an a curve $\mathcal{C}_{\chi'_v(y_1)}$ for some v and $\mathcal{C}_{\chi''_w(y_1)}$ for some w. Parallel transport of one of these curves along the other will sweep out a two-dimensional surface of constant c_{ijk}. When this surface is translated back to y_1, one obtains a two-dimensional neighbourhood of constant c_{ijk} about the point y_1. This procedure can be repeated until an n-dimensional neighbourhood of y_1 of constant c_{ijk} is constructed. If F is compact, it may be covered by a finite number of these neighbourhoods and $\mathcal{C}_{y_1}$ contains all of F. Otherwise, it has dimensionality less than one.

This lemma has been used previously [12] to demonstrate that the standard fibre is G/D, where G is a Lie group and D is a discrete subgroup. When the standard fibre is a group, and the gauge matrix A represents the adjoint action of g^{-1}, then if $g = exp(t_j X_j)$,

$$\begin{aligned} exp(-t_j X_j)X_i exp(t_j X_j) \ &= \ X_m(\delta_{im} - t_j c_{ijm} + \frac{1}{2}t_j t_k c_{ikl}c_{ljm} + ...) \\ A_{mi} \ &= \ \delta_{mi} - t_j c_{ijm} + \frac{1}{2}t_j t_k c_{ikl}c_{ljm} + ... \end{aligned} \qquad (A24)$$

Substituting the formula for A_{mi} into equation (A10) and equating the coefficients at each order in t gives

$$\begin{aligned} O(t^0): \quad c_{ijm} \ &= \ c_{ijm} \\ O(t): \quad c_{ijk}c_{klm} \ &= \ c_{ilk}c_{kjm} \ + \ c_{ikm}c_{kjl} \\ O(t^2): \quad c_{ijk}c_{klp}c_{lnm} \ &= \ c_{kjm}c_{ipt}c_{tnk} \ + \ c_{ilm}c_{jpt}c_{tnl} \ - \ 2c_{klm}c_{ink}c_{jpl} \\ &\vdots \end{aligned} \qquad (A25)$$

The relations for $O(t^r)$, $r \geq 1$ all follow from the Jacobi identity for the Lie algebra structure constants.

One may wish to consider a more general linear transformation A of the form

$$A_{mi} = \delta_{mi} - t_j d_{1ijm} + \frac{1}{2} t_j t_k d_{2ikl} d_{3ljm} + ... \qquad (A26)$$

leading to the relations

$$c_{ijk} d_{1klm} = d_{1ijk} c_{kjm} + c_{ikm} d_{1kjl} + ... \qquad (A27)$$

$$c_{ijk} d_{2klp} d_{3lnm} = c_{kjm} d_{2ipt} d_{3tnk} + c_{ilm} d_{2jpt} d_{3tnl} - 2 c_{klm} d_{1ink} d_{1jpl} \qquad (A28)$$

$$\vdots$$

Since the structure constants c_{ijk} are anti-symmetric, equation (A27) comprises of $\frac{n(n-1)}{2}$ relations for the n^3 components of the tensor d_{1ijk}, so that it does not fully determine d_1. However, equation (A28) consists of $\frac{n(n-1)}{2} \cdot n^3$ relations for the $3n^3$ components of d_{1ijk}, d_{2ijk} and d_{3ijk}. Similarly, higher orders of t give rise to equations which will include a larger number of conditions on the coefficients d_{Mijk}. Some of these constraints may be redundant, but they should eliminate the arbitrariness in the choice of d_{Mijk} and leave coefficients that are equal to the structure constants c_{ijk}.

B. Gauge Theories and the Seven-Sphere

The success of the standard model, based on the symmetry group $SU(3) \times SU(2) \times U(1)$, immediately suggests the problem of finding an explanation for the gauge groups and the number of forces. One of the simplest and yet potentially fundamental ways of deriving the standard model from theoretical principles would be through an identification of the four forces with the existence of only four normed real division algebras, corresponding to the sequence of parallelizable spheres, with the first two spheres coinciding with the gauge groups U(1) and SU(2). This correspondence leads one to conjecture the existence of a gauge theory based on S^7, which nearly possesses a Lie group structure. In earlier attempts to construct an octonionic gauge theory [66][67][68], there was no invariance under gauge transformations of the potential taking values only in the base space. The reasons for the lack of a gauge principle have been illuminated in the previous two sections in the study of the fibre-coordinate dependence of the connection form transformation rule for S^7 bundles.

The extension of the fibre bundle description of gauge theories to a Kaluza-Klein unification of gravity with the elementary particle interactions through eleven-dimensional

N=1 supergravity also involves the seven-sphere, as the ground state solution for the metric field, $(AdS)_4 \times S^7$ provides a mechanism for the spontaneous compactification of seven of the eleven dimensions. Because S^7 admits 8 Killing spinors, generating 8 supersymmetries, dimensional reduction of the d=11 theory gives N=8 gauged supergravity in four dimensions [69], and although spontaneous compactification over S^7 is initially promising for the identification of the elementary particle interactions and the sequence of parallelizable spheres, the gauge group SO(8) represents the invariances of a theory described by a principal bundle with both standard fibre and structure group SO(8), without reference to S^7. Moreover, the number of gauge bosons is considerably larger than that associated with any of the known elementary particle interactions and specifically the strong nuclear force. It therefore represents a considerable departure from the original scheme of using the sequence of parallelizable spheres as the basis for a description of the electromagnetic, weak and strong interactions.

Since the construction of the N=1, d=11 supergravity Lagrangian, several other investigations have been undertaken regarding the possibility of finding a gauge theory based on S^7. Indeed, it has been shown in [12] and the previous section that it is not feasible to build a purely octonionic generalization of Yang-Mills theory. Nevertheless, an octonionic gauge theory has been proposed in a recent study [24], using a bimodule representation of the octonion algebra.

Recalling that the octonion algebra is defined by the multiplication relations

$$\begin{aligned} e_0^2 &= e_0 \qquad e_0 e_i \ = \ e_i e_0 \ = \ e_i \\ e_i e_j &= -\delta_{ij} e_0 \ + \ \epsilon_{ijk} e_k \\ \epsilon_{ijk} &= 1 \ for \ (ijk) \ = \ (123),(145),(167),(264),(257),(347),(356) \end{aligned} \tag{B1}$$

using the conventions of [70], which can be summarized as $e_\mu e_\nu = C^\lambda_{\mu\nu} e_\lambda$, $\mu, \ \nu, \ \lambda = 0,1,...,7$. The left representation is $(L_\mu)_{\lambda\nu} = (C_\mu)^\lambda_\nu \equiv C^\lambda_{\mu\nu}$ and the right representation is $(R_\mu)_{\lambda\nu} = (\tilde{C}_\mu)^\lambda_\nu \equiv C^\lambda_{\nu\mu}$ with $L_0 = R_0 = Id$. Then $\lambda_i = i\frac{L_i}{4}$ and $\rho_j = i\frac{R_j}{4}$ satisfy the commutation relations

$$\begin{aligned} [\lambda_i, \ \lambda_j] &= \frac{i}{2}\epsilon_{ijk} \ \lambda_k \ + \ 2[\rho_j, \lambda_i] \\ [\rho_i, \ \rho_j] &= -\frac{i}{2}\epsilon_{ijk} \ \rho_k \ + \ 2[\lambda_j, \rho_i] \end{aligned} \tag{B2}$$

and the trace relations

$$Tr(\lambda_i \lambda_j) \ = \ Tr(\rho_i \rho_j) \ = \ \frac{1}{2}\,\delta_{ij} \tag{B3}$$

Suppose a gauge potential is defined to be $A_\mu(x) = A^i_\mu(x)\lambda_i$, leading to the covariant derivative $\mathcal{D}_\mu = \partial_\mu + ig\, A_\mu$. Under a gauge transformation corresponding to the element $\Omega(x) = e^{i\alpha^i(x)\lambda_i}$, $\mathcal{D}_\mu\psi(x) \to e^{i\alpha^i\lambda_i}\, \mathcal{D}_\mu\psi(x)$ for any fermion field $\psi(x)$, and $[\mathcal{D}_\mu, \mathcal{D}_\nu]$ also transforms covariantly, allowing one to immediately find an invariant $-\frac{1}{2}Tr(F_{\mu\nu}F^{\mu\nu})$.

However, the gauge tranformation which leaves this action invariant

$$\begin{aligned} A_\mu(x) \;\to\; A'_\mu(x) &= \Omega(x)A_\mu(x)\Omega^{-1}(x) \;+\; \frac{i}{g}(\partial_\mu\Omega(x))\Omega^{-1}(x) \\ &= e^{i\alpha\cdot\lambda}A_\mu e^{-i\alpha\cdot\lambda} \;-\; \frac{1}{g}\partial_\mu\alpha^i\lambda_i \\ &= A_\mu \;+\; [i\alpha\cdot\lambda, A_\mu] \;-\; \frac{1}{2}[\alpha\cdot\lambda,[\alpha\cdot\lambda, A_\mu]] \;+\; ... - \frac{1}{g}\partial_\mu\alpha^i\lambda_i \end{aligned} \tag{B4}$$

contains not only the generators λ_i associated with the left representation of the octonion algebra but also generators arising from multiple commutators of λ_i and ρ_j. To linear order in α,

$$\begin{aligned} A'_\mu &= A'^i_\mu\lambda_i \;+\; 2i\alpha^j\, A^k_\mu\, [\rho_k, \lambda_j] \\ A^{i\prime}_\mu &= A^i_\mu \;-\; \frac{1}{2}\epsilon_{ijk}\alpha^j A^k_\mu \;-\; \frac{1}{g}\partial_\mu\alpha^i \end{aligned} \tag{B5}$$

and

$$\begin{aligned} F'_{\mu\nu} &= F'^i_{\mu\nu}\,\lambda_i \;-\; 2g\alpha^k(A'^i_\mu A'^j_\nu \;-\; A'^i_\nu A'^j_\mu)[\lambda_i,[\rho_k,\lambda_j]] \\ &\quad+\; 2i[\partial_\mu(\alpha^j A'^k_\nu) \;-\; \partial_\nu(\alpha^j A'^k_\nu)][\rho_k,\lambda_j] \\ F'^i_{\mu\nu} &= \partial_\mu A'^i_\nu \;-\; \partial_\nu A'^i_\mu \;-\; \frac{1}{2}g\epsilon_{ijk}A'^j_\mu A'^k_\nu \end{aligned} \tag{B6}$$

The trace relations

$$\begin{aligned} Tr(\lambda_i[\rho_j,\lambda_k]) &= \frac{i}{8}\epsilon_{ijk} \\ Tr([\lambda_i,\rho_j][\lambda_k,\rho_l]) &= \frac{-1}{32}[\epsilon_{ijkl} \;+\; 2(\delta_{ik}\delta_{jl} \;-\; \delta_{il}\,\delta_{jk})] \end{aligned} \tag{B7}$$

can be used to express the Lagrangian in component form

$$\mathcal{L} \;=\; -\frac{1}{4}(\partial_\mu A^i_\nu \;-\; \partial_\nu A^i_\mu)(\partial^\mu A^{\nu i} \;-\; \partial^\nu A^{\mu i}) - \frac{1}{16}g^2 A^j_\mu A^k_\nu(A^{j\mu}A^{k\nu} \;-\; A^{k\mu}A^{j\nu}) \tag{B8}$$

but the action is not invariant under the substitution $A^i_\mu \to A'^i_\mu$ as (B8) becomes

$$\begin{aligned} &-\frac{1}{4}(\partial_\mu A^i_\nu \;-\; \partial_\nu A^i_\mu)(\partial^\mu A^{\nu i} \;-\; \partial^\nu A^{\mu i}) \;-\; \frac{1}{16}g^2 A^j_\mu A^k_\nu(A^{j\mu}A^{k\nu} \;-\; A^{k\mu}A^{j\nu}) \\ &+\; \frac{1}{2}[\partial_\mu(\epsilon_{ijk}\alpha^j A^k_\nu \;+\; \frac{1}{g}\partial_\nu\alpha^i) \;-\; \partial_\nu(\epsilon_{ijk}\alpha^j A^k_\mu \;+\; \frac{1}{g}\partial_\mu\alpha^i)]\cdot\partial^\mu A^{\nu i} \\ &+\; \frac{1}{8}g^2(A^j_\mu A^k_\nu \;-\; A^j_\nu A^k_\mu)(\frac{1}{2}\epsilon^j_{pq}\alpha^p A^q_\mu \;+\; \frac{1}{g}\partial^\mu\alpha^j)\cdot A^\nu_k \end{aligned} \tag{B9}$$

and the extra terms in (B9) are not total derivatives. Consequently, the transformation of the potential $A_\mu \rightarrow A'_\mu$ is required and this involves generators other than those corresponding to the left representation of the octonion algebra.

The appearance of the non-closed algebraic part of the variation involving the generators ρ_k is a manifestation of the fibre-coordinate dependence in the transformation rule of the connection form (A1). One may recall that $T\psi^y_{\beta\alpha}(\xi_x)$ is $\phi_{y'}X_1$ for some $X_1 \in V$ when the standard fibre admits a parallelism, and thus the y-dependence of this term can be eliminated for an S^7 bundle. From equation (A7) and proof of the subsequent lemma, it follows that the y-dependence of $T\psi^x_{\beta\alpha} \cdot \phi_y \Gamma^\alpha(\xi_x)$ can be eliminated only when the standard fibre is a group manifold. For an S^7 bundle, the remaining y-dependence should be associated with this term in equation (B5). Moreover, the analysis of §3 leads one to conjecture that the Lagrangian is part of a larger theory possessing G_2 invariance. Writing $A'_\mu = A'^{AB}_\mu J_{AB}$, a transformation $A'_\mu \rightarrow A_\mu = \Omega^{-1} A_\mu \Omega - \frac{i}{g}\Omega^{-1}(\partial_\mu \Omega)$, $\Omega \in G_2$ can be found such that A_μ has only non-zero components A^i_μ multiplying the generators λ_i. From (B6), $F_{\mu\nu} = F^{AB}_{\mu\nu} J_{AB}$ has non-zero components corresponding to the generators of G_2 and the Lagrangian $-\frac{1}{2}Tr(F_{\mu\nu}F^{\mu\nu})$ actually should be invariant under the entire group of G_2 transformations. This symmetry is broken only when one specializes to a particular choice for the vanishing components of the gauge potential.

This result may also be understood from the context of non-associative deformations of gauge theories [71]. These generalizations are based on an algebraic structure consisting of M set of generators $\{T_{pi}\}$, $p = 1, ..., M$. Together, the entire set of generators form an associative Lie algebra structure. Restriction to one set of generators, T^p_i, p fixed, leads to a problem with closure of the algebra.

$$[T^p_i, T^p_j] = f^p_{ijk}\, T^p_k + \sum_{n=1}^{M} \sigma^p_n\, [T^n_j, T^p_i] \qquad (B10)$$

The deviation from associativity can be measured by the associator

$$\begin{aligned} J(T^p_i,\, T^p_j,\, T^p_k) &= \epsilon^{ijk}(T^p_i,\, T^p_j,\, T^p_k) = \epsilon^{ijk}\,[\,(T^p_i\, T^p_j)\, T^p_k - T^p_i(T^p_j\, T^p_k)\,] \\ &= \sigma^p_n([T^p_i, [T^n_k, T^n_j]] + [T^p_j, [T^n_i, T^p_k]] + [T^p_k, [T^n_j, T^p_i]]) \end{aligned} \qquad (B11)$$

The lack of closure of the algebra associated with a single set of generators, or equivalently, the coupling of the gauge potentials corresponding to distinct sets of generators, leads to a Lagrangian, based on only the field strengths $F^p_{\mu\nu}$, which contains extra nonlinear terms. Therefore, when the algebra T^p_i, p *fixed* represents the octonions, the entire algebra is given by the matrices associated with left and right multiplication introduced earlier. The theory obtained is therefore a special case of this general procedure of deforming gauge theories, where the Lagrangian is part of an action with the larger G_2 symmetry generated by the combined set of fourteen generators.

It is also of interest to note that a general procedure for constructing a non-associative gauge theory has been developed in [72], where the potential takes values in a non-associative algebra A. The gauge symmetry of the action, however, is the automorphism group of A, G_A, so that given a symmetric, bi-linear non-degenerate form $\langle\ u\ |\ v\ \rangle$, $u,\ v\ \in\ A$, satisfying the invariance condition $\langle\ gu\ |\ gv\ \rangle$, a Lagrangian

$$L_0\ =\ \frac{1}{4}\ \langle\ F_{\mu\nu}\ |\ F_{\mu\nu}\ \rangle \tag{B12}$$

may be constructed. If Λ_p is the set of A-valued p-forms, then it is necessary to assume that $g \in gl(\Lambda_1),\ \xi\ \in\ \Lambda_1$ and

$$\begin{aligned} &(i) \quad g(\omega\omega)\ =\ (g\omega)(g\omega) \\ &(ii) \quad d\xi\ +\ \xi\ \xi\ =\ 0 \\ &(iii) \quad d(g\omega)\ =\ g(d\omega)\ -\xi(g\omega)\ -\ (g\omega)\xi \end{aligned} \tag{B13}$$

Although it is not known if there is a solution for g and ξ for any given non-associative algebra, it can be assumed that one exists when A is the octonion algebra. The symmetry group G_A is then G_2, and the Lagrangian resembles the one given in equation (B8).

A common property of all of these theories is that the action possesses a Lie group symmetry even though it has been constructed so that it seems to include only components in the non-associative octonion algebra. This provides further support for the assertion proven in [12] and Appendix A that a pure gauge theory with a symmetry defined only by the non-associative algebra does not exist. In addition, it may be noted that although $(g^{-1}A_\mu)g\ =\ g^{-1}(A_\mu g)$ for alternative algebras by Artin's theorem,

$$\{g^{-1}(A_\mu g)\}\{(g^{-1}A_\nu)g\}\ \neq\ g^{-1}\ (A_\mu\ A_\nu)\ g \tag{B14}$$

in general for non-associative algebras, so that the validity of invariance under finite gauge transformations as a consequence of invariance under infinitesimal transformations cannot be proven for non-associative algebras [72]. Since an extensive investigation of the possibility of constructing a pure gauge theory based only on the non-associative algebra has revealed that a Lie group structure is essential, this property will be assumed in the following sections, although the seven-sphere shall continue to used in this geometrical approach to the internal symmetry spaces.

C. The Fermion Term in a Yang-Mills Theory based on Octonions

The Lagrangian for pure Yang-Mills theories is

$$\begin{aligned} -\frac{1}{4}\ Tr(F_{\mu\nu}F^{\mu\nu}) \ &= \ -\frac{1}{4}F^a_{\mu\nu}F^{b\mu\nu}\ Tr(T^aT^b) \ = \ -\frac{1}{4}\ Tr(F^a_{\mu\nu}F^{b\mu\nu}) \\ F^a_{\mu\nu} \ &= \ \partial_\mu A^a_\nu \ - \ \partial_\nu A^a_\mu \ + \ f^{abc}A^b_\mu A^c_\nu \end{aligned} \tag{C1}$$

where f^{abc} are structure constants of the Lie group. The pure Yang-Mills action $\int_M\ d^4x\ Tr(F\wedge F^*)$ is bounded by the functional, $\int_M\ d^4x\ Tr(F\wedge F)$, given by the second Chern class for a principal bundle in the expansion

$$\begin{aligned} c(\Omega) \ &= \ Det(I+\frac{i}{2\pi}\Omega) \ = \ 1 \ + \ c_1(\Omega) \ + \ c_2(\Omega) \ + \ ... \\ c_0 \ &= \ 1 \\ c_1 \ &= \ \frac{i}{2\pi}Tr\ \Omega \\ c_2 \ &= \ \frac{1}{8\pi^2}\ [Tr\ \Omega\wedge\Omega \ - \ Tr\ \Omega\wedge Tr\ \Omega] \\ &\vdots \end{aligned} \tag{C2}$$

where Ω is the curvature form. While the second Chern class may be generalized to the manifold S^7, the analysis of §3 shows that a similar generalization of the Yang-Mills functional may not be invariant under gauge transformations of potentials taking values in the tangent space at a chosen origin of S^7.

For a Lie group, the fermion term

$$i\bar{\psi}\not{D}\psi \ = \ i\bar{\psi}_\alpha(\gamma^\mu)_{\alpha\beta}(\partial_\mu \ + \ igA^a_\mu T^a)\psi_\beta \tag{C3}$$

assumes that the generators can be expressed in matrix form. Since the octonion algebra is not a group, it cannot be mapped isomorphically onto a matrix algebra.

As an example [73] of the problems that arise in a matrix representation of the octonions, let X, Y, $\Lambda\in M_3$ where $M_3 \ = \ \{3\times 3\ \ traceless\ matrices\ over\ a\ field\ F\}$ with

Tr $\Lambda^3 \neq 0$. Define $\phi(X) = \frac{Tr\Lambda^2 X}{Tr\Lambda^3}$ and

$$\begin{aligned} \langle X, Y\rangle &= \phi(X)\phi(Y) + \frac{1}{\beta^2}\{Tr(X\Lambda)Tr(Y\Lambda) - 2\ Tr(X\Lambda Y\Lambda)\} \\ X \circ Y &= \phi(X)Y \ + \ \phi(Y)X \ - \ \langle X, Y\rangle\Lambda + \beta\{[X, Y, \Lambda] - \phi([X, Y, \Lambda])\Lambda\} \\ [X, Y, \Lambda] &= \frac{1}{2}\{[X, Y]\Lambda \ + \ [Y, \Lambda]X \ + \ [\Lambda, X]Y\} \ - \ \frac{1}{2}Tr(XY\Lambda - \Lambda YX)E \\ \beta &\in F \qquad E \ = \ diag(1, 1, 1) \end{aligned} \tag{C4}$$

Consider the Gell-Mann matrices λ_j, j = 1,...,8 with product relations

$$\begin{aligned} \lambda_j\lambda_k &= \frac{2}{3}\delta_{jk}E \ + \ \sum_{l=1}^{8}(d_{jkl} + if_{jkl})\lambda_l \\ d_{jkl} &= \frac{1}{4}Tr(\lambda_j\{\lambda_k, \lambda_l\}) \qquad f_{jkl} \ = \ -\frac{i}{4}Tr\{\lambda_j[\lambda_k, \lambda_l]\} \end{aligned} \tag{C5}$$

and inner products

$$\begin{aligned} \langle\lambda_j, \lambda_k\rangle &= -\beta^2(2\sqrt{3}d_{8jj} - 1)\delta_{jk} \qquad (j, k \neq 8) \\ \lambda_j \circ \lambda_k &= -\langle\lambda_j, \lambda_k\rangle\Lambda \ + \ 2i\beta f_{ik8}\lambda_8 \ + \ \sqrt{3}i\beta\sum_{l,p=1}^{8}[f_{jkl}d_{l8p} + f_{k8l}d_{ljp} + f_{8jl}d_{lkp}]\lambda_p \\ with\ \Lambda &= \sqrt{3}\lambda_8 \end{aligned} \tag{C6}$$

Setting $e_0 = \Lambda = \sqrt{3}\lambda_8$, $e_j = -\frac{i}{\beta}\lambda_j, j = 1, 2, 3$, $e_j = \frac{1}{\sqrt{2}\beta}\lambda_j, j = 4, 5$, and $e_j = -\frac{1}{\sqrt{2}\beta}\lambda_j, j = 6, 7$ it follows that $\langle e_j, e_k\rangle = \delta_{jk}$ and

$$e_j \circ e_k \ = \ -\delta_{jk}e_0 \ + \ \sum_{l=1}^{7} a_{jkl}e_l \tag{C7}$$

which represents the octonion algebra.

Thus, octonions may be represented by traceless 3×3 matrices and they act on $\mathbb{C}^3$ via matrix multiplication. However $\mathbb{C}^3$ is probably not a bimodule for $\mathbb{O}$ as both left and right multiplication by the octonions must be defined and the following identities should be satisfied

$$\begin{aligned} &R_{a^2} = R_a^2 \qquad L_{a^2} = L_a^2 \qquad \forall\ a \in \mathbb{O} \\ &R_{a\circ b} - R_bR_a = -[L_a, R_b] \qquad L_{a\circ b} - L_aL_b = [L_a, R_b] \qquad \forall\ a,\ b \in \mathbb{O} \end{aligned} \tag{C8}$$

where R_a, L_a are linear operators given by $m \to R_a m = ma$ and $m \to L_a m = am$ for all $m \in \mathbb{C}^3$. Right multiplication can then be defined as $\psi' = \psi\, A$ or $\psi'^T = A^T\, \psi^T$ so that

$$\begin{pmatrix} \psi_1' \\ \psi_2' \\ \psi_3' \end{pmatrix} = \begin{pmatrix} a_{11}\psi_1 + a_{21}\psi_2 + a_{31}\psi_3 \\ a_{12}\psi_1 + a_{22}\psi_2 + a_{32}\psi_3 \\ a_{13}\psi_1 + a_{23}\psi_2 + a_{33}\psi_3 \end{pmatrix} \qquad (C9)$$

An second possible definition exists if a non-standard matrix multiplication rule is used. However, in both cases, the representation is not faithful because $R_{e_0}\psi \neq \psi$ as $e_0 = diag(1, 1, -2)$. The significant point is that $L_a(L_b\psi)$ should be related to $L_{a\circ b}\psi$; otherwise, one is considering a 3×3 matrix algebra with no reference to the octonions.

Despite this lack of a faithful matrix representation of the octonion algebra, the principle of triality can be used to define products elements between the 8-dimensional vector space M representing the octonions and the two spaces of even and odd half spinors. Following the treatment of Chevalley [74], if Q is a quadratic form on $\mathbb{O}$, S_p and S_i are the spaces of even and odd half-spinors for Q, and ρ denotes the spin representation of the corresponding Clifford group Γ, there is bilinear form β on $S \times S$, where $S = S_p + S_i$, such that β is symmetric, vanishes on $S_p \times S_i$ and $S_i \times S_p$ and its restrictions to $S_p \times S_p$ and $S_i \times S_i$ are non-degenerate. Defining the space $A = M \times S$, a law of composition in A can be defined. Let

$$\begin{aligned} \Phi(\xi, \eta, \zeta) &= F(\xi + \eta + \zeta) + F(\xi) + F(\eta) + F(\zeta) \\ &\quad - [F(\xi + \eta) + F(\eta + \zeta) + F(\zeta + \xi)] \qquad (C10) \\ F(x + u + u') &= \beta(\rho(x) \cdot u, u') \end{aligned}$$

Then $\omega = \xi \circ \eta$ is defined by $\Phi(\xi, \eta, \zeta) = \Lambda(\xi \circ \eta, \zeta)$ where

$$\Lambda(x + u', x' + u') = B(x, x') + \beta(u, u') \qquad (C11)$$

where B is a bilinear form on $M \times M$. One finds that $(\xi, \eta) \to \xi \circ \eta$ is the law of composition of a nonassociative algebra on A. In particular, $M \circ S_p \subset S_i$, $M \circ S_i \subset S_p$ and $S_p \circ S_i \subset M$ and, by the principle of triality, there is an automorphism J of order 3 of the vector space A , which maps M onto S_p, S_p onto S_i and S_i onto M, and a law of composition on $M \times M$ representing the algebra of octonions

$$\begin{aligned} x * y &= (x \circ u_1') \circ (y \circ u_1) \\ u_1 \in S_p,\ \gamma(u_1) &= 1,\ u_1' = x_1 \circ u_1,\ Q(x_1) = 1 \end{aligned} \qquad (C12)$$

where γ is a quadratic form on S given by $\gamma(u + v) = \gamma(u) + \gamma(v) + \beta(u, v)$. This composition law also suggests a method for combining two elements of the space of spinors

S and obtaining an element of M. This product could then be used to construct the fermion term in the Lagrangian, which is essentially a bilinear product of two spinors. A similar approach to fermions and octonions has been made in an analysis of products of vertex operators in superstring theory [75].

D. An Application to Quantum Principal Bundles

The concepts of principal bundles and gauge transformations can be generalized using quantum groups. The bundle $P = P(B, A)$ is a quantum principal bundle [76] with quantum structure group A and base B if

(i) A is a Hopf algebra with co-product $\Delta : A \rightarrow A \otimes A$, co-unit $\epsilon : A \rightarrow k$, and antipode map $S : A \rightarrow A$.
(ii) $(P, \Delta_R : P \rightarrow P \otimes A)$ is a right A-co-module algebra
(iii) $B = P^A = \{u \in P : \Delta_R u = u \otimes 1\}$
(iv) $(\cdot \otimes id)(id \otimes \Delta_R) : P \otimes P \rightarrow P \otimes A$ is a surjection
(v) $ker^{\sim} = \Gamma_{hor}$ where $\Gamma_{hor} = Pj(\Gamma_B)P \subset \Gamma_P$, with $j : \Omega B \hookrightarrow \Omega P$ being an inclusion and Γ_B being the space of one-forms on B, and $\sim = (\cdot \otimes id) \circ (id \otimes \Delta_R)|_{P^2} : \Gamma_P \rightarrow P \otimes A$.

Although the Hopf algebra acts on the co-module algebra P in the principal quantum bundle, the map $(\cdot \otimes id)(id \otimes \Delta_R) : P \otimes P \rightarrow P \otimes A$ descends to an isomorphism $P \otimes_B P \rightarrow P \otimes A$. By analogy with classical bundles, the action of A on P in a quantum principal bundle is determined locally by the action of the quantum group on itself or the space of polynomial functions on this group, which is essentially its dual [77].

The polynomial function space for a group such as SU(N) is generated by the set of coordinate functions $u^i_j : g \rightarrow g^i_j$, where g^i_j is the (i,j)th matrix element of g in the fundamental representation. Similarly, for a quantum group $\mathcal{U}$, its dual A is generated by the non-commutative coordinate functions $u^i_j : \mathcal{U} \rightarrow \mathbb{C}$ and the co-product is defined to be $\Delta u^i_j = u^i_k \otimes u^k_j$. This suggests that one can choose the operation of left multiplication to be a map from A to A

$$\begin{aligned} L_g &: u^i_j \rightarrow u^i_k \ \langle u^k_j, g\rangle \\ L_g &= (id \otimes |g\rangle) \circ \Delta \end{aligned} \tag{D1}$$

and the operation of right multiplication is a map from A to A defined by

$$\begin{aligned} R_g &: u^i_j \rightarrow \langle u^i_k, g\rangle \ u^k_j \\ R_g &= (|g\rangle \otimes id) \circ \Delta \end{aligned} \tag{D2}$$

where $\langle\ ,\ \rangle$ represents the inner product between elements of the quantum group $\mathcal{U}$ and its dually paired Hopf algebra A, so that $\langle u^i_j, g\rangle = g^i_j \in \mathbb{C}$. Thus, $L_g u^i_j = g^k_j u^i_k$ implying the following rule for left multiplication

$$L_g L_{\tilde{g}}\ u^i_j = g^k_l \tilde{g}^l_j u^i_k = L_{g\tilde{g}} u^i_j \tag{D3}$$

Similarly, for right multiplication, $R_g R_{\tilde{g}} = R_{\tilde{g}g}$. Thus, the noncommutative function spaces, $A = \mathbb{C}\langle u^i_j\rangle/(R_{12}u_1u_2 - u_2u_1R_{12})$ are dually paired to the quantum groups $\mathcal{U} = \frac{\mathbb{C}\langle l^{+i}_j, l^{-i}_j\rangle}{\left\{ {(R_{12}l^{\pm}_2 l^{\pm}_1 - l^{\pm}_1 l^{\pm}_2 R_{12}) \atop (R_{12}l^{+}_2 l^{-}_1 - l^{-}_1 l^{+}_2 R_{12})} \right\}}$, where the $l^{\pm i}_j$ are the generators of the universal enveloping algebra of the quantum group in the Chevalley basis [78]. The inner products satisfy

$$\begin{aligned} \langle u^i_j, l^{+k}_l\rangle &= R^{ik}_{jl} \qquad \langle u^i_j, l^{-k}_l\rangle = {R^{-1}}^{ki}_{lj} \\ \langle ab,\ c\rangle &= \langle a\otimes b,\ c_{(1)}\otimes c_{(2)}\rangle \\ \langle a,\ cd\rangle &= \langle a_{(1)}\otimes a_{(2)},\ c\otimes d\rangle \end{aligned} \tag{D4}$$

The adjoint action on a Hopf algebra is taken to be a map $Ad_R : A \to A\otimes A$, $Ad_R(a) = \sum a_{(2)}\otimes (Sa_{(1)})a_{(3)},\ a\in A$. An adjoint action can also be defined as a map from A to A

$$\begin{aligned} Ad_{S(g)}a &= \langle a_{(1)},\ g_{(1)}\rangle\ a_{(2)}\ \langle a_{(3)}, S(g_{(2)})\rangle = R_g(a_{(1)}\langle a_{(1)}, S(g_{(1)})\rangle) = R_g L_{g^{-1}} \\ & g\in\mathcal{U},\ a\in A \end{aligned} \tag{D5}$$

Given multiplication in the quantum group $\mathcal{U}$, it follows that if one considers a left translation from y to $y' = g\cdot y$,

$$R_{y*}R_{g*} = R_{y'*} = L_{y'*}\ Ad_{S(y')} \tag{D6}$$

for tangent mappings [79] induced by transformations from the Hopf algebra A to A, whereas a right translation from y to $y' = y\cdot g$ implies $L_{y*}L_{g*} = L_{y'*}$. It follows that, for a specific choice of parallelism, such as that defined by left multiplication, on the standard fibre in the quantum principal bundle, transformations in the structure group should only be defined using right translations rather than left translations on the Hopf algebra. Moreover, if the connection forms are required to take values on the base space subalgebra B, then the group of *gauge* transformations is an automorphism group acting on the total space P, which preserves the base space B. Previous considerations of the fibre coordinate dependence of the connection form transformations imply that it is this group which must necessarily be used in the definition of gauge transformations in the quantum principal bundle [76]. In the classical limit, for principal bundles, the automorphism group, the structure group and the standard fibre all coincide.

Specifically, if $\beta : A \to \Gamma_B$ is a linear map such that $\beta(1) \ = \ 0$,

$$\omega(a) \ = \ \sum \Phi^{-1}(a_{(1)}) j(\beta(a_{(2)})\Phi(a_{(3)}) \ + \ \sum \Phi^{-1}(a_{(1)} d\Phi(a_{(2)})) \qquad (D7)$$

is a connection one-form in the principal bundle $P(B, A, \Phi)$ having a trivialization $\Phi : A \to P$ [76]. Given two linear maps f_1 and f_2 on the quantum group $\mathcal{U}$ from A to B, the convolution is $g \ = \ f_1 * f_2$, where $g(a) \ = \ \sum f_1(a_{(1)}) f_2(a_{(2)})$. If $\gamma : A \to B$ is a convolution invertible map such that $\gamma(1) \ = \ 1$, then the connection form (D7) transforms as

$$\omega^\gamma \ = \ (\Phi^\gamma)^{-1} * j(\beta) * \Phi^\gamma \ + \ (\Phi^\gamma)^{-1} * d\Phi^\gamma \qquad (D8)$$

under a change of trivialization $\Phi \to \Phi^\gamma$ [76].

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