Mathematical Applications of String Theory: Spin Structures on Riemann Surfaces and the Perfect Numbers

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Abstract. The equality between the number of odd spin structures on a Riemann surface of genus g, with $2^g - 1$ being a Mersenne prime, and the even perfect numbers, is an indication that the action of the modular group on the set of spin structures has special properties related to the sequence of perfect numbers. A primality test for Mersenne numbers is developed by using a geometrical representation of the numbers for a particular set of values of the Mersenne index n. Non-existence of finite odd perfect numbers is demonstrated to be equivalent to the irrationality of the square root of a product of a sequence of repunits multiplied by twice the base of one of the repunits.

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1. Introduction

Previous applications of physics to number theory include representation of the Riemann zeta funtion using partition functions for specific statistical mechanical systems [1][2], proof of the inversion formula of the Möbius function [3] and fundamental theorem of algebra [4] using supersymmetric quantum mechanics, expressions of Gaussian sums based on the partition function for $\frac{\mathbb{Z}}{n\mathbb{Z}}$ conformal field theories [5], and modular group symmetry of partition functions for compactified string theories [6]. In this talk, a further connection between string theory and number theory is established by relating the number of odd spin structures on genus-g Riemann surfaces and perfect numbers. Both conjectures regarding the existence of an infinite sequence of even perfect numbers and the non-existence of finite odd perfect numbers shall be considered. The counting of odd spin structures to obtain the sequence of even perfect numbers is given in §3 while a condition for the existence of odd perfect numbers is given in §4.

2. Primality Tests Based on a Geometrical Representation of Mersenne Numbers

The primality tests for Mersenne numbers are known to be satisfied by 38 primes $2^p - 1$ [7]. The two main types of tests are

(1) Lucas - Lehmer test

Consider the sequence defined by the relation $V_{(n+1)} = V_n^2 - 2$, with $V_0 = 4$. If p is the index of a Mersenne prime $M_p = 2^p - 1$, then $V_{p-2} \equiv 0 \ (mod \ M_p)$ [8][9].

(2) Primality tests based on the factorization of $N - 1$ or $N + 1$

If one wishes to test the primality of N, it is useful to study the factorization of the numbers of $N - 1$ and $N + 1$, particularly for numbers of the form $c \ 2^n \pm 1$.

The use of Lucas sequences

$$\begin{aligned} U_{k+2} &= PU_{k+1} - QU_k \qquad U_0 = 0, \ U_1 = 1 \\ V_{k+2} &= PV_{k+1} - QV_k \qquad V_0 = 2, \ V_1 = P \end{aligned} \tag{1}$$

occurs in the following theorem [10]:

Theorem (Brillhart, Lehmer, Selfridge 1975) - Let $N+1 = q_i^{n_i}$, and consider the set of Lucas sequences $\{U_k^{(i)}\}$. If for q_i, there exists a Lucas sequence in this set such that $N|U_{(N+1)}^{(i)}$, but $N \not| \frac{U_{(N+1)}^{(i)}}{q_i}$ then N is prime.

There are various geometrical representations of perfect numbers and Mersenne primes. Even perfect numbers are known to be both triangular and hexagonal, whereas Mersenne numbers can be viewed as a triangular array for finite n, placing 2^m sites at the m^{th} level, with m taking values between 0 and $n-1$. The following lemma [11] related to primality of integer $N = 2^n - 1$ can then be used in connection with this geometrical representation of Mersenne number.

Lemma (de la Rosa 1978) - A positive integer is a prime or a power of 2 if and only if it cannot be expressed as the sum of at least three consecutive positive integers.

The application of this lemma to the geometrical representation of the Mersenne number is achieved by partitioning the triangular array into K parts. Since there $2^m - 1$ intervals between the 2^m sites at the m^{th} level, the division of the triangular array into K approximately equal parts will involve shared sites between neighbouring triangles if $K|2^m - 1$. The base level increases linearly with the level number and has length $l_m = ml_1$, where l_1 is the distance between the two sites at the $m = 1$ level. The distance between neighbouring sites at the m^{th} level is $\frac{l_m}{2^m-1} = m\frac{l_1}{2^m-1}$. If $K \not| 2^m - 1$ for $1 < m \leq n-1$, then $\frac{l_m}{K}$ is not an integer multiple of $\frac{l_m}{2^m-1}$, there are no shared sites in the interior of the triangle and the triangles T_j only intersect at the apex. The number of overcounted sites is $K - 1$.

When there are shared sites between the triangles, the counting is slightly different (Fig. 1).

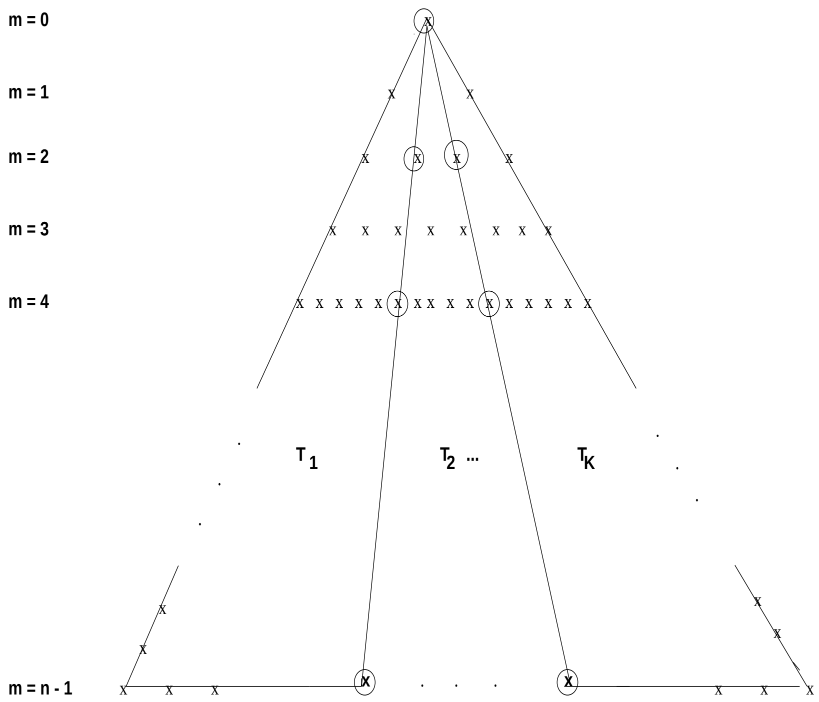

Fig. 1. Triangular Representation of the Mersenne Number $2^n - 1$. The circled sites are shared by adjacent triangles.

For example, in the model case of shared sites at the last level, $K|2^{n-1} - 1$, the total number of shared sites may be computed by defining the divisor function $\tau_2(n-1, K)$ to be the number of values of m such that either $m = 0$ or $m|n-1$ and $K|2^m - 1$. The number of shared sites is $1 + (K-1)(\tau_2(n-1,K) - 1)$ and the number of overcounted sites is $(K-1)\tau_2(n-1,K)$.

If the partition includes a site on the i^{th} level, where $i \leq n-1$, then $K|2^m - 1$ for some $m|i$, and the divisor function can be defined as $\tau_2(i,K) = 1 + ord\{m|m \neq 0, m|i, K|2^i - 1\}$. The notation $[m]$ will be used to denote the class of integers which are multiples of m less than n; the sequence begins with m and ends with an integer i, so that the number of shared sites is $1 + (K-1)(\tau_2(i,K) - 1)$ and the number of overcounted sites is $(K-1)\tau_2(i,K)$.

Consider the Lucas sequence $U_n(\alpha,\beta) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ so that $U_n(2,1) = 2^n - 1$. A well-known property of Lucas sequences is that $gcd(U_m, U_n) = U_\nu$ where $\nu = gcd(m,n)$ [12][13]. If $K|U_m$ and $K|U_n$, then $K|U_\nu$. Thus, a partition of the triangle into K equal regions will intersect the site at the ν^{th} level as well as the m^{th} level and n^{th} level and all three sites belong to the same class. Continuing this process, it follows that there is a minimum value m_0 such that the class $[m_0]$, representing a sequence of multiples of m_0, contains all integers $1 < m \leq n-1$ for which $K|2^m - 1$ such that $K|2^i - 1$.

At each level, the sites are distributed approximately equally amongst the K triangles; however, there will be extra sites. Let m_K be the first integer such that $2^m - 1 > K$ or $2^{m_K} > K + 1 > 2^{m_K-1}$ so that $m_K = \{log_2(K + 1)\}$ and

$$2^{m_K} - 1 \equiv K_{m_K} \ (mod\ K) \qquad 1 \leq K_{m_K} < K \tag{2}$$

Thus, at level m_K, the 2^{m_K} sites distributed amongst the K triangles produce an extra $K_{m_K} + 1$ sites.

Moreover, given a sequence of congruence relations $2^{m_K} \equiv K_{m_K} + 1 \ (mod\ K)$, $2^{m_K+1} \equiv 2(K_{m_K}+1) \ (mod\ K)$, ..., $2^{n-1} \equiv 2^{n-1-m_K}(K_{m_K}+1) \ (mod\ K)$, the number of extra sites from levels m_K, ..., $n-1$ is

$$\sum_{i=m_K}^{n-1} 2^{(i-m_K)}2^{m_K} = 2^{m_K}\Bigg[(2^{m_K}-1) + 2^{m_K}((2^{m_K}-1)+ \\ ... + 2^{m_K}\left((2^{m_K}-1) + 2^{m_K}\sum_{i=rm_K}^{n-1} 2^{i-rm_K}\right)\Bigg] \tag{3}$$

where r is an integer such $rm_K < n-1 < (r+1)m_K$, which is congruent to

$$(K_{m_K}+1)\Big[K_{m_K} + (K_{m_K}+1)(K_{m_K} + (K_{m_K}+1)(K_{m_K} + (K_{m_K}+1)(K_{m_K} + \\ ... + (K_{m_K} + (K_{m_K}+1) \cdot (2^{n-rm_K}-1))\Big] \qquad (mod\ K) \tag{4}$$
$$= (K_{m_K}+1)^r \left(K_{m_K} + 2^{n-rm_K} - 1\right) - K_{m_K}(K_{m_K}+1) \qquad (mod\ K)$$

When $2^i > K$, there will be either 0 or 1 site in the j^{th} triangle and the number of extra sites from levels 0 to $m_K - 1$ is

$$\sum_{i=0}^{m_K-1} 2^i = 2^{m_K} - 1 \equiv K_{m_K} \qquad (mod\ K) \tag{5}$$

so that from levels 0 to $n-1$, they total

$$(K_{m_K}+1)^r(K_{m_K}+2^{n-rm_K}-1) \;-\; K_{m_K}^2 \qquad\qquad (mod\ K) \qquad\qquad (6)$$

Compositeness of the Mersenne number requires that the entire sum is a sum of consecutive integers. The Mersenne number therefore will be composite if either $(K_{m_K}+1)^r(K_{m_K}+2^{n-rm_K}-1) \;-\; K_{m_K}^2$ is congruent to the number $\frac{K(K+1)}{2}$.

At a particular level m, a certain set of triangles $\{T_{j'}\} \subset \{T_j\}$ will contain an extra site. If $2^m-1 \;\equiv\; K_m\ (mod\ K)$, there will be indices $j_{m,s},\ s=0,1,...,K_m$ such that $j_{m,0} \;=\; 1$, $j_{m,2} \;=\; \{\frac{K}{K_m}\},\ j_{m,2} \;=\; \{\frac{2K}{K_m}\},\ ...,\ j_{m,K_m-1} \;=\; \{\frac{(K_m-1)K}{K_m}\},\ j_{m,K_m} \;=\; K$. Since $2^{m+1}-1 \;\equiv\; 2K_m+1\ (mod\ K)$, $j_{m+1,0} \;=\; 1$, $j_{m+1,1} \;=\; \{\frac{K}{2K_m+1}\}$, $j_{m+1,2} \;=\; \{\frac{2K}{2K_m+1}\}, ...,$ $j_{m,2K_m} \;=\; \{\frac{2K_mK}{2K_m+1}\}, ..., j_{2K_m+1,K} \;=\; K$.

If h congruence cycles of the doubling map are completed between levels m_K and $n-1$, then the total number of extra sites * is

$$\begin{aligned}\sum_{i=0}^{m_K-1} 2^i \;+\; \sum_{t=0}^{h\ ord_K(2)} 2^t(K_{m_K}+1) \;&=\; 2^{m_K} \;-\; 1 \;+\; (2^{hord_K(2)+1}-1)\ (K_{m_K} \;+\; 1) \\ &\equiv\; 2K_{m_K} \;+\; 1 \quad (mod\ K)\end{aligned} \qquad (7)$$

when K is odd.

Since $\frac{K(K+1)}{2} \;\equiv\; 0$ when K is odd, the congruence relation is

$$2^{1+\{log_2 K\} \;-\; log_2 K} \;-\; 1 \;\equiv\; 0 \qquad\qquad (mod\ K) \qquad\qquad (8)$$

which implies that either $2^{1+\{log_2 K\}-log_2 K} \;=\; \frac{2K+1}{K}$ or $2^{1+\{log_2 K\}-log_2 K} \;=\; \frac{3K+1}{K}$. The first condition has no solution, while the second constraint is solved by $K \;=\; \frac{2^m-1}{3}$ for all even $m \;\geq\; 4$.

When K is even, let $K \;=\; 2^{t_K} \;\cdot\; K_0$ where K_0 is an odd integer. If the partition of the triangle does not contain shared sites, the congruence relation becomes

$$(2^{t_K}+1)2^{\{log_2 K\}-log_2 K} \;\cdot\; K \;-\; 1 \;\equiv\; \frac{K}{2} \quad (mod\ K) \qquad\qquad (9)$$

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* The notation $ord_p(q)$ has been used everywhere in this version of the report to denote the minimum exponent e such that $q^e \equiv 1\ (mod\ p)$.

Thus $(2^{t_K}+1)\ 2^{\{log_2 K\}\ -\ log_2 K}\ \cdot\ K\ -\ 1$ is required to take one of the following set of values: $(2^{t_K}+1)K+\frac{K}{2}, (2^{t_K}+1)K+\frac{3K}{2}, ..., 2(2^{t_K}+1)K-\frac{K}{2}$.

Consider the first value. It follows that

$$\{log_2\ K\}\ =\ log_2\left(K\ +\ \frac{K+2}{2(2^{t_K}+1)}\right) \tag{10}$$

and

$$K\ +\ \frac{K+2}{2(2^{t_K}+1)}\ =\ 2^{t_K}\ \cdot\ K_0\ +\ \frac{2^{t_K-1}\cdot K_0+1}{2^{t_K}+1}\ =\ 2^m \tag{11}$$

which requires that

$$K_0\ =\ 2N\ +\ \frac{N-1}{2^{t_K-1}} \tag{12}$$

for some integer N. This is an integer if $N-1\ =\ \alpha\ 2^{t_K-1}$ with α integer, which implies that

$$2^{t_K}\cdot K_0\ +\ \alpha\ 2^{t_K-1}\ +\ 1\ =\ 2^m \tag{13}$$

which could only have solutions when $t_K=1$. From equations (12) and (13), the relation

$$2K_0\ +\ N\ =\ 5N\ +\ 2\alpha\ =\ 7\alpha\ +\ 5\ =\ 2^m \tag{14}$$

However, the congruence relation $2^m\ \equiv\ 5\ \ (mod\ 7)$ has no solutions, so that there is no value of K satisfying the condition (10).

More generally, the congruence relation is

$$\begin{aligned}(2r+1)\ \cdot\ 2^m\ &\equiv\ 2r+5 \qquad (mod\ 2r+7) \qquad r\ \in\ \mathbb{Z}\\ 6\ \cdot\ 2^m\ &\equiv\ 2 \qquad\quad\ \ (mod\ 2r+7)\end{aligned} \tag{15}$$

which does not have a solution if $2r+7$ is an odd multiple of 3. If $t_K=1$, $r\ =\ 0,\ 1,\ 2$ and $5\cdot 2^m\ \equiv\ 9\ \ (mod\ 11)$ is the only congruence relation with a solution $m\ =\ 2\ +\ 10s$, which implies that $\alpha\ =\ \frac{20\cdot 2^{10s}\ -\ 9}{11}$ and $K\ =\ \frac{2}{11}(12\cdot 2^{10s}\ -\ 1)$. Since K is even, it cannot divide $2^i\ -\ 1$ for any i, so that there are no shared sites in a partition into K triangles. The Mersenne number 2^n-1 is composite when the index n equals $1+\{log_2(K+1)\}+h\ ord_K(2)$ if h congruence cycles are completed between levels m_K and $n-1$. Otherwise the Mersenne number is prime.

When $n-1-m_K = h\, ord_K(2) + r_2(n-1,K)$, $1 \leq r_2(n-1,K) \leq ord_K(2) - 1$, the total number of extra sites is

$$\begin{aligned} \sum_{i=0}^{m_K-1} 2^i + \sum_{t=0}^{hord_K(2)+r_2(n-1,K)} & 2^t(K_{m_K}+1) \\ &= 2^{m_K} - 1 + (2^{hord_K(2)+r_2(n-1,K)+1} - 1)(K_{m_K}+1) \\ &\equiv 2^{r_2(n-1,K)+1}\,(K_{m_K}+1) - 1 \quad (mod\ K) \end{aligned} \tag{16}$$

It is now possible for the number of extra sites to be congruent to an even number $2w = 2, 4, 6, ...$ if $(2^{r_2(n-1,K)+1} - 1)K_{m_K} \equiv 2w+1 \ \ (mod\ K)$, where, for the first non-trivial case, $2w = 4$, the expression equivalent to the number of extra sites, $(2^{r_2(n-1,K)+1} - 1)\ K_{m_K} - 1$ would have a minimum value of $K + 4$.

It may also be noted briefly that since any prime is either $6n-1$ or $6n+1$, the Mersenne prime must be of the latter type, and it also should have the form $8k+7$. Combining these two conditions, it follows that Mersenne primes must have the form $24k+7$.

While these congruence relations do not represent particularly strong constraints, it is of interest to note that $6n+1$ is prime if n does not have the forms $6xy+(x+y)$ or $6xy-(x+y)$, with x and y integer [9][14]. The existence of an infinite number of solutions to similar congruence relations such as $2^{n-2} \equiv 1\ (mod\ n)$ [15] and $x^n \equiv a\ (mod\ m)$ where $gcd(\phi(m), n) = 1$ [16] has been investigated also.

3. Spin Structures on Riemann Surfaces and the Even Perfect Numbers

Further elaborating on the geometrical representation of perfect numbers, it may be noted that there is a numerical equivalence between the even perfect number $2^{g-1}(2^g-1)$ and the number of odd spin structures on a genus-g Riemann surface. Spin structures may be defined as sections of the square root of the cotangent bundle on the surface. The choice of sign of the square root for each handle of the surface, reflected in a choice of boundary condition for spin-$\frac{1}{2}$ for each A-cycle and B-cycle of the surface gives rise to two choices of sign for each cycle and thus 2^{2g} spin structures. A distinction can be made between even and odd spin structures, in the sense of the overall parity of theta characteristic or the number of Dirac zero modes on the surface mod 2.

At genus g, there $2^{g-1}(2^g+1)$ even spin structures and $2^{g-1}(2^g+1)$ structures. Each set of spin structures transforms into itself under the action of the mapping class group, the discontinuous group, which, after factorization, reduces Teichmuller space to moduli space. At genus one, there is one odd spin structure associated with the choice of signs $(++)$ for the A-cycle and B-cycle and there are three even spin structres $(+-)$, $(-+)$, $(--)$. The number of Dirac zero modes is additive when surfaces of genus g_1 and g_2 are combined. Because of the underlying singlet-triplet structure for each genus-one component, the total number of odd spin structures can be counted.

$$1 + \binom{g}{2} 3^2 + \binom{g}{4} 3^4 + ... \binom{g}{g-1} 3^{g-1} = \frac{(1+3)^g + (1-3)^g}{2} = 2^{g-1}(2^g - 1) \quad (17)$$

when g is odd.

The set of odd spin structures therefore produces an infinite sequence of numbers which contains the 38 even perfect numbers. It may be conjectured that an analysis of the action of the modular group on the set of odd spin structures may determine whether the sequence of even perfect numbers continues indefinitely. In particular, properties of spin-structure sectors, sufficiently large to generate the entire set of odd spin structures, and the corresponding subgroups of finite subgroups of finite index in the modular group, when the genus index equals the index of a Mersenne prime, can be examined.

4. Odd Perfect Numbers

The other conjecture regarding perfect numbers is the non-existence of any odd numbers N satisfying $\frac{\sigma(N)}{N} = 2$, where $\sigma(N)$ is the sum of integer divisors of N. A well-known result [17] concerning odd perfect numbers is they must take the form

$$N = (4k + 1)^{4m+1} s^2 \quad (18)$$

where $4k+1$ is a prime number and $gcd(4k+1, s) = 1$. Given the prime decomposition $s = q_1^{\alpha_1} ... q_l^{\alpha_l}$, the formula

$$\sigma(q_i^{\alpha_i}) = 1 + q_i + q_i^2 + q_i^3 + ... q_i^{\alpha_l} = \frac{q_i^{\alpha_i+1} - 1}{q_i - 1} \quad (19)$$

and the multiplicative property of the sum-of-divisors function, $\sigma(MN) = \sigma(M) \cdot \sigma(N)$, it follows that

$$\sigma(s^2) = \sigma(q_1^{2\alpha_1}) ... \sigma(q_l^{2\alpha_l}) = \prod_{i=1}^{l} \frac{q_i^{2\alpha_i+1} - 1}{q_i - 1} \quad (20)$$

Similarly,

$$\sigma((4k+1)^{4m+1}) = \frac{(4k + 1)^{4m+2} - 1}{(4k + 1) - 1} \tag{21}$$

so that the condition for N to be a perfect number is

$$\frac{\sigma(N)}{N} = \left[\frac{(4k+1)^{4m+2}-1}{4k(4k+1)^{4m+1}}\right] \frac{\sigma(s^2)}{s^2} = \left[\frac{(4k+1)^{4m+2} - 1}{4k(4k+1)^{4m+1}}\right] \left[\frac{\sigma(s^2)}{\sigma(s)^2}\right] \left[\frac{\sigma(s)}{s}\right]^2 \tag{22}$$
$$= 2$$

or equivalently

$$\prod_{i=1}^{l} \frac{1}{(q_i^{\alpha_i+1} - 1)} \frac{\sigma(s)}{s} = \prod_{i=1}^{l} \frac{1}{(q_i-1)q_i^{\alpha_i}} = \sqrt{2} \prod_{i=1}^{l} \frac{1}{(q_i^{2\alpha_i+1}-1)^{\frac{1}{2}}(q_i-1)^{\frac{1}{2}}} \times \left[\frac{4k(4k+1)^{4m+1}}{(4k+1)^{4m+2}-1}\right]^{\frac{1}{2}} \tag{23}$$

Consistency of this equation requires rationality of the right-hand side.

There are only a few repunits $\frac{x^n - 1}{x - 1}$ which are perfect squares and they are given in the following theorem [18][19]

Theorem (Nagell 1921; Ljunggren 1943) - The integer solutions to the equation

$$\frac{x^n - 1}{x - 1} = y^2 \tag{24}$$

are

$$\begin{aligned} &n = 2,\ x = y^2 - 1,\ y \in \mathbb{Z} \\ &\qquad\qquad if\ x\ is\ prime,\ then\ x = 3,\ y = \pm 2 \\ &n = 3,\ x = 0,\ y \pm 1;\ x = -1,\ y = \pm 1 \\ &n = 4,\ x = 7,\ y = 20 \\ &n = 5,\ x = 3,\ y = 11 \end{aligned} \tag{25}$$

It can be deduced from general arguments, however that even if the repunits $\frac{x^n - 1}{x_1 - 1}$ and $\frac{x_2^n - 1}{x_2 - 1}$ are not perfect squares, their products are perfect squares; this may be verified when

$n = 3$ using integer solutions to the quadratic Diophantine equation $z^2 - D\,r^2 = -3$ [20][21]. Solutions to the equations

$$\frac{x^n - 1}{x - 1} = y^m \tag{26}$$

are known not to exist for $x = 10$, $n \geq 2$, and they have been obtained for $x = z^t$, $2 \leq z \leq 10000$, $t \geq 1$ [22]. For equations of the type

$$a\,\frac{x^n - 1}{x - 1} = y^m \tag{27}$$

integer solutions have been listed when $1 < a < x \leq 10$, $n > 2$, $m \geq 2$ [23]. Characteristics of the divisors of the repunits $\frac{x^n - 1}{x - 1}$ [24]-[29] are required to determine whether the square root expression in equation (23) can be rational. If the expression is always an irrational number, there will be no solutions for q_i, α_i, k, m in equation (23), implying the non-existence of finite odd perfect numbers.

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