Sharp estimates of the rate of convergence for U-statistics

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The rate of convergence for non-degenerate $U-$statistics of n i.i.d. random variables is investigated in the conditions of the central limit theorem. The basic result sharpens and improves earlier results of many papers devoted to this problem for such random sums. The Berry-Esséen bounds of order $O(n^{-1/2})$ hold for non-degenerate $U-$statistics of degree m with canonical Hoeffding representation

$$U_n - \theta = \frac{m}{n} \sum_{i_1=1}^{n} g_1(X_{i_1}) + \frac{m(m-1)}{n(n-1)} \sum_{1 \le i_1 < i_2 \le n} g_2(X_{i_1}, X_{i_2}) + etc.$$

under following minimal sufficient conditions on canonical functions g_c, $c = 1, \ldots, m$: for $c = 1$ there exists absolute moment of order 3 of random variable g_1, i.e. $E|g_1|^3 < \infty$ and for $2 \le c \le m$ there exist weak moments of order δ_c of random variables $g_2, \ldots, g_m$, i.e. $\sup_{t \ge 0}(t^{\delta_c} P(|g_c| > t)) < \infty$ where $\delta_c = (2c+1)/(2c-1)$.

1. Introduction.

Let $X_1, \ldots, X_n$ be independent random variables taking values in a measurable space $(\mathfrak{X}, \mathfrak{B})$, and all with an identical distribution P. Consider $U-$statistics (see Hoeffding (1948))

$$U_n = \binom{n}{m}^{-1} \sum_{1 \le i_1 < \cdots < i_m \le n} \Phi(X_{i_1}, \ldots, X_{i_m}) \tag{1}$$

with a symmetric kernel $\Phi : \mathfrak{X}^m \mapsto R$ such that $E|\Phi(X_1, \ldots, X_m)| < \infty$. Define for $c = 0, 1, \ldots, m$ the conditional expectations

$$\Phi_c(x_1, \ldots, x_c) = E\, \Phi(x_1, \ldots, x_c, X_{c+1}, \ldots, X_m)$$

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setting $\Phi_0 = E\,\Phi(X_1,\dots,X_m)$ and $\Phi_m = \Phi$, and then define for $c = 1,\dots,m$ the functions $g_c : \mathfrak{X}^c \mapsto R$ by the formula

$$g_c(x_1,\dots,x_c) = \sum_{d=1}^{c}(-1)^{c-d} \sum_{1\le j_1<\dots<j_d\le c} (\Phi_d(x_{j_1},\dots,x_{j_d}) - \theta),$$

where $\theta = E\,\Phi(X_1,\dots,X_m)$. The functions g_c possess the property of degeneracy $E\,g_c(x_1,\dots,x_{c-1},X_c) = 0$, $c = 1,\dots,m$, and are called canonical. According to the canonical Hoeffding representation

$$U_n - \theta = \frac{m}{n}\sum_{i_1=1}^{n} g_1(X_{i_1}) + \frac{m(m-1)}{n(n-1)} \sum_{1\le i_1<i_2\le n} g_2(X_{i_1},X_{i_2})$$

$$+\dots+ \frac{m(m-1)\cdots 2\cdot 1}{n(n-1)\cdots(n-m+1)} \sum_{1\le i_1<\dots<i_m\le n} g_m(X_{i_1},\dots,X_{i_m}), \tag{2}$$

i.e. any U-statistic is a linear combination of pairwise uncorrelated U-statistical sums of the form (1) with degenerate kernels g_c, $c = 1,\dots,m$. Hoeffding (1948, 1961) developed a comprehensive theory of U-statistics. In particular, using the decomposition (2) he proved the asymptotic normality of normalized U-statistics. In the equation (2) the first sum $m\,n^{-1}\big(g_1(X_1) + \dots + g_1(X_n)\big)$ is known as the Hoeffding projection of the U-statistic, and is clearly the usual sum of independent identically distributed random variables.

In what follows we shall consider U-statistics with a symmetric non-degenerate kernel $\Phi : \mathfrak{X}^m \to R$. It is assumed that $0 < Eg_1^2(X_1) < \infty$. Without loss of generality $Eg_1^2(X_1) = 1$ can be taken. Denote

$$F_n(x) := P(\sqrt{n}\,(U_n - \theta)/m \le x), \quad F(x) := P(\tau \le x), \quad x \in R,$$

where τ is a standard normal random variable. By the central limit theorem for non-degenerate U-statistics, as $n \to \infty$ (see Arcones (1994), Borovskikh (1989, 1996))

$$\Delta_n := \sup_x |F_n(x) - F(x)| \to 0,$$

if

$$t^{\gamma_c} P(|g_c| > t) \to 0 \quad \text{as} \quad t \to \infty$$

for each $c = 2, \dots, m$, where $\gamma_c = 2c/(2c-1)$. Note that

$$1 + \frac{1}{2m-1} = \gamma_m < \gamma_{m-1} < \dots < \gamma_3 < \gamma_2 = \frac{4}{3}\,.$$

Here and below, it is assumed that $g_c = g_c(X_1,\dots,X_c)$ for all $c = 1, 2,\dots, m$.

The problem of estimation of convergence rate in this central limit theorem was examined by many authors (see Borovskikh (1996), Lee (1990)). In every succeeding paper, conditions imposed on the kernel Φ were weakened or the order of decreasing the variables Δ_n to zero with $n \to \infty$ was refined for different values of the parameter m. The first estimates of order $O(n^{-l/(2(l+1))})$ were obtained by Malevich and Abdalimov (1970) and Grams and Serfling (1973) under the assumption $E|\Phi|^l < \infty$

as $l \geq 3$, for any fixed $m \geq 2$. Then, the case of $m = 2$ was under active investigation. For example, Bickel (1974) obtained an estimate of the convergence rate of the order $O(n^{-1/2})$ under the assumption that the kernel Φ is bounded. Chan and Wierman (1977) established that the estimate $O(n^{-1/2})$ holds true if Φ has a finite fourth moment; they also showed that if Φ only has finite third absolute moment, then the estimate $O(n^{-1/2}(\ln n)^{1/3})$ holds. The estimate of the order $O(n^{-1/2})$ under the assumption $E|\Phi|^3 < \infty$ was obtained by Callaert and Janssen (1978); here, the authors follow the method of characteristic functions and develop the key idea of Chan and Wierman (1977) about a partition of the U-statistical sum, and moreover, employ martingale properties of U-statistics. Borovskikh (1979) weakened the condition $E|\Phi|^3 < \infty$ and proved the estimate $O(n^{-1/2})$ under the conditions $E|g_1|^3 < \infty$ and $E\Phi^2 < \infty$, i.e., an absolute moment of the third order is required only for g_1. Helmers and van Zwet (1982) obtained the estimate $O(n^{-1/2})$ under the condition $E|\Phi|^{5/3+\varepsilon} < \infty$ for $\varepsilon > 0$. Borovskikh and Korolyuk (1984, 1985) weakened this condition to $E|\Phi|^{5/3} < \infty$. This result can also be found in Friedrich (1989), Bolthausen and Götze (1989), Lee (1990). Lower bounds of the order $O(n^{-1/2})$ for the convergence rate under conditions $E|g_1|^3 < \infty$ and $E\Phi^2 < \infty$ (and for moments of higher orders) were obtained by Maesono (1991) (see also Maesono and Yamato (1994)). Bentkus, Götze and Zitikis (1994) constructed lower bounds in the case $m = 2$ under the conditions $E|g_1|^3 < \infty$ and $E|\Phi|^{5/3-\varepsilon} < \infty$ for $\varepsilon > 0$, and gave an example of kernel Φ and random variables $X_j = X_j^{(n)}$, $j = 1, \ldots, n$, such that the estimate $\Delta_n \geq c(\varepsilon)\, n^{-1/2+3\varepsilon/2}$ holds with a constant $c(\varepsilon) > 0$. Indeed, this estimate does not mean that the condition $E|\Phi|^{5/3} < \infty$ is necessary and minimal for the estimate of order $O(n^{-1/2})$ to hold. Bentkus and Götze (1995) generalized this result to the case of arbitrary $m \geq 2$ with analogous lower bounds for Δ_n.

2. Results.

We shall consider the problem of estimating of the convergence rate both under general conditions $Eg_1^2(X_1) = 1$ and $E|\Phi(X_1, \ldots, X_m)| < \infty$, and under minimal conditions imposed on canonical functions g_c, $c = 1, \ldots, m$, which are sufficient for the estimate $O(n^{-1/2})$ with any natural $m \geq 1$. In what follows, we shall need the following **key functions**.

1. Truncated canonical functions

$$g_c^n\colon (x_1, \ldots, x_c) \to g_c^n(x_1, \ldots, x_c),\ c = 1, \ldots, m,$$

where

$$g_c^n(x_1, \ldots, x_c) = g_c(x_1, \ldots, x_c)\, I(|g_c(x_1, \ldots, x_c)| \leq n^{(2c-1)/2}).$$

Here $n \geq 2$ and $I(A)$ is the indicator function of a set $A \in \mathfrak{B}$.

2. Truncated moment functions

$$\begin{aligned}
\beta_n &= \sum_{c=2}^{m} \binom{m}{c} \{\sqrt{n}\, E|g_c|\, I(|g_c| > n^{(2c-1)/2})\},\\
\gamma_n(p) &= \sum_{c=2}^{m} \binom{m}{c} \{n^{(2c-p(2c-1))/2}\, E|g_c|^p\, I(|g_c| \leq n^{(2c-1)/2})\},
\end{aligned}$$

where $1 \le p \le 2$.

3. Weak moment functions

$$B_c(x,p) = \sup_{t \ge x} (t^p P(|g_c| > t)), \ c = 1, \dots, m,$$

where $x \ge 0$, and $p > 0$.

4. Weak moments of order p

$$B_c(0,p) = \sup_{t \ge 0} (t^p P(|g_c| > t)), \ c = 1, \dots, m.$$

Theorem 1. *Let the following conditions hold*

$$Eg_1^2(X_1) = 1, \quad E|\Phi(X_1, \dots, X_m)| < \infty.$$

Then for all $m \ge 1$, $n \ge m$ *and any* p, $1 \le p < 2$,

$$\begin{aligned}\Delta_n \le\ & 132.8\, E|g_1|^3\, I(|g_1| \le \sqrt{n}) \cdot \frac{1}{\sqrt{n}} + 2.85\, Eg_1^2\, I(|g_1| > \sqrt{n}) \\ & + \sqrt{6\pi}\, e^{m/3}\, 6^{m/2} m! \sum_{c=2}^{m} \binom{m}{c} E|g_c^n| \prod_{s=1}^{c} |g_1^n(X_s)|^{2/c} \cdot \frac{1}{\sqrt{n}} \\ & + 2^{2m^2+4m} 6^m m^{2m} m!\, 3^{6/(2-p)} \Gamma(6/(2-p)) \cdot \{\beta_n + \gamma_n(p)\}, \qquad (3)\end{aligned}$$

where $\Gamma(z)$ *is the gamma function.*

Estimate (3) holds true under the conditions of the central limit theorem, too, i.e., (3) implies the following statement.

Corollary 1. *If* $Eg_1^2(X_1) = 1$, *and* B*-conditions*

$$B_c(x, \gamma_c) \to 0, \quad c = 2, \dots, m,$$

hold as $x \to \infty$, *then*

$$\sup_x |F_n(x) - F(x)| \to 0 \quad \text{as} \quad n \to \infty.$$

It appears that the weak moment B-functions not only express sufficient conditions for the central limit theorem for U-statistics to hold, but moreover, they control the rate of this convergence. Actually, the following theorem holds involving weak moments $B_c(0, \delta_c)$ of order δ_c, where

$$\delta_c = (2c+1)/(2c-1), \quad c = 1, \dots, m.$$

Note that

$$1 + \frac{2}{2m-1} = \delta_m < \delta_{m-1} < \cdots < \delta_3 < \delta_2 = \frac{5}{3}\ .$$

Theorem 2. *Let the moment conditions*

$$Eg_1^2(X_1) = 1, \quad E|g_1(X_1)|^3 < \infty,$$

and B-conditions

$$B_c(0, \delta_c) < \infty, \quad c = 2, \dots, m,$$

be satisfied. Then for all $m \ge 1$ and $n \ge m$, the inequality

$$\Delta_n \le A_m \Big\{ \sum_{c=1}^{m} \binom{m}{c} \Big(E|g_c| \prod_{s=1}^{c} |g_1(X_s)|^{2/c} + B_c(0, \delta_c) \Big) \Big\} \cdot \frac{1}{\sqrt{n}} \tag{4}$$

holds with some positive constant A_m which depends only on m, in addition,

$$E|g_c| \prod_{s=1}^{c} |g_1(X_s)|^{2/c} \le (E|g_1|^3)^c + 3cB_c(0, \delta_c).$$

Particularly, in the case of $m = 2$, for all $n \ge 2$

$$\Delta_n \le \{215.6\, E|g_1|^3 + 46.2\, (E|g_1|^3 E|g_2|^{3/2})^{2/3} + 412940\, B_2(0, 5/3)\} \cdot \frac{1}{\sqrt{n}},$$

where

$$E|g_2|^{3/2} \le \min\{1 + 9B_2(0, 5/3), 10\, B_2^{9/10}(0, 5/3)\}.$$

Corollary 2. *Let $Eg_1^2(X_1) = 1$ and the moment conditions*

$$E\,|g_c|^{\delta_c} < \infty,\ c = 1, \dots, m$$

be satisfied. Then for all $m \ge 1$ and $n \ge m$

$$\Delta_n \le A_m \Big\{ \sum_{c=1}^{m} \binom{m}{c} E\,|g_c|^{\delta_c} \Big\} \cdot \frac{1}{\sqrt{n}}$$

with some positive constant A_m which depends only on m. In particularly, if $m = 2$ then for all $n \ge 2$

$$\Delta_n \le \{233.88\, E|g_1|^3 + 412967.72\, E|g_2|^{5/3}\} \cdot \frac{1}{\sqrt{n}}.$$

Theorem 3. *Let the moment conditions*

$$Eg_1^2(X_1) = 1, \quad E|g_1(X_1)|^3 < \infty,$$

and B-conditions

$$B_c(0, \delta_c) < \infty, \quad c = 2, \dots, m,$$

hold. Then for all $m \ge 1$ and $n \ge m$

$$\Delta_n \le A_m \Big\{ \sum_{c=1}^{m} \binom{m}{c} \Big(E|g_c| \prod_{s=1}^{c} |g_1(X_s)|^{2/c} + B_c(n^{1/8}, \delta_c) \Big) \Big\} \cdot \frac{1}{\sqrt{n}} \tag{5}$$

with some positive constant A_m which depends only on m.

Note that $B_c(n^{1/8}, \delta_c) \le B_c(0, \delta_c)$, $c = 1, \dots, m$.

Theorem 4. *Let the moment conditions*

$$Eg_1^2(X_1) = 1, \quad E|g_1(X_1)|^3 < \infty,$$

and B-conditions

$$B_c(0, \delta_c) < \infty, \quad c = 2, \dots, m,$$

hold. Then for all $m \ge 1$ and $n \ge m$ the following inequality holds

$$\Delta_n \le A_m \left\{E|g_1|^3 + 3(m-1)\, E|g_2(X_1, X_2)\, g_1(X_1)\, g_1(X_2)| + \rho_n\right\} \cdot \frac{1}{\sqrt{n}}, \qquad (6)$$

where

$$\rho_n = \sum_{c=2}^{m} B_c(n^{1/8}, \delta_c) + \sum_{c=3}^{m} B_c^{2/3}(0, \delta_c) \Big(E|g_1|^3 + \sum_{d=2}^{m} B_d(n^{1/8}, \delta_d)\Big)^{c/3} \cdot n^{-(c-1)/6},$$

and $A_m > 0$ is a constant which depends only on m.

Here $\rho_n \to 0$ as $n \to \infty$, provided that $B_c(x, \delta_c) \to 0$, $c = 2, \dots, m$, as $x \to \infty$.

On asymptotic expansion. If the Cramer condition for the random variable $g_1(X_1)$ is assumed in addition to the conditions of Theorem 4, then the estimate (6) can be improved. Denote

$$\begin{aligned} \kappa_3 &= Eg_1^3(X_1) + 3(m-1)\, E(g_2(X_1, X_2)\, g_1(X_1)\, g_1(X_2)), \\ \chi(t) &= E\exp(it\, g_1(X_1)), \quad t \in R. \end{aligned}$$

Theorem 5. *Let the moment conditions*

$$Eg_1^2(X_1) = 1, \quad E|g_1(X_1)|^3 < \infty,$$

the Cramer condition

$$\lim_{|t| \to \infty} \sup |\chi(t)| < 1,$$

and B-conditions

$$B_c(x, \delta_c) \to 0, \quad c = 2, \dots, m,$$

as $x \to \infty$ hold. Then the following relation holds as $n \to \infty$

$$\sup_x \left| F_n(x) - F(x) - \frac{\kappa_3}{6} \cdot \frac{1}{\sqrt{2\pi n}} (1 - x^2)\, e^{-x^2/2} \right| = o(n^{-1/2}). \qquad (7)$$

Theorem 5 can be found in Maesono and Yamato (1994) with condition $E|\Phi|^3 < \infty$ instead of B-conditions $B_c(x, \delta_c) \to 0$, $c = 2, \dots, m$ as $x \to \infty$.

Comparison of conditions. Concerning B-conditions for which (4) holds, note the following inequalities which are always true:

$$B_c(0,\delta_c) \le E|g_c|^{\delta_c}, \quad c = 1,\dots,m.$$

The inverse inequalities are not true in general. We can only assert that for any positive p, $p < \delta_c$,

$$E|g_c|^p \le \delta_c(\delta_c - p)^{-1} B_c^{p/\delta_c}(0,\delta_c).$$

Example 1. *Assume that the random variable g_c has the probability density*

$$p(x) = \delta_c(2(1+\delta_c))^{-1} \min\{1, |x|^{-\delta_c - 1}\}, \quad x \in R.$$

and, hence, $x^{\delta_c} P(|g_c| > x) = (1+\delta_c)^{-1}, \quad x \ge 1.$

Then for all $c = 2,\dots,m$

$$B_c(0,\delta_c) = (1+\delta_c)^{-1} \quad \text{and} \quad E|g_c|^{\delta_c} = \infty.$$

Example 2. *Suppose that g_c is δ_c-stable random variable such that for any $t \in R$*

$$E\exp(itg_c) = \exp(-|t|^{\delta_c}), \quad c = 2,\dots m$$

It is well-known that in this situation for all $x \ge 2$

$$\alpha_1 \le x^{\delta_c} P(|g_c| > x) \le \alpha_2$$

and

$$\alpha_3 \ln x \le E|g_c|^{\delta_c} I(1 \le |g_c| \le x) \le \alpha_4 \ln x$$

with some positive constants α_j.

That is

$$\alpha_1 \le B_c(0,\delta_c) \le \alpha_2, \quad \text{but} \quad E|g_c|^{\delta_c} = \infty$$

for all $c = 2,\dots,m$.

Therefore, the B-conditions $B_c(0,\delta_c) < \infty$, $c = 2,\dots,m$, of Theorem 2 are weaker then the moment conditions $E|g_c|^{\delta_c} < \infty$, $c = 2,\dots, m$, of Corollary 2.

3. Proofs.

Theorems 2, 3 and 4 are essentially consequences of Theorem 1, since we derive estimates (4)–(6) from (3). The proof of Theorem 5 uses also the estimates of characteristic functions, obtained in the proof of Theorem 1. In turn, the proof of Theorem 1 is based on Hoeffding decomposition, on truncated inequalities and method of characteristic functions. Analysing the characteristic function of a normalized U-statistic with truncated canonical functions g_c^n, $c = 1,\dots, m$, we develop the idea of Chan and Wierman (1977) on a partition of U-statistical sum and apply martingale structure of U-statistics (see Borovskikh and Korolyuk (1997)).

Proof of Theorem 1. The following Lemma contains general truncated inequality, which enables one to introduce truncated random variables $g_c^n(X_1,\dots,X_c)$ instead of $g_c(X_1,\dots,X_c)$, $c = 1,\dots,m$. For this purpose, first denote

$$S_{nm}(\Phi) = \sum_{1 \le i_1 < \cdots < i_m \le n} \Phi(X_{i_1},\dots,X_{i_m}) + f(X_1,\dots,X_n),$$

where $\Phi\colon \mathfrak{X}^m \to R$ is a symmetric function of m variables, $f\colon \mathfrak{X}^n \to R$ is an arbitrary measurable function of n variables. Let Φ^t denote the truncated at the level $t > 0$ function Φ i.e.,

$$\Phi^t(x_1,\dots,x_m) = \Phi(x_1,\dots,x_m)\, I(|\Phi(x_1,\dots,x_m)| \le t)$$

with the indicator function $I(A)$ of a set $A \in \mathfrak{B}$.

Lemma 1. *For all $x \in R$ and any $t > 0$*

$$|P(S_{nm}(\Phi) \le x) - P(S_{nm}(\Phi^t) \le x)|$$

$$\le \frac{n(n-1)\cdots(n-m+1)}{m!}\, P(|\Phi(X_1,\dots,X_m)| > t). \tag{8}$$

Proof. First, consider the case $m = 2$. For each $k = 1,\dots,n$, denote

$$\alpha_k(\Phi) = \sum_{i=k+1}^{n} \Phi(X_k, X_i),$$

$$\zeta_k = \sum_{j=1}^{k-1}\sum_{i=j+1}^{n} \Phi^t(X_i,X_j) + \sum_{j=k+1}^{n-1}\sum_{i=j+1}^{n} \Phi(X_i,X_j) + f(X_1,\dots,X_n),$$

and put

$$\theta_l = \sum_{i<l} \Phi(X_k,X_i) + \sum_{i>l} \Phi^t(X_k,X_i)$$

for $l = k+1,\dots,\, n$. The following formulas hold

$$\begin{aligned}
P(S_{n2}(\Phi) &\le x) - P(S_{n2}(\Phi^t) \le x)\\
&= E\{I(S_{n2}(\Phi) \le x) - I(S_{n2}(\Phi^t) \le x)\}\\
&= E\{I(\alpha_1(\Phi) + \zeta_1 \le x) - I(\alpha_1(\Phi^t) + \zeta_1 \le x)\}\\
&+ E\{I(\alpha_2(\Phi) + \zeta_2 \le x) - I(\alpha_2(\Phi^t) + \zeta_2 \le x)\}\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&+ E\{I(\alpha_{n-1}(\Phi) + \zeta_{n-1} \le x) - I(\alpha_{n-1}(\Phi^t) + \zeta_{n-1} \le x)\}\\
&= \sum_{k=1}^{n-1} E\{I(\alpha_k(\Phi) + \zeta_k \le x) - I(\alpha_k(\Phi^t) + \zeta_k \le x)\}.
\end{aligned}$$

By similar arguments, we get the formula

$$E(I(\alpha_k(\Phi) + \zeta_k \le x) - I(\alpha_k(\Phi^t) + \zeta_k \le x))$$

$$= \sum_{l=k+1}^{n} E(I(\Phi(X_k,X_l) + \theta_l + \zeta_k \le x) - I(\Phi^t(X_k,X_l) + \theta_l + \zeta_k \le x)).$$

Therefore,

$$P(S_{2n}(\Phi) \le x) - P(S_{2n}(\Phi^t) \le x)$$

$$= \sum_{k=1}^{n-1} \sum_{l=k+1}^{n} E\{I(\Phi(X_k, X_l) + \theta_l + \zeta_k \le x) - I(\Phi^t(X_k, X_l) + \theta_l + \zeta_k \le x)\}.$$

Here

$$\begin{aligned} EI(\Phi^t(X_k, X_l) \;&+\; \theta_l + \zeta_k \le x) \\ &= EI(\Phi^t(X_k, X_l) + \theta_l + \zeta_k \le x)\, I(|\Phi(X_k, X_l)| \le t) \\ &\quad + EI(\Phi^t(X_k, X_l) + \theta_l + \zeta_k \le x)\, I(|\Phi(X_k, X_l)| > t) \\ &= EI(\Phi(X_k, X_l) + \theta_l + \zeta_k \le x) \\ &\quad - EI(\Phi(X_k, X_l) + \theta_l + \zeta_k \le x)\, I(|\Phi(X_k, X_l)| > t) \\ &\quad + EI(\theta_l + \zeta_k \le x)\, I(|\Phi(X_k, X_l)| > t). \end{aligned}$$

Since

$$|I(\Phi(X_k, X_l) + \theta_k + \zeta_k \le x) - I(\theta_l + \zeta_k \le x)| \le 1,$$

we have as a result the following inequality

$$|E(I(\Phi(X_k, X_l) + \theta_l + \zeta_k \le x) - I(\Phi^t(X_k, X_l) + \theta_l + \zeta_k \le x))|$$

$$\le P(|\Phi(X_1, X_2)| > t),$$

which is valid for any $t > 0$.

Consequently, for all $x \in R$ and $t > 0$

$$|P(S_{n2}(\Phi) \le x) - P(S_{n2}(\Phi^t) \le x)| \le \frac{n(n-1)}{2} P(|\Phi(X_1, X_2)| > t),$$

which proves (8) for $m = 2$.

Now let us consider the general case $m \ge 2$. According to the definition

$$S_{nm}(\Phi) = \sum_{i_1=1}^{n-m+1} \sum_{i_2=i_1+1}^{n-m+2} \cdots \sum_{i_m=i_{m-1}+1}^{n} \Phi(X_{i_1}, X_{i_2}, \ldots, X_{i_m}) + f(X_1, \ldots, X_n).$$

For all integer numbers r_1, r_2,..., r_c, such that $1 \le r_1 < r_2 < \ldots < r_c \le n-m+c$, $c = 1, \ldots, m$, denote

$$\alpha_{r_1 r_2 \ldots r_c}(\Phi) = \sum_{i_{c+1}=r_c+1}^{n-m+c+1} \cdots \sum_{i_m=i_{m-1}+1}^{n} \Phi(X_{r_1}, X_{r_2}, \ldots, X_{r_c}, X_{i_{c+1}} \ldots, X_{i_m}),$$

so that $\alpha_{r_1 r_2 \ldots r_m}(\Phi) = \Phi(X_{r_1}, X_{r_2}, \ldots, X_{r_m})$;

$$\zeta(r_1, r_2, \ldots, r_c)$$

$$= \sum_{i_c=r_{c-1}+1}^{r_c-1} \sum_{i_{c+1}=i_c+1}^{n-m+c+1} \cdots \sum_{i_m=i_{m-1}+1}^{n} \Phi^t(X_{r_1}, X_{r_2}, \ldots, X_{r_{c-1}}, X_{i_c}, \ldots, X_{i_m})$$

$$+ \sum_{i_c=r_c+1}^{n-m+c} \sum_{i_{c+1}=i_c+1}^{n-m+c+1} \cdots \sum_{i_m=i_{m-1}+1}^{n} \Phi(X_{r_1}, X_{r_2}, \ldots, X_{r_{c-1}}, X_{i_c}, \ldots, X_{i_m}),$$

and

$$\zeta_c = \sum_{s=1}^{c} \zeta(r_1, r_2, \ldots, r_s) + f(X_1, \ldots, X_n).$$

In these notations, for all $c = 1, \ldots, m$, the following formulas are valid

$$P(S_{nm}(\Phi) \le x) - P(S_{nm}(\Phi^t) \le x)$$

$$= \sum_{r_1=1}^{n-m+1} \sum_{r_2=r_1+1}^{n-m+2} \cdots \sum_{r_c=r_{c-1}+1}^{n-m+c} E\{I(\alpha_{r_1\ldots r_c}(\Phi) + \zeta_c \le x) - I(\alpha_{r_1\ldots r_c}(\Phi^t) + \zeta_c \le x)\}$$

$$= \sum_{1 \le r_1 < r_2 < \ldots < r_c \le n-m+c} E\{I(\alpha_{r_1\ldots r_c}(\Phi) + \zeta_c \le x) - I(\alpha_{r_1\ldots r_c}(\Phi^t) + \zeta_c \le x)\}.$$

Put $c = m$ here. Then

$$P(S_{nm}(\Phi) \le x) - P(S_{nm}(\Phi^t) \le x)$$

$$= \sum_{1 \le r_1 < \ldots < r_m \le n} E\{I(\alpha_{r_1\ldots r_m}(\Phi) + \zeta_m \le x) - I(\alpha_{r_1\ldots r_m}(\Phi^t) + \zeta_m \le x)\}.$$

Since

$$|E\{I(\alpha_{r_1\ldots r_m}(\Phi) + \zeta_m \le x) - I(\alpha_{r_1\ldots r_m}(\Phi^t) + \zeta_m \le x)\}| \le P(|\Phi(X_1, \ldots, X_m)| > t),$$

we get the proof of Lemma 1 for any $m \ge 2$.

In what follows, we shall apply the truncated inequality (8). Besides, we shall need the following representation of the kernel $\Phi(x_1, \ldots x_m)$ via canonical functions g_c:

$$\Phi(x_1, \ldots, x_m) = \theta + \sum_{c=1}^{m} \sum_{1 \le i_1 < \cdots < i_c \le m} g_c(x_{i_1}, \ldots, x_{i_c}),$$

just as the inverse representation of canonical functions g_c via the kernel Φ

$$g_c(x_1, \ldots, x_c) = \int_X \cdots \int_X \Phi(y_1, \ldots y_m) \prod_{s=1}^{c} (\delta_{x_s}(dy_s) - P(dy_s)) \prod_{s=c+1}^{m} P(dy_s), \qquad (9)$$

and the Hoeffding representation (2) in the form

$$U_n(\Phi) - \theta = \sum_{c=1}^{m} \binom{m}{c} U_n(g_c).$$

Then, we introduce the notations

$$\begin{aligned} g_{nc}(x_1, \ldots, x_c) &= \int_X \cdots \int_X g_c(y_1, \ldots y_c)\, I(|g_c| \le n^{(2c-1)/2}) \prod_{s=1}^{c} (\delta_{x_s}(dy_s) - P(dy_s)), \\ &\quad c = 1, 2, \ldots, m; \\ g_{ncd}(x_1, \ldots, x_d) &= -\int_X \int_X g_c(y_1, \ldots, y_c)\, I(|g_c| > n^{(2c-1)/2}) \\ &\quad \times \prod_{s=1}^{d} (\delta_{x_s}(dy_s) - P(dy_s)) \cdot \prod_{s=d+1}^{c} P(dy_s), \qquad c \ge d+1; \end{aligned}$$

$$
\begin{aligned}
h_c &= g_{nc1}, \qquad h_c^n = h_c\, I(|h_c| \le \sqrt{n}),\\
h_{nc} &= h_c^n - Eh_c^n, \quad c = 2, 3, \dots, m;\\
S_n &= \xi_1 + \cdots + \xi_n,\\
\xi_j &= \frac{1}{\sigma_n} \cdot \frac{1}{\sqrt{n}} \Big(g_{n1}(X_j) + \sum_{c=2}^{m} \binom{m}{c} \frac{c}{m}\, h_{nc}(X_j) \Big), \quad j = 1, \dots, n;\\
\sigma_n^2 &= E\Big(g_{n1}(X_1) + \sum_{c=2}^{m} \binom{m}{c} \frac{c}{m}\, h_{nc}(X_1) \Big)^2;\\
v_n &= \frac{\sqrt{n}}{\sigma_n m} \cdot \sum_{c=2}^{m} \binom{m}{c} U_n(g_{nc}),\\
w_n &= \frac{\sqrt{n}}{\sigma_n m} \sum_{d=2}^{m-1} \sum_{c=d+1}^{m} \binom{m}{c} \binom{c}{d} U_n(g_{ncd}),
\end{aligned}
$$

so that $w_n = 0$ as $m = 1,\, 2$.

Lemma 2. *The inequality holds*

$$
\begin{aligned}
\sup_x &|P(\sqrt{n}\,(U_n - \theta)/m \le x) - F(x)|\\
&\le \sup_x |P(S_n + v_n + w_n \le x) - F(x)|\\
&\quad + \Big(1 + \frac{1}{\sqrt{2\pi}}\Big) E|g_1|^2\, I(|g_1| > \sqrt{n})\\
&\quad + 3 \cdot \beta_n + \frac{1}{\sqrt{2\pi e}}\, |\sigma_n^2 - 1|/\sigma_n. \qquad (10)
\end{aligned}
$$

Proof. First apply Lemma 1 (inequality (8)) m times: at the first step we introduce random variables $g_1^n(X_{i_1})$; at the second step introduce random variables $g_2^n(X_{i_1}, X_{i_2})$, $i_1 < i_2$, etc., at the m-th step introduce random variables $g_m^n(X_{i_1}, \dots, X_{i_m})$, $i_1 < \cdots < i_m$; then we obtain the inequality

$$
\begin{aligned}
\sup_x &\Big| P(\sqrt{n}\,(U_n - \theta)/m \le x) - P\Big(\frac{\sqrt{n}}{m} \cdot \sum_{c=1}^{m} \binom{m}{c} U_n(g_c^n) \le x \Big) \Big|\\
&\le \sum_{c=1}^{m} \frac{1}{c!}\, n^{[c]}\, P(|g_c| > n^{(2c-1)/2}). \qquad (11)
\end{aligned}
$$

Write the Hoeffding decomposition for the U-statistic with kernel the g_c^n:

$$
U_n(g_c^n) = \theta_c^n + \sum_{d=1}^{c} \binom{c}{d} U_n(g_{ncd}),
$$

where

$$
\begin{aligned}
\theta_c^n &= Eg_c(X_1,\ldots,X_c)\, I(|g_c|\le n^{(2c-1)/2}),\\
g_{ncd}(x_1,\ldots,x_d) &= \int_X \cdots \int_X g_c(y_1,\ldots,y_c)\, I(|g_c|\le n^{(2c-1)/2})\\
&\qquad\qquad \times \prod_{s=1}^{d}(\delta_{x_s}(dy_s)-P(dy_s)) \prod_{s=d+1}^{c} P(dy_s).
\end{aligned}
$$

Here $g_{ncd}=g_{ncc}=g_{nc}$ for $c=d$. Denote

$$
\bar\theta_c^n := -Eg_c(X_1,\ldots,X_c)\, I(|g_c|> n^{(2c-1)/2}).
$$

Also by the degeneracy property of $g_c(x_1,\ldots,x_c)$, we see that $\theta_c^n=\bar\theta_c^n$.

Therefore

$$
\begin{aligned}
\sum_{c=1}^{m}\binom{m}{c} U_n(g_c^n) &= \sum_{c=1}^{m}\binom{m}{c}\bar\theta_c^n + \sum_{c=1}^{m}\binom{m}{c}\sum_{d=1}^{c}\binom{c}{d} U_n(g_{ncd})\\
&= \sum_{c=1}^{m}\binom{m}{c}\bar\theta_c^n + \sum_{c=1}^{m}\binom{m}{c}\binom{c}{1} U_n(g_{nc1})\\
&\quad +\sum_{c=2}^{m}\binom{m}{c} U_n(g_{nc}) + \sum_{d=2}^{m-1}\sum_{c=d+1}^{m}\binom{m}{c}\binom{c}{d} U_n(g_{ncd}).
\end{aligned}
$$

In this formula

$$
\sum_{c=1}^{m}\binom{m}{c}\binom{c}{1} U_n(g_{nc1}) = \frac{m}{n}\cdot\sum_{j=1}^{n} g_{n1}(X_j) + \sum_{c=2}^{m}\binom{m}{c} c\cdot\frac{1}{n}\sum_{j=1}^{n} h_c(X_j).
$$

Therefore, in (11)

$$
\begin{aligned}
\frac{\sqrt{n}}{m}\cdot\sum_{c=1}^{m}\binom{m}{c} U_n(g_c) &= \frac{\sqrt{n}}{m}\sum_{c=1}^{m}\binom{m}{c}\bar\theta_c^n + \frac{1}{\sqrt{n}}\sum_{j=1}^{n} g_{n1}(X_j)\\
&\quad +\sum_{c=2}^{m}\binom{m}{c}\frac{c}{m}\cdot\frac{1}{\sqrt{n}}\sum_{j=1}^{n} h_c(X_j) + (v_n+w_n)\cdot\sigma_n.
\end{aligned}
$$

Using this formula and Lemma 1, we introduce the truncated random variables $h_c^n(X_j)$, $j=1,\ldots,n$, i.e., we get the inequality

$$
\begin{aligned}
\sup_x\Big|P\Big(\frac{\sqrt{n}}{m}\cdot\sum_{c=1}^{m}\binom{m}{c} U_n(g_c^n)\le x\Big) - P\Big(\sum_{c=2}^{m}\binom{m}{c}\frac{c}{m}\cdot\frac{1}{\sqrt{n}}\sum_{j=1}^{n} h_c^n(X_j)+f_n\le x\Big)\Big|&\\
\le n\sum_{c=2}^{m} P(|h_c(X_1)|>\sqrt{n}),& \qquad (12)
\end{aligned}
$$

where

$$
f_n = \frac{\sqrt{n}}{m}\sum_{c=1}^{m}\binom{m}{c}\bar\theta_c^n + \frac{1}{\sqrt{n}}\sum_{j=1}^{n} g_{n1}(X_j)
$$

$$+\sum_{c=2}^{m}\binom{m}{c}\frac{c}{m}\cdot\frac{1}{\sqrt{n}}\sum_{j=1}^{n}h_c^n(X_j)+(v_n+w_n)\cdot\sigma_n.$$

In (12) we have the following representation

$$\sum_{c=2}^{m}\binom{m}{c}\frac{c}{m}\frac{1}{\sqrt{n}}\sum_{j=1}^{n}h_c^n(X_j)+f_n=\sigma_n\left(S_n+v_n+w_n\right)+\eta_n,$$

where

$$\eta_n=\frac{\sqrt{n}}{m}\sum_{c=1}^{m}\binom{m}{c}\bar{\theta}_c^n+\sqrt{n}\sum_{c=2}^{m}\binom{m}{c}\frac{c}{m}Eh_c^n(X_1).$$

Join (11) and (12) taking into account the latter representation; we get

$$\begin{aligned}&\sup_x|P(\sqrt{n}\,(U_n-\theta)/m\le x)-F(x)|\\ \le\;&\sup_x|P(\sigma_n(S_n+v_n+w_n)+\eta_n\le x)-F(x)|+d_n\\ =\;&\sup_x|P(S_n+v_n+w_n\le x)-F(x\sigma_n+\eta_n)|+d_n,\end{aligned}$$

where

$$d_n=n\cdot\sum_{c=2}^{m}P(|h_c(X_1)|>\sqrt{n})+\sum_{c=1}^{m}\frac{1}{c!}n^{[c]}P(|g_c|>n^{(2c-1)/2}).$$

This yields

$$\sup_x|P(\sqrt{n}\,(U_n-\theta)/m\le x)-F(x)|\le\sup_x|P(S_n+v_n+w_n\le x)-F(x)|$$

$$+\sup_x|F(x+\eta_n)-F(x)|+\sup_x|F(\sigma_n x)-F(x)|+d_n,$$

where, by the properties of the function $F(x)$,

$$\sup_x|F(x+\eta_n)-F(x)|\le\frac{1}{\sqrt{2\pi}}|\eta_n|,$$

and

$$\sup_x|F(\sigma_n x)-F(x)|\le\frac{1}{\sqrt{2\pi e}}|\sigma_n^2-1|/\sigma_n.$$

It remains to note that

$$d_n\le\sum_{c=1}^{m}\frac{1}{c!}\{\sqrt{n}\,E|g_c|\,I(|g_c|>n^{(2c-1)/2})\}$$

$$+\sum_{c=2}^{m}\{\sqrt{n}\,E|g_c|\,I(|g_c|>n^{(2c-1)/2})\}\le 2\beta_n,$$

$$|\eta_n|\le\frac{1}{m}\sum_{c=1}^{m}\binom{m}{c}\{\sqrt{n}\,E|g_c|\,I(|g_c|>n^{(2c-1)/2})\}$$

$$+\sum_{c=2}^{m}\binom{m}{c}\frac{c}{m}\{\sqrt{n}\,E|g_c|\,I(|g_c|>n^{(2c-1)/2})\}\le 2\beta_n.$$

Lemma 2 is proved, since

$$1+\frac{1}{c!}+\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{m}\binom{m}{c}+\frac{1}{\sqrt{2\pi}}\binom{m}{c}\frac{c}{m}\le 3\binom{m}{c}$$

for $2\le c\le m$, and moreover,

$$\sqrt{n}\,E|g_1|\,I(|g_1|>\sqrt{n})\le Eg_1^2I(|g_1|>\sqrt{n}).$$

Our further aim is to estimate the first term on the right-hand side of (10) using the Esseen inequality. Denote by $\varphi(t)$ the characteristic function of the random variable $g_{n1}(X_1)+\bar{h}_n(X_1)$, where $\bar{h}_n(X_1)=\sum_{c=2}^m\binom{m}{c}\frac{c}{m}h_{nc}(X_1)$, and denote by $\varphi_n(t)$ the characteristic function of the random variable $S_n+v_n+w_n$. Therefore, $\varphi^n(\frac{t}{\sigma_n\sqrt{n}})$ is the characteristic function of S_n. Then, by the Esseen inequality we have in (10)

$$\begin{aligned}&\sup_x|P(S_n+v_n+w_n\le x)-F(x)|\\ \le\;&\frac{1}{\pi}\int_{-\sqrt{n}/a}^{\sqrt{n}/a}|\varphi_n(t)-e^{-t^2/2}|\,|t|^{-1}\,dt+\frac{24\cdot a}{\pi\sqrt{2\pi}}\cdot n^{-1/2}\\ \le\;&\frac{1}{\pi}\int_{-\sqrt{n}/a}^{\sqrt{n}/a}\left|\varphi^n\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)-e^{-t^2/2}\right||t|^{-1}\,dt+\frac{24\cdot a}{\pi\sqrt{2\pi}}\cdot n^{-1/2}\\ &+\frac{1}{\pi}\int_{-\sqrt{n}/a}^{\sqrt{n}/a}\left|\varphi_n(t)-\varphi^n\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\right||t|^{-1}\,dt,\end{aligned}\tag{13}$$

where

$$a=\sigma_n^{-3}\,E|g_{n1}(X_1)+\bar{h}_n(X_1)|^3.$$

First, we estimate the integral on the right-hand side of (13), which contains the difference $\varphi^n(\frac{t}{\sigma_n\sqrt{n}})-e^{-t^2/2}$. Expanding the exponent into the Taylor series, we get

$$\left|E\exp\Big\{\frac{it}{\sigma_n\sqrt{n}}\,(g_{n1}(X_1)+\bar{h}_n(X_1))\Big\}\right|\le 1-\frac{t^2}{2n}+\frac{a\cdot|t|^3}{6\cdot n^{3/2}}.$$

Therefore, in the domain $|t|\le\sqrt{n}/a$

$$\left|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\right|\le\exp\Big(-\frac{t^2}{3n}\Big).\tag{14}$$

Further, putting $\psi(\frac{t}{\sqrt{n}})=\exp(-\frac{t^2}{2n})$, we have

$$\begin{aligned}\left|\varphi^n\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)-e^{-t^2/2}\right|&=\left|\varphi^n\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)-\psi^n\Big(\frac{t}{\sqrt{n}}\Big)\right|=\left|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)-\psi\Big(\frac{t}{\sqrt{n}}\Big)\right|\\ &\times\left|\varphi^{n-1}\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)+\varphi^{n-2}\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\psi\Big(\frac{t}{\sqrt{n}}\Big)+\ldots+\psi^{n-1}\Big(\frac{t}{\sqrt{n}}\Big)\right|.\end{aligned}$$

Here, for all $t \in R$,

$$\Big|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big) - \psi\Big(\frac{t}{\sqrt{n}}\Big)\Big| \le \Big|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big) - 1 + \frac{t^2}{2n}\Big| + \Big|\psi\Big(\frac{t}{\sqrt{n}}\Big) - 1 + \frac{t^2}{2n}\Big|$$

$$\le \frac{1}{6}\, a\cdot|t|^3\cdot n^{-3/2} + \frac{1}{8}\, t^4\, n^{-2}.$$

Therefore, taking into account (14) with $|t| \le \sqrt{n}/a$, we have

$$\Big|\varphi^n\Big(\frac{t}{\sigma_n\sqrt{n}}\Big) - e^{t^2/2}\Big| \le \Big(\frac{a\cdot|t|^3}{6\cdot n^{3/2}} + \frac{t^4}{8n^2}\Big)\sum_{k=0}^{n-1}\exp\Big\{-\frac{(n-1-k)}{3n}\,t^2 - \frac{k}{2n}\,t^2\Big\}.$$

In addition, in the domain $|t| \le \sqrt{n}/a$ we can write

$$\sum_{k=0}^{n-1}\exp\Big\{-\frac{(n-1-k)}{3n}\,t^2 - \frac{k}{2n}\,t^2\Big\} = e^{-\frac{(n-1)}{3n}t^2}\sum_{k=0}^{n-1}\exp\Big\{-\frac{t^2}{6n}\,k\Big\}$$

$$= e^{-\frac{(n-1)}{3n}t^2}\,\frac{(1-\exp(-\frac{t^2}{6}))}{(1-\exp(-\frac{t^2}{6n}))} = \exp\Big\{\frac{t^2}{2n} - \frac{t^2}{3}\Big\}\frac{(1-\exp(-\frac{t^2}{6}))}{(\exp(\frac{t^2}{6n})-1)}$$

$$\le 6\,e^{1/2}\,ne^{-t^2/3}\,\frac{1-\exp(-\frac{t^2}{6})}{t^2}.$$

Thus, in the domain $|t| \le \sqrt{n}/a$ the following inequality holds

$$\Big|\varphi^n\Big(\frac{t}{\sigma_n\sqrt{n}}\Big) - e^{-t^2/2}\Big| \le e^{1/2}\cdot\frac{7}{4}\cdot\frac{a}{\sqrt{n}}\cdot|t|\cdot e^{-t^2/3}. \tag{15}$$

Applying (15), we get the estimate for (13)

$$\frac{1}{\pi}\int_{-\sqrt{n}/a}^{\sqrt{n}/a}\Big|\varphi^n\Big(\frac{t}{\sigma_n\sqrt{n}}\Big) - e^{-t^2/2}\Big|\,|t|^{-1}\,dt \le \frac{7}{4}\cdot\sqrt{\frac{3e}{\pi}}\cdot\frac{a}{\sqrt{n}}; \tag{16}$$

here, according to the definition of a,

$$a \le 8\cdot\sigma_n^{-3}\,E|g_1^n(X_1)|^3 + 2^{2m+6}\cdot\sigma_n^{-3}\cdot\beta_n\cdot\sqrt{n}. \tag{17}$$

Now we proceed with the estimating of the second integral on the right-hand side of (13). First, consider the characteristic function

$$\varphi_n(t) = E\exp\{itS_n + itv_n + itw_n\},$$

when $t \in R$.

For this purpose, for all $n \ge 2$ and any $t \in R$, choose an integer number $k = k(t)$ by the formula

$$k = [n/(2+|t|^{p+\varepsilon})] + 1, \tag{18}$$

where the symbol $[z]$ denotes the integer part of z, and the fixed numbers p and ε are such that $1 \le p < 2$, $0 < \varepsilon < 2-p$. It is obvious that $1 \le k \le n-1$, and

$$\frac{1}{2+|t|^{p+\varepsilon}} < \frac{k}{n} \le \frac{1}{2+|t|^{p+\varepsilon}} + \frac{1}{n} \tag{19}$$

for all $n \geq 2$ and $t \in R$. Moreover, one can see that in the domain $|t| \leq \sqrt{n}/a$

$$\frac{1}{2+|t|^{p+\varepsilon}} < \frac{k}{n} \leq \frac{3}{2+|t|^{p+\varepsilon}} \tag{20}$$

for all $n \geq 2$.

With k so chosen, represent v_n as a sum

$$v_n = v_{nk} + \bar{v}_{nk},$$

where

$$\begin{aligned}
v_{nk} &= \frac{\sqrt{n}}{\sigma_n m} \sum_{c=2}^{m} \binom{m}{c} \binom{n}{c}^{-1} S_{nk}(g_{nc}), \\
S_{nk}(g_{nc}) &= \sum_{i_1=1}^{k} \sum_{i_2=i_1+1}^{n-c+2} \cdots \sum_{i_c=i_{c-1}+1}^{n} g_{nc}(X_{i_1}, X_{i_2}, \ldots, X_{i_c}); \\
\bar{v}_{nk} &= \frac{\sqrt{n}}{\sigma_n m} \sum_{c=2}^{m} \binom{m}{c} \binom{n}{c}^{-1} \bar{S}_{nk}(g_{nc}),
\end{aligned}$$

where

$$\bar{S}_{nk}(g_{nc}) = \sum_{k+1 \leq i_1 < \cdots < i_c \leq n} g_{nc}(X_{i_1}, \ldots, X_{i_c}).$$

We can also rewrite the variable w_n as a sum

$$w_n = w_{nk} + \bar{w}_{nk},$$

where

$$\begin{aligned}
w_{nk} &= \frac{\sqrt{n}}{\sigma_n m} \sum_{d=2}^{m-1} \sum_{c=d+1}^{m} \binom{m}{c} \binom{c}{d} \binom{n}{d}^{-1} S_{nk}(g_{ncd}), \\
S_{nk}(g_{ncd}) &= \sum_{i_1=1}^{k} \sum_{i_2=i_1+1}^{n-d+2} \cdots \sum_{i_d=i_{d-1}+1}^{n} g_{ncd}(X_{i_1}, X_{i_2}, \ldots, X_{i_d}); \\
\bar{w}_{nk} &= \frac{\sqrt{n}}{\sigma_n m} \sum_{d=2}^{m-1} \sum_{c=d+1}^{m} \binom{m}{c} \binom{c}{d} \binom{n}{d}^{-1} \bar{S}_{nk}(g_{ncd}), \\
\bar{S}_{nk}(g_{ncd}) &= \sum_{k+1 \leq i_1 < \cdots < i_d \leq n} g_{ncd}(X_{i_1}, \ldots, X_{i_d}).
\end{aligned}$$

Now we can examine the estimation of the characteristic function $\varphi_n(t)$, $t \in R$.

Estimation of the influence of w_{nk}. By the inequality $|e^{ix} - 1| \leq |x|$, $x \in R$,

$$\varphi_n(t) = E \exp\{itS_n + itv_n + it\bar{w}_{nk}\} + R_{n1}(t), \tag{21}$$

where

$$|R_{n1}(t)| \leq |t| \, E|w_{nk}|,$$

and also

$$E|w_{nk}| \le \frac{\sqrt{n}}{\sigma_n m} \cdot \sum_{d=2}^{m-1} \sum_{c=d+1}^{m} \binom{m}{c}\binom{c}{d}\binom{n}{d}^{-1} E|S_{nk}(g_{ncd})|.$$

On the right-hand side

$$E|S_{nk}(g_{ncd})| \le \left\{ \binom{n}{d} - \binom{n-k}{d} \right\} E|g_{ncd}|,$$

where

$$\binom{n}{d} - \binom{n-k}{d} \le \frac{n^d}{d!} \cdot \frac{k}{n} \tag{22}$$

According to formula (9)

$$g_{ncd}(x_1, \ldots, x_d) = \sum_{s=0}^{d} (-1)^{d-s} \sum_{1 \le j_1 < \ldots < j_s \le d} g_{cs}^n(x_{j_1}, \ldots, x_{j_s}),$$

where

$$g_{cs}^n(x_1, \ldots, x_s) = E g_c(x_1, \ldots, x_s, X_{s+1}, \ldots, X_c)\, I(|g_c| > n^{(2c-1)/2});$$

therefore,

$$E|g_{ncd}| \le 2^d\, E|g_c|\, I(|g_c| > n^{(2c-1)/2}).$$

Moreover,

$$\sum_{d=2}^{m-1} \sum_{c=d+1}^{m} \binom{m}{c}\binom{c}{d}\binom{n}{d}^{-1} \frac{n^d}{d!}\, 2^d\, E|g_c|\, I(|g_c| > n^{(2c-1)/2})$$

$$\le \sum_{c=3}^{m} \binom{m}{c} E|g_c|\, I(|g_c| > n^{(2c-1)/2}) \sum_{d=2}^{c-1} 2^d \binom{c}{d}\binom{n}{d}^{-1} \cdot \frac{n^d}{d!}$$

$$\le (2m-1)^m \cdot \sum_{c=3}^{m} \binom{m}{c} E|g_c| \cdot I(|g_c| > n^{(2c-1)/2});$$

here we use the fact that for $n \ge d$

$$\binom{n}{d}^{-1} n^d \le d^d \le (c-1)^d.$$

Thus, in (21) for all $t \in R$

$$|R_{n1}(t)| \le \frac{(2m-1)^m}{\sigma_n m} \cdot \beta_n \cdot |t| \cdot \frac{k}{n}. \tag{23}$$

Estimation of influence of v_{nk}. According to (21), we first write

$$E \exp\{itS_n + itv_n + it\bar{w}_{nk}\} = E \exp\{itS_n + it\bar{v}_{nk} + it\bar{w}_{nk}\}$$

$$+it\, E \exp\{itS_n + it\bar{v}_{nk} + it\bar{w}_{nk}\}\, v_{nk} + R_{n2}(t), \tag{24}$$

where, by the inequality $|e^{ix}-1-ix| \le 2|x|^p$ for $1 \le p \le 2$, $x \in R$,

$$|R_{n2}(t)| \le 2\,|t|^p\ E|v_{nk}|^p.$$

Taking into account the inequality $|a_1+\cdots+a_m|^p \le m^{p-1}\,(|a_1|^p+\cdots+|a_m|^p)$, $a_i \in R$, we can write the following estimate

$$E|v_{nk}|^p \le (m-1)^{p-1}\cdot m^{-p}\sigma_n^{-p}n^{p/2}\sum_{c=2}^{m}\binom{m}{c}\binom{n}{c}^{-p} E|S_{nk}(g_{nc})|^p,$$

where, according to the moment inequality for U-statistics (see, Borovskikh(1996)),

$$E|S_{nk}(g_{nc})|^p \le 2^{c(2-p)}\,E|g_{nc}|^p\left\{\binom{n}{c}-\binom{n-k}{c}\right\}.$$

Note that

$$g_{nc}(x_1,\ldots,x_c)=\sum_{d=0}^{c}(-1)^{c-d}\sum_{1\le j_1<\cdots<j_d\le c} g_d^n(x_{j_1},\ldots,x_{j_d}), \tag{25}$$

where

$$g_d^n(x_1,\ldots,x_d)=Eg_c(x_1,\ldots,x_d,X_{d+1},\ldots,X_c)\,I(|g_c|\le n^{(2c-1)/2}),$$

and moreover, for all $c \le m \le n$,

$$\begin{aligned}\binom{n}{c}^{-p} &= \frac{(c!)^p}{(n(n-1)\cdots(n-c+1))^p} \le (c!)^p\cdot n^{-cp}\cdot m^{cp}\cdot m^{-[c]p},\\ \binom{m}{c}^{p} &= m^{[c]p}\,(c!)^{-p}.\end{aligned}$$

We have from (25)

$$\begin{aligned}E|g_{nc}|^p &\le (c+1)^{p-1}\sum_{d=0}^{c}\binom{c}{d}^p E|g_d^n|^p\\ &\le (c+1)^{p-1}\Big(\sum_{d=0}^{c}\binom{c}{d}^p\Big)\,E|g_c|^p\,I(|g_c|\le n^{(2c-1)/2}),\end{aligned} \tag{26}$$

where

$$(c+1)^{p-1}\sum_{d=0}^{c}\binom{c}{d}^p \le ((c+1)!)^{p-1}\cdot 2^c.$$

Thus, for all $t \in R$ we get

$$|R_{n2}(t)| \le 3\cdot 2^{2m}\,m^{mp}((m+1)!)^{p-1}\,\sigma_n^{-p}\cdot\gamma_n(p)\cdot|t|^p\cdot\frac{k}{n}. \tag{27}$$

Then we consider the second term on the right-hand side of (24). Taking into account the definition of v_{nk} and the degeneracy property of the function $g_{nc}(x_1,\ldots,x_c)$, write the following representation

$$E\exp\{itS_n+it\bar{v}_{nk}+it\bar{w}_{nk}\}\,v_{nk}$$

$$= \frac{\sqrt{n}}{\sigma_n m}\,\varphi^{k-1}\left(\frac{t}{\sigma_n\sqrt{n}}\right)\cdot\sum_{i_1=1}^{k} E\{e^{it\xi_{i_1}+it(\bar S_{nk}+\bar v_{nk}+\bar w_{nk})}\Theta^n_{i_1}\} = \zeta_1(t)+\zeta_2(t)+\zeta_3(t),$$

where

$$\begin{aligned}
\zeta_1(t) &= \frac{\sqrt{n}}{\sigma_n m}\,\varphi^{k-1}\Big(\frac{t}{\sigma_n m}\Big)\sum_{i_1=1}^{k} E\{(e^{it\xi_{i_1}}-1)(e^{it\bar v_{nk}}-1)\,e^{it(\bar S_{nk}+\bar w_{nk})}\,\Theta^n_{i_1}\},\\
\zeta_2(t) &= \frac{\sqrt{n}}{\sigma_n m}\,\varphi^{k-1}\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\sum_{i_1=1}^{k} E\{(e^{it\xi_{i_1}}-1)(e^{it\bar w_{nk}}-1)\,e^{it\bar S_{nk}}\,\Theta^n_{i_1}\},\\
\zeta_3(t) &= \frac{\sqrt{n}}{\sigma_n m}\,\varphi^{k-1}\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\sum_{i_1=1}^{k} E\{(e^{it\xi_{i_1}}-1)\,e^{it\bar S_{nk}}\,\Theta^n_{i_1}\};
\end{aligned}$$

now,

$$\bar S_{nk} = \xi_{k+1}+\cdots+\xi_n,$$

$$\Theta^n_{i_1} = \sum_{c=2}^{m}\binom{m}{c}\binom{n}{c}^{-1}\sum_{i_2=i_1+1}^{n-c+2}\cdots\sum_{i_c=i_{c-1}+1}^{n} g_{nc}(X_{i_1},X_{i_2},\ldots,X_{i_c}).$$

Estimation of $\zeta_1(t)$. First we write

$$|\zeta_1(t)| \le \frac{\sqrt{n}}{\sigma_n m}\Big|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\Big|^{k-1}\sum_{i_1=1}^{k} E|e^{it\xi_1}-1|\,|e^{it\bar v_{nk}}-1|\,|\Theta^n_{i_1}|.$$

By the Hölder inequality for $1<p\le 2$

$$\begin{aligned}
&E|e^{it\xi_1}-1|\,|e^{it\bar v_{nk}}-1|\,|\Theta^n_{i_1}|\\
&\le (E|e^{it\xi_1}-1|^q\,|e^{it\bar v_{nk}}-1|^q)^{1/q}\,(E|\Theta^n_{i_1}|^p)^{1/p},
\end{aligned}$$

where $q=p/(p-1)$. Applying inequalities $|e^{ix}-1|\le|x|$ and $|e^{ix}-1|\le 2|x|^{p-1}$, $x\in R$, $1\le p\le 2$, and taking into account the independence of ξ_1 and v_{nk}, we get

$$\begin{aligned}
(E|e^{it\xi_1}-1|^q\,|e^{it\bar v_{nk}}-1|^q)^{1/q} &= (E|e^{it\xi_1}-1|^q\cdot E|e^{it\bar v_{nk}}-1|^q)^{1/q}\\
&\le 2\,|t|^p\,(E|\xi_1|^q)^{1/q}\,(E|\bar v_{nk}|^p)^{1-1/p},
\end{aligned}$$

where

$$(E|\xi_1|^q)^{1/q} = \frac{1}{\sigma_n\sqrt{n}}\,(E|Z_1|^q)^{1/q},$$

$$Z_1 = g_{n1}(X_1)+\sum_{c=2}^{m}\binom{m}{c}\frac{c}{m}\,h_{nc}(X_1).$$

Then we have

$$E|\bar v_{nk}|^p \le \frac{(m-1)^{p-1}}{m^p}\,\sigma_n^p\,n^{p/2}\sum_{c=2}^{m}\binom{m}{c}^{p}\binom{n}{c}^{-p}E|\bar S_{nk}(g_{nc})|^p.$$

Then, in the same way as we deduced (27), we get

$$E|\bar S_{nk}(g_{nc})|^p \le 2^{(2-p)c}\sum_{k+1\le i_1<\cdots<i_c\le n} E|g_{nc}(X_{i_1},\ldots,X_{i_c})|^p$$

$$\leq 2^{(2-p)c}\binom{n}{c} E|g_{nc}|^p \leq 2^{2c}((c+1)!)^{p-1}\binom{n}{c} E|g_c|^p\, I(|g_c| \leq n^{(2c-1)/2}).$$

Therefore,

$$E|\bar{v}_{nk}|^p \leq \frac{(m-1)^{p-1}}{m^p}\, \sigma_n^{-p} \cdot n^{p/2}$$

$$\times \sum_{c=2}^{m} \binom{m}{c}^p \binom{n}{c}^{-p+1} 2^{2c} \cdot ((c+1)!)^{p-1} \cdot E|g_c|^p \cdot I(|g_c| \leq n^{(2c-1)/2}).$$

Here

$$\binom{n}{c}^{-p+1} = \frac{(c!)^{p-1}}{(n(n-1)\cdots(n-c+1))^{p-1}} \leq \frac{(c!)^{p-1}}{n^{c(p-1)}} \cdot \frac{m^{c(p-1)}}{(m^{[c]})^{p-1}},$$

$$\binom{m}{c}^p = \binom{m}{c} \cdot \binom{m}{c}^{p-1} = \binom{m}{c} \cdot \frac{(m^{[c]})^{p-1}}{(c!)^{p-1}}.$$

Thus,

$$E|\bar{v}_{nk}|^p \leq \frac{(m-1)^{p-1}}{m^p}\, \sigma_n^{-p}$$

$$\times \sum_{c=2}^{m} \binom{m}{c} m^{c(p-1)}\, 2^{2c} \cdot ((c+1)!)^{p-1}\, n^{(2c-p(2c-1))/2}\, E|g_c|^p \cdot I(|g_c| \leq n^{(2c-1)/2}).$$

Then, we can write the estimates for all $1 \leq i_1 \leq k$

$$\begin{aligned}
E|\Theta_{i_1}^n|^p &\leq (m-1)^{p-1} \sum_{c=2}^{m} \binom{m}{c}^p \binom{n}{c}^{-p} \\
&\quad \times E\Big|\sum_{i_1<i_2<\cdots<i_c\leq n} g_{nc}(X_1, X_{i_2}, \ldots, X_{i_c})\Big|^p \\
&\leq (m-1)^{p-1} \cdot \sum_{c=2}^{m} \binom{m}{c}^p \binom{n}{c}^{-p} 2^{(2-p)(c-1)} \\
&\quad \times \sum_{i_1<i_2<\cdots<i_c\leq n} E|g_{nc}(X_1, X_{i_2}, \ldots, X_{i_c})|^p \\
&\leq (m-1)^{p-1} \cdot \sum_{c=2}^{m} \binom{m}{c} \binom{n}{c}^{-p} \binom{n}{c-1} E|g_{nc}|^p \\
&\leq (m-1)^{p-1} \sum_{c=2}^{m} \binom{m}{c}^p \binom{n}{c}^{-p+1} \cdot \frac{c}{n} \cdot 2^{c+1+(2-p)(c-1)} \\
&\quad \times ((c+1)!)^{p-1}\, E|g_c|^p\, I(|g_c| \leq n^{(2c-1)/2}).
\end{aligned}$$

Thus, for all $1 \leq i_1 \leq k$

$$E|\Theta_{i_1}^n|^p \leq (m-1)^{p-1} \sum_{c=2}^{m} \binom{m}{c} m^{c(p-1)}\, c\, 2^{2c}\, ((c+1)!)^{p-1} \cdot n^{(2c-p(2c-1))/2}$$

$$\times n^{-1-p/2}\, E|g_c|\, I(|g_c| \leq n^{(2c-1)/2}).$$

Therefore, we have shown that for $t \in R$, for all $1 < p \le 2$, and $q = p/(p-1)$ the following holds

$$|\zeta_1(t)| \le 2^{2m} \cdot m^{mp} \, ((m+1)!)^{p-1} \, \sigma_n^{-p-1} \, (E|Z_1|^q)^{1/q} \, |t|^p$$
$$\times \left|\varphi\Big(\frac{t}{\sigma_n \sqrt{n}}\Big)\right|^{k-1} n^{-(2-p)/(2p)} \, \gamma_n(p). \tag{28}$$

Estimation of $\zeta_2(t)$. First we write the inequality

$$|\zeta_2(t)| \le \frac{\sqrt{n}}{\sigma_n m} \left|\varphi\Big(\frac{t}{\sigma_n \sqrt{n}}\Big)\right|^{k-1} \sum_{i_1=1}^{k} E|e^{it\xi_1} - 1| \, |e^{it\bar{w}_{nk}} - 1| \, |\Theta^n_{i_1}|.$$

For $1 < p \le 2$, by the Hölder inequality and taking into account inequalities $|e^{ix} - 1| \le 2\,|x|^{(p-1)/p}$, we have

$$E|e^{it\xi_1} - 1| \, |e^{it\bar{w}_{nk}} - 1| \, |\Theta^n_{i_1}|$$
$$\le (E|e^{it\xi_1} - 1|^q)^{1/q} \cdot (E|e^{it\bar{w}_{nk}} - 1|^q)^{1/q} \, (E|\Theta^n_{i_1}|^p)^{1/p}$$
$$\le \frac{2 \cdot |t|^{2-1/p}}{\sigma_n \sqrt{n}} \, (E|Z_1|^q)^{1/q} \, (E|\bar{w}_{nk}|)^{1-1/p} \, (E|\Theta^n_{i_1}|^p)^{1/p}.$$

Here

$$E|\bar{w}_{nk}| \le \frac{\sqrt{n}}{\sigma_n m} \sum_{d=2}^{m-1} \sum_{c=d+1}^{m} \binom{m}{c} \binom{c}{d} \binom{n}{d}^{-1} E|\bar{S}_{nk}(g_{ncd})|,$$

and moreover,

$$E|\bar{S}_{nk}(g_{ncd})| \le \sum_{k+1 \le i_1 < \ldots < i_d \le n} E|g_{ncd}| \le \binom{n}{d} E|g_{ncd}|$$
$$\le \ 2^d \binom{n}{d} E|g_c| \, I(|g_c| > n^{(2c-1)/2}),$$
$$\sum_{d=2}^{m-1} \sum_{c=d+1}^{m} \binom{m}{c} \binom{c}{d} \, 2^d \, E|g_c| \, I(|g_c| > n^{(2c-1)/2})$$
$$= \ \sum_{c=3}^{m} \binom{m}{c} E|g_c| \, I(|g_c| > n^{(2c-1)/2}) \cdot \Big(\sum_{d=2}^{c-1} \binom{c}{d} \, 2^d\Big)$$
$$\le \ \sum_{c=3}^{m} \binom{m}{c} \, 3^c \, E|g_c| \, I(|g_c| > n^{(2c-1)/2}).$$

Therefore,

$$E|\bar{w}_{nk}| \le \frac{1}{\sigma_n m} \sum_{c=3}^{m} \binom{m}{c} \, 3^c \, \{\sqrt{n} \, E|g_c| \, I(|g_c| > n^{(2c-1)/2})\}.$$

For $t \in R$ and all $1 < p \le 2$ and $q = p/(p-1)$ this leads to the inequality

$$|\zeta_2(t)| \le 2^{2m/p-1} \, 3^{m(1-1/p)} \cdot m^{m-1} \, ((m+1)!)^{1-1/p}$$

$$\times\, \sigma_n^{-3+1/p} \cdot (E|Z_1|^q)^{1/q}\, |t|^{2-1/p} \cdot \Big|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\Big|^{k-1} n^{-(2-p)/p} \cdot \gamma_n^{1/p}(p) \cdot \beta_n^{1-1/p}. \tag{29}$$

Estimation of $\zeta_3(t)$. According to the definition, and taking into account the independence of $X_1, \ldots, X_n$, and the degeneracy of the functions $g_{nc}(x_1, \ldots, x_c)$, $c = 2, \ldots, m$, we can write

$$\begin{aligned}
\zeta_3(t) \;&=\; \frac{\sqrt{n}}{\sigma_n m} \sum_{c=2}^{m} \binom{m}{c} \binom{n}{c}^{-1} \varphi^{n-c}\Big(\frac{t}{\sigma_n\sqrt{n}}\Big) \\
&\qquad \times \sum_{i_1=1}^{k} \sum_{i_2=i_1+1}^{n-c+2} \cdots \sum_{i_c=i_{c-1}+1}^{n} E\{(e^{it\xi_1}-1)\, e^{it\xi_2+\ldots+it\xi_c}\, g_{nc}(X_1, X_2, \ldots, X_c)\} \\
&=\; \frac{\sqrt{n}}{\sigma_n m} \sum_{c=2}^{m} \binom{m}{c} \binom{n}{c}^{-1} \Big[\binom{n}{c} - \binom{n-k}{c}\Big] \varphi^{n-c}\Big(\frac{t}{\sigma_n\sqrt{n}}\Big) \\
&\qquad \times E\Big\{g_{nc}(X_1, \ldots, X_c) \prod_{s=1}^{c} (e^{it\xi_s}-1)\Big\}.
\end{aligned} \tag{30}$$

From (28)–(30) we derive the necessary estimate in (24).

Estimation of the influence of $\bar{w}_{nk}$. Now consider the first expression on the right-hand side of (24). First, we can write

$$E\exp\{itS_n + it\bar{v}_{nk} + it\bar{w}_{nk}\} = E\exp\{itS_n + it\bar{v}_{nk}\} + \varphi^k\Big(\frac{t}{\sigma_n\sqrt{n}}\Big) R_{n3}(t), \tag{31}$$

then, using the inequality $|e^{ix}-1| \le |x|$, $x \in R$, we have

$$|R_{n3}(t)| \le |t| \cdot E|\bar{w}_{nk}|,$$

i.e., for all $t \in R$,

$$|R_{n3}(t)| \le \frac{|t|}{\sigma_n m}\, 3^m \cdot \beta_n. \tag{32}$$

Estimation of the influence of $\bar{v}_{nk}$. According to the definition in (31),

$$E\exp\{itS_n + it\bar{v}_{nk}\} = E\exp\{itS_n\} + it\, Ee^{itS_n}\, \bar{v}_{nk} + \varphi^k\Big(\frac{t}{\sigma_n\sqrt{n}}\Big) R_{n4}(t), \tag{33}$$

where for $1 \le p \le 2$,

$$|R_{n4}(t)| \le 2\, |t|^p\, E|\bar{v}_{nk}|^p,$$

holds, i.e., for $t \in R$,

$$|R_{n4}(t)| \le 2^{2m+1} \cdot m^{mp-1}\, ((m+1)!)^{p-1}\, \sigma_n^{-p} \cdot |t|^p \cdot \gamma_n(p). \tag{34}$$

In order to estimate the second term on the right-hand side of (33), write the representation

$$Ee^{itS_n}\, \bar{v}_{nk} = \frac{\sqrt{n}}{\sigma_n m} \sum_{c=2}^{m} \binom{m}{c} \binom{n}{c}^{-1} \sum_{k+1 \le i_1 < \cdots < i_c \le n} E\{e^{itS_n}\, g_{nc}(X_{i_1}, \ldots, X_{i_c})\}$$

$$= \frac{\sqrt{n}}{\sigma_n m} \sum_{c=2}^{m} \binom{m}{c} \binom{n}{c}^{-1} \binom{n-k}{c} \varphi^{n-c}\Big(\frac{t}{\sigma_n \sqrt{n}}\Big)$$

$$\times E\Big\{ g_{nc}(X_1, \dots, X_c) \cdot \prod_{s=1}^{c} (e^{it\xi_s} - 1) \Big\}. \tag{35}$$

Write

$$\zeta_4(t) = \zeta_3(t) + E e^{itS_n}\, \bar{v}_{nk}.$$

Then one can see from (30) and (35), that

$$\zeta_4(t) = \frac{\sqrt{n}}{\sigma_n m} \sum_{c=2}^{m} \binom{m}{c} \varphi^{n-c}\Big(\frac{t}{\sigma_n \sqrt{n}}\Big)\, E\Big\{ g_{nc}(X_1, \dots, X_c) \prod_{s=1}^{c} (e^{it\xi_s} - 1) \Big\}. \tag{36}$$

First, by the degeneracy property of the function $g_{nc}(x_1, \dots, x_c)$, we can write

$$E\Big\{ g_{nc}(X_1, \dots, X_c) \prod_{s=1}^{c} (e^{it\xi_s} - 1) \Big\} = E\{ g_{nc}(X_1, \dots, X_c) \cdot e^{it\sum_{s=1}^{c} \xi_s} \}$$

$$= e^{-\frac{itc}{\sigma_n \sqrt{n}}(Eg_1^n(X_1) + E\bar{h}^n(X_1))}\, E\{ g_{nc}(X_1, \dots, X_c)\, e^{\frac{it}{\sigma_n \sqrt{n}} \sum_{s=1}^{c} (g_1^n(X_s) + \bar{h}^n(X_s))} \}$$

$$= e^{-\frac{itc}{\sigma_n \sqrt{n}}(Eg_1^n(X_1) + E\bar{h}^n(X_1))}\, E\Big\{ g_{nc}(X_1, \dots, X_c) \prod_{s=1}^{c} (e^{\frac{it}{\sigma_n \sqrt{n}}(g_1^n(X_s) + \bar{h}^n(X_s))} - 1) \Big\}, \tag{37}$$

where $\bar{h}^n(X_s) = \sum_{c=2}^{m} \binom{m}{c} \frac{c}{m}\, h_c^n(X_s)$.

According to (25),

$$g_{nc}(x_1, \dots, x_c) = g_c^n(x_1, \dots, x_c) + \sum_{d=0}^{c-1} (-1)^{c-d} \sum_{1 \le j_1 < \dots < j_d \le c} g_d^n(x_{j_1}, \dots, x_{j_d}),$$

where for all $d = 0,\ 1, \dots,\ c-1$, we have the formula

$$g_d^n(x_1, \dots, x_d) = -Eg_c(x_1, \dots, x_d, X_{d+1}, \dots, X_c)\, I(|g_c| > n^{(2c-1)/2}).$$

Therefore, in (37), we can write

$$E\Big\{ g_{nc}(X_1, \dots, X_c) \prod_{s=1}^{c} (e^{it\xi_s} - 1) \Big\}$$

$$= E\Big\{ g_c^n(X_1, \dots, X_c) \prod_{s=1}^{c} (e^{\frac{it}{\sigma_n \sqrt{n}}(g_1^n(X_s) + \bar{h}^n(X_s))} - 1) \Big\} + R_{n5}(t), \tag{38}$$

where

$$|R_{n5}(t)| \le 2^c \cdot E|g_c|\, I(|g_c| > n^{(2c-1)/2}).$$

Using the inequality $|e^{ix} - 1| \le |x|$, $x \in R$, and the well-known formula

$$\prod_{s=1}^{c} (a_s + b_s) = \sum_{d=0}^{c} \sum_{1 \le j_1 < \cdots < j_d \le c} \prod_{s=1}^{d} b_{j_s} \prod_{s=d+1}^{c} a_{j'_s}, \quad a_s, b_s \in R,$$

where $\{j'_{d+1},\dots,j'_c\} = \{1,\dots,c\}\backslash\{j_1,\dots,j_d\}$, the following relations can be obtained

$$
\begin{aligned}
&\Big|E\Big\{g_c^n(X_1,\dots,X_c)\prod_{s=1}^{c}(e^{\frac{it}{\sigma_n\sqrt{n}}(g_1^n(X_s)+\bar h^n(X_s))}-1)\Big\}\Big|\\
\le\ & |t|^c\cdot n^{-c/2}\cdot\sigma_n^{-c}\,E|g_c^n(X_1,\dots,X_c)|\prod_{s=1}^{c}(|g_1^n(X_s)|+|\bar h^n(X_s)|)\\
=\ & |t|^c\cdot n^{-c/2}\,\sigma_n^{-c}\sum_{d=0}^{c}\sum_{1\le j_1<\dots<j_d\le c}E|g_c^n(X_1,\dots,X_c)|\\
&\quad\times\prod_{s=1}^{d}|\bar h^n(X_{j_s})|\cdot\prod_{s=d+1}^{c}|g_1^n(X_{j'_s})|\\
=\ & |t|^c\,n^{-c/2}\,\sigma_n^{-c}\sum_{d=0}^{c}\binom{c}{d}E|g_c^n|\prod_{s=1}^{d}|\bar h^n(X_s)|\cdot\prod_{s=d+1}^{c}|g_1^n(X_s)|\\
=\ & |t|^c\cdot n^{-c/2}\,\sigma_n^{-c}\,E|g_c^n|\prod_{s=1}^{c}|g_1^n(X_s)|\\
&\quad+|t|^c\cdot n^{-c/2}\,\sigma_n^{-c}\sum_{d=1}^{c}\binom{c}{d}E|g_c^n|\prod_{s=1}^{d}|\bar h^n(X_s)|\prod_{s=d+1}^{c}|g_1^n(X_s)|. \qquad (39)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|\zeta_4(t)| \le\ & \frac{1}{\sigma_n m}\cdot\sum_{c=2}^{m}\binom{m}{c}2^c\Big|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\Big|^{n-c}\cdot\{\sqrt{n}\,E|g_c|\,I(|g_c|>n^{(2c-1)/2})\}\\
&+\frac{1}{\sigma_n m}\cdot\sum_{c=2}^{m}\binom{m}{c}\sigma_n^{-c}\,|t|^c\Big|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\Big|^{n-c}\cdot n^{-\frac{c}{2}+\frac12}\cdot E|g_c^n|\prod_{s=1}^{c}|g_1^n(X_s)|\\
&+\frac{1}{\sigma_n m}\sum_{c=2}^{m}\binom{m}{c}\sigma_n^{-c}\,|t|^c\Big|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\Big|^{n-c}n^{-\frac{c}{2}+\frac12}\\
&\times\sum_{d=1}^{c}\binom{c}{d}E|g_c^n|\prod_{s=1}^{d}|\bar h^n(X_s)|\prod_{s=d+1}^{c}|g_1^n(X_s)|. \qquad (40)
\end{aligned}
$$

If $2\le c\le m$, and $n\ge m$, then

$$\Big|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\Big|^{n-c}\le\Big|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\Big|^{n-m}.$$

Since

$$\prod_{s=1}^{c}|g_1^n(X_s)|^{1-2/c}\le n^{c/2-1},$$

then

$$n^{-\frac{c}{2}+\frac12}\,E|g_c^n|\prod_{s=1}^{c}|g_1^n(X_s)|\le E|g_c^n|\prod_{s=1}^{c}|g_1^n(X_s)|^{2/c}\cdot\frac{1}{\sqrt{n}}.$$

By the Hölder inequality, and taking into account that $Eg_1^2=1$,

$$E|g_c^n|\prod_{s=1}^{d}|\bar h^n(X_s)|\prod_{s=d+1}^{c}|g_1^n(X_s)|\le(E|g_c^n|^2)^{1/2}(E|\bar h^n(X_1)|^2)^{1/2},$$

and therefore, we can write the following inequality

$$\sum_{d=1}^{c} \binom{c}{d} E|g_c^n| \prod_{s=1}^{d} |\bar{h}^n(X_s)| \prod_{s=d+1}^{c} |g_1^n(X_s)|$$

$$\leq c\,(E|\bar{h}^n(X_1)|^2)^{1/2}(1+(E|\bar{h}^n(X_1)|^2)^{1/2})^{c-1}(E|g_c^n|^2)^{1/2}.$$

Therefore, inequality (40) takes the form

$$|\zeta_4(t)| \leq \frac{2^m}{\sigma_n m}\cdot\left|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\right|^{n-m}\cdot\beta_n$$

$$+\frac{1}{\sigma_n m}\sum_{c=2}^{m}\binom{m}{c}\sigma_n^{-c}\,|t|^c\left|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\right|^{n-c} E|g_c^n|\prod_{s=1}^{c}|g_1^n(X_s)|^{2/c}\cdot\frac{1}{\sqrt{n}}$$

$$+\frac{1}{\sigma_n m}\sum_{c=2}^{m}\binom{m}{c}\sigma_n^{-c}\,|t|^c\left|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\right|^{n-c}\cdot c\cdot(1+(E|\bar{h}^n(X_1)|^2)^{1/2})^{c-1}$$

$$\times\,(E|\bar{h}^n(X_1)|^2)^{1/2}\cdot(n^{-c+1}\,E|g_c^n|^2)^{1/2}. \tag{41}$$

Note that in (41)

$$E|\bar{h}^n(X_1)|^2 \leq 2^m\cdot\beta_n. \tag{42}$$

In the following two cases may occur, $\beta_n \geq 1$, or $\beta_n \leq 1$. If $\beta_n \geq 1$, then always $\Delta_n \leq 1 \leq \beta_n$. This implies that $\Delta_n \leq c_1\beta_n + c_2$ for any $c_1 \geq 1$ and $c_2 \geq 0$. This proves inequality (3) in the case $\beta_n \geq 1$. Below, we consider the case $\beta_n \leq 1$ starting with (41). Using (42), we have from (41)

$$|\zeta_4(t)| \;\leq \frac{2^m}{\sigma_n m}\cdot\left|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\right|^{n-m}\cdot\beta_n$$

$$+\frac{1}{\sigma_n m}\sum_{c=2}^{m}\binom{m}{c}\sigma_n^{-c}\,|t|^c\left|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\right|^{n-c} E|g_c^n|\prod_{s=1}^{c}|g_1^n(X_s)|^{2/c}\cdot\frac{1}{\sqrt{n}}$$

$$+\frac{2^{m/2}\cdot\beta_n^{1/2}}{\sigma_n m}\sum_{c=2}^{m}\binom{m}{c}\sigma_n^{-c}\,|t|^c\cdot\left|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\right|^{n-c}$$

$$\times\,c\cdot(1+2^{m/2})^{c-1}\cdot(n^{-c+1}\,E|g_c^n|^2)^{1/2}. \tag{43}$$

Thus, we have proved that for $t \in R$,

$$\varphi_n(t)-\varphi^n\Big(\frac{t}{\sigma_n\sqrt{n}}\Big) = R_{n1}(t)+R_{n2}(t)+it\,(\zeta_1(t)+\zeta_2(t))$$

$$+it\,\varphi^k\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\,(R_{n3}(t)+R_{n4}(t))+it\,\zeta_4(t), \tag{44}$$

where each term on the right-hand side is already estimated in (23), (27)–(29), (32), (34), and (43), respectively. Further, let $1 \leq p < 2$ in (27), and let $p = 2$ in (28), (29), and (34). Then, using (44), and taking into account (14), we obtain

$$\int_{-\sqrt{n}/a}^{\sqrt{n}/a}\left|\varphi_n(t)-\varphi^n\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\right|\,|t|^{-1}\,dt$$

$$\le \int_{-\sqrt{n}/a}^{\sqrt{n}/a} (|R_{n1}(t)| + |R_{n2}(t)|)\, |t|^{-1}\, dt + \int_{-\sqrt{n}/a}^{\sqrt{n}/a} (|\zeta_1(t)| + |\zeta_2(t)|)\, dt$$

$$+ \int_{-\sqrt{n}/a}^{\sqrt{n}/a} (|R_{n3}(t)| + |R_{n4}(t)|)\, e^{-\frac{t^2}{3}\cdot\frac{k}{n}}\, dt + \int_{-\sqrt{n}/a}^{\sqrt{n}/a} |\zeta_4(t)|\, dt. \tag{45}$$

Simple calculations show that

$$\begin{aligned}
&\int_{-\sqrt{n}/a}^{\sqrt{n}/a} (|R_{n1}(t)| + |R_{n2}(t)|)\, |t|^{-1}\, dt \\
\le\ & \frac{3(2m-1)^m}{\sigma_n m}\, \beta_n \int_{-\sqrt{n}/a}^{\sqrt{n}/a} \frac{dt}{2+|t|^{p+\varepsilon}} \\
&\quad + 9\cdot 2^{2m}\, m^{mp}\, ((m+1)!)^{p-1}\, \sigma_n^{-p}\, \gamma_n(p) \int_{-\sqrt{n}/a}^{\sqrt{n}/a} \frac{|t|^{p-1}\, dt}{2+|t|^{p+\varepsilon}} \\
\le\ & \frac{9(2m-1)}{\sigma_n m}\, (p+\varepsilon-1)^{-1}\, \beta_n \\
&\quad + 36\cdot 2^{2m}\, m^{mp}\, ((m+1)!)^{p-1}\, \varepsilon^{-1}\, \sigma_n^{-p}\cdot \gamma_n(p),
\end{aligned}$$

where $1 \le p < 2$, and $0 < \varepsilon < 2-p$.

In order to estimate other integrals in (45), we shall use the following inequality

$$e^{-\frac{t^2}{3n}\,(k-1)} \le e\cdot e^{-\frac{1}{3}\,|t|^{2-p-\varepsilon}},$$

which holds for all $|t| \le \sqrt{n}/a$, and k from (18).

Write the following estimates

$$\begin{aligned}
&\int_{-\sqrt{n}/a}^{\sqrt{n}/a} (|\zeta_1(t)| + |\zeta_2(t)|)\, dt \le 2^{2m+1}\, m^{2m}\, (m+1)!\, \sigma_n^{-3} \int_{-\sqrt{n}/a}^{\sqrt{n}/a} dt\cdot t^2\, e^{-\frac{t^2}{3n}(k-1)} \\
&+2^{m+1}\, m^{m-1}\, ((m+1)!)^{1/2} 3^{m/2}\, \sigma_n^{-5/2}\, \beta_n^{1/2}\, \gamma_n^{1/2}(2) \int_{-\sqrt{n}/a}^{\sqrt{n}/a} dt\cdot |t|^{3/2}\, e^{-\frac{t^2}{3n}(k-1)} \\
&\le \frac{e}{(2-p-\varepsilon)}\cdot 3^{\frac{3}{2-p-\varepsilon}}\cdot \Gamma\Big(\frac{3}{2-p-\varepsilon}\Big)\, 2^{2m+2}\, m^{2m}\, (m+1)!\, \sigma_n^{-3}\cdot \gamma_n(2) \\
&\qquad + \frac{e}{(2-p-\varepsilon)}\cdot 3^{\frac{m}{2}+\frac{5}{2(2-p-\varepsilon)}}\cdot \Gamma\Big(\frac{5}{2\,(2-p-\varepsilon)}\Big) \\
&\qquad \times 2^{m+2}\, m^{m-1}\, ((m+1)!)^{1/2}\, \sigma_n^{-5/2}\cdot \beta_n^{1/2}\, \gamma_n^{1/2}(2),
\end{aligned}$$

where $\Gamma(z)$ is the gamma function;

$$\begin{aligned}
&\int_{-\sqrt{n}/a}^{\sqrt{n}/a} dt\, (|R_{n3}(t)| + |R_{n4}(t)|) \left|\varphi\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\right|^k \le \frac{3^m\, e}{\sigma_n m}\, \beta_n \int_{-\infty}^{\infty} dt\, |t|\, e^{-\frac{1}{3}|t|^{2-p-\varepsilon}} \\
&\qquad +2^{2m+1}\, m^{2m-1}\, (m+1)!\, e\, \sigma_n^{-2}\, \gamma_n(2) \int_{-\infty}^{\infty} dt\, t^2\, e^{-\frac{1}{3}|t|^{2-p-\varepsilon}}
\end{aligned}$$

$$\leq \frac{2\,e\,3^m}{\sigma_n m}\,3^{\frac{2}{2-p-\varepsilon}}\,\Gamma\Big(\frac{2}{2-p-\varepsilon}\Big)\,\beta_n$$

$$+\frac{2^{2m+2}\,m^{2m}\,(m+1)!\,e}{(2-p-\varepsilon)}\,3^{\frac{3}{2-p-\varepsilon}}\cdot\Gamma\Big(\frac{3}{2-p-\varepsilon}\Big)\,\sigma_n^{-2}\cdot\gamma_n(2);$$

$$\int_{-\sqrt{n}/a}^{\sqrt{n}/a}|\zeta_4(t)|\,dt \leq \frac{1}{\sigma_n m}\,2^m\,e^{m/3}\,\beta_n\int_{-\sqrt{n}/a}^{\sqrt{n}/a}dt\,e^{-\frac{t^2}{3}}$$

$$+\frac{1}{\sigma_n m}\,e^{m/3}\sum_{c=2}^{m}\binom{m}{c}\,\sigma_n^{-c}\Big\{\int_{-\sqrt{n}/a}^{\sqrt{n}/a}dt\,|t|^c\,e^{-\frac{t^2}{3}}\Big\}E|g_c^n|\prod_{s=1}^{c}|g_1^n(X_s)|^{2/c}\cdot\frac{1}{\sqrt{n}}$$

$$+\frac{2^{m/2}\,e^{m/3}}{\sigma_n m}\cdot\beta_n^{1/2}\sum_{c=2}^{m}\binom{m}{c}\,\sigma_n^{-c}\Big\{\int_{-\sqrt{n}/a}^{\sqrt{n}/a}dt\,|t|^c\,e^{-\frac{t^2}{3}}\Big\}c\cdot(1+2^{m/2})^{c-1}\,(n^{-c+1}\,E|g_c^n|^2)^{1/2}$$

$$\leq \frac{1}{\sigma_n m}\,2^m\,e^{m/3}\,3\sqrt{3\pi}\,\beta_n+\frac{1}{\sigma_n m}\,e^{m/3}\sum_{c=2}^{m}\binom{m}{c}\,\sigma_n^{-c}\,3^{(c+1)/2}\,\Gamma\Big(c+\frac{1}{2}\Big)$$

$$\times E|g_c^n|\prod_{s=1}^{c}|g_1^n(X_s)|^{2/c}\cdot\frac{1}{\sqrt{n}}+\frac{1}{\sigma_n m}\,2^{m/2}\,e^{m/3}\,\beta_n^{1/2}\sum_{c=2}^{m}\binom{m}{c}\,\sigma_n^{-c}\,3^{(c+1)/2}\,\Gamma\Big(c+\frac{1}{2}\Big)$$

$$\times\, c\,(1+2^{m/2})^{c-1}\cdot(n^{-c+1}\,E|g_c^n|^2)^{1/2}\leq\frac{3\sqrt{3\pi}}{\sigma_n m}\cdot 2^m\,e^{m/3}\,\beta_n$$

$$+\frac{\sqrt{\pi}}{\sigma_n m}\,e^{m/3}\,3^{(m+1)/2}\,m!\sum_{c=2}^{m}\binom{m}{c}\,\sigma_n^{-c}\,E|g_c^n|\prod_{s=1}^{c}|g_1^n(X_s)|^{2/c}\cdot\frac{1}{\sqrt{n}}$$

$$+\sigma_n^{-1}\,\sqrt{\pi}\,m!\,6^{m/2}\,e^{m/3}(1+\sigma_n^{-2}\,(1+2^{m/2})^2)^{m/2}\,\beta_n^{1/2}\,\gamma_n^{1/2}(2),$$

since $2\leq c\leq m$, $\Gamma(c+\frac{1}{2})\leq\sqrt{\pi}\,c!$, and

$$\sum_{c=2}^{m}\binom{m}{c}\,\sigma_n^{-c}\,3^{(c+1)/2}\,\Gamma\Big(c+\frac{1}{2}\Big)\,c\,(1+2^{m/2})^{c-1}\,(n^{-c+1}\,E|g_c^n|^2)^{1/2}$$

$$\leq\sqrt{\pi}\,m!\,m\,3^{m/2}\,(1+\sigma_n^{-2}(1+2^{m/2})^2)^{m/2}\,\gamma_n^{1/2}(2).$$

Thus, the estimation of all the integrals on the right-hand side of (45), leads to the inequality

$$\int_{-\sqrt{n}/a}^{\sqrt{n}/a}\Big|\varphi_n(t)-\varphi^n\Big(\frac{t}{\sigma_n\sqrt{n}}\Big)\Big|\,|t|^{-1}\,dt\leq\beta_n\Big\{\frac{9\,(2m-1)}{\sigma_n m}\,(p+\varepsilon-1)^{-1}$$

$$
\begin{aligned}
&\qquad + \frac{1}{\sigma_n m}\, 2^m\, e^{m/3}\, 3\sqrt{3\pi} + \frac{2\, e\cdot 3^m}{\sigma_n m}\, 3^{\frac{2}{2-p-\varepsilon}}\, \Gamma\Big(\frac{2}{2-p-\varepsilon}\Big)\Big\} \\
&+ \quad \gamma_n(p)\,\{36\cdot 2^{2m}\, m^{mp}\, ((m+1)!)^{p-1}\cdot\varepsilon^{-1}\cdot\sigma_n^{-p}\} \\
&+ \quad \gamma_n(2)\,\Big\{\frac{e}{(2-p-\varepsilon)}\, 3^{\frac{3}{2-p-\varepsilon}}\, \Gamma\Big(\frac{3}{2-p-\varepsilon}\Big)\, 2^{2m+2}\, m^{2m}\, (m+1)! \\
&\qquad + \frac{e}{(2-p-\varepsilon)}\cdot 2^{2m+2}\, m^{2m}\, (m+1)!\, 3^{\frac{3}{2-p-\varepsilon}}\, \Gamma\Big(\frac{3}{2-p-\varepsilon}\Big)\cdot\sigma_n^{-2}\Big\} \\
&+ \quad \beta_n^{1/2}\cdot\gamma_n^{1/2}(2)\,\Big\{\sigma_n^{-5/2}\, \frac{e}{(2-p-\varepsilon)}\cdot 3^{\frac{m}{2}+\frac{5}{2(2-p-\varepsilon)}} \\
&\qquad\qquad \times\Gamma\Big(\frac{5}{2(2-p-\varepsilon)}\Big)\cdot 2^{m+2}\, m^{m-1}\, ((m+1)!)^{1/2} \\
&\qquad\qquad + \sigma_n^{-1}\sqrt{\pi}\, 6^{m/2}\, e^{m/3}\cdot m!\, (1+\sigma_n^{-2}(1+2^{m/2})^2)^{m/2}\Big\} \\
&+ \quad \frac{\sqrt{\pi}}{\sigma_n m}\, 2^{m/2}\, e^{m/3}\, 3^{(m+1)/2}\, m! \sum_{c=2}^{m}\binom{m}{c} E|g_c^n| \prod_{s=1}^{c} |g_1^n(X_s)|^{2/c}\cdot\frac{1}{\sqrt{n}}. \qquad (46)
\end{aligned}
$$

The estimation of σ_n^2. According to the definition

$$
\sigma_n^2 = Eg_{n1}^2(X_1) + 2\sum_{c=2}^{m}\binom{m}{c}\frac{c}{m}\, Eg_{n1}(X_1)h_{nc}(X_1) + E\Big(\sum_{c=2}^{m}\binom{m}{c}\frac{c}{m}\, h_{nc}(X_1)\Big)^2.
$$

Here

$$
Eg_{n1}^2 = 1 - Eg_1^2\, I(|g| > \sqrt{n}) - (Eg_1\, I(|g_1| > \sqrt{n}))^2;
$$

$$
|Eg_{n1}(X_1)\, h_{nc}(X_1)| \le \sqrt{n}\, E|h_{nc}| \le 2\sqrt{n}\cdot E|g_c|\, I(|g_c| > n^{(2c-1)/2});
$$

$$
\begin{aligned}
E\Big(\sum_{c=2}^{m}\binom{m}{c}\frac{c}{m}\, h_{nc}(X_1)\Big)^2 &\le 2\sqrt{n}\cdot\Big(\sum_{c=2}^{m}\binom{m}{c}\frac{c}{m}\Big)\cdot\Big(\sum_{c=2}^{m}\binom{m}{c}\frac{c}{m}\, E|h_{nc}|\Big) \\
&\le 2^{m+1}\cdot\sum_{c=2}^{m}\binom{m}{c}\{\sqrt{n}\cdot E|g_c|\, I(|g_c| > n^{(2c-1)/2})\} = 2^{m+1}\cdot\beta_n.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sigma_n^2 &\ \ge\ 1 - 2\, Eg_1^2\, I(|g_1| > \sqrt{n}) - 2\,\beta_n, \\
\sigma_n^2 &\ \le\ 1 - Eg_1^2\, I(|g_1| > \sqrt{n}) + 2\,(2^m+1)\,\beta_n. \qquad (47)
\end{aligned}
$$

This implies

$$
|\sigma_n^2 - 1| \le \alpha,
$$

where $\alpha = 2 \cdot Eg_1^2\, I(|g_1| > \sqrt{n}) + 2\,(2^m+1)\,\beta_n$. One of the two cases may occur, $2\alpha \le 1$, and $2\alpha \ge 1$. If it is $2\alpha \ge 1$, then the needed inequality (3) is obvious, since $\Delta_n \le 1 \le 2\alpha$. In the case of $2\alpha \le 1$ we can use (10) and (46). First note that

$$\frac{1}{\sigma_n^2} \le \frac{1}{1-\alpha} \le 1 + 2\alpha \le 2,$$

and therefore, in (10)

$$\sigma_n^{-1}\,|1-\sigma_n^2| \le 4\,Eg_1^2\,I(|g_1| > \sqrt{n}) + 4\,(2^m+1)\,\beta_n. \tag{48}$$

Join estimates (10), (13), (16), (46) and (48). As a result, we get for any $1 \le p < 2$ and $0 < \varepsilon < 2-p$ the following inequality

$$\begin{aligned}
\Delta_n \quad \le \quad & \Big(28 \cdot \sqrt{\frac{6e}{\pi}} + \frac{24 \cdot 16}{\pi\sqrt{\pi}}\Big)\, E|g_1^n(X_1)|^3 \cdot \frac{1}{\sqrt{n}} \\
& + \Big(1 + \frac{1}{\sqrt{2\pi}} + \frac{4}{\sqrt{2\pi e}}\Big)\, Eg_1^2\, I(|g_1| > \sqrt{n}) \\
& + \Big(3 + \frac{1}{\sqrt{2\pi e}}\, 4(2^m+1) + 7 \cdot \sqrt{\frac{6e}{\pi}} \cdot 2^{2m+5} + 18 \cdot \sqrt{2} \cdot (p+\varepsilon-1)^{-1} \\
& \qquad + 2^m\, e^{m/3}\, 3\sqrt{6\pi} + 2\sqrt{2}\, e \cdot 3^{m+2/(2-p-\varepsilon)} \cdot \Gamma\Big(\frac{2}{2-p-\varepsilon}\Big)\Big) \cdot \beta_n \\
& + (3^2 \cdot 2^{2m+2+p/2} \cdot m^{mp}\, ((m+1)!)^{p-1} \cdot \varepsilon^{-1}) \cdot \gamma_n(p) \\
& + \Big(\frac{e}{2-p-\varepsilon} \cdot 3^{\frac{3}{2-p-\varepsilon}+1} \cdot \Gamma\Big(\frac{3}{2-p-\varepsilon}\Big) \cdot 2^{2m+2} \cdot m^{2m}\, (m+1)!\Big) \cdot \gamma_n(2) \\
& + \Big(\frac{\sqrt{2}\cdot e}{2-p-\varepsilon}\, 3^{\frac{m}{2}+\frac{5}{2(2-p-\varepsilon)}} \cdot \Gamma\Big(\frac{5}{2(2-p-\varepsilon)}\Big) \cdot 2^{m+3} \cdot m^{m-1}\, ((m+1)!)^{1/2} \\
& \qquad + \sqrt{2\pi}\, 6^{m/2}\, e^{m/3}\, m!\, (1 + 2(1+2^{m/2})^2)^{m/2}\Big) \cdot \beta_n^{1/2} \cdot \gamma_n^{1/2}(2) \\
& + \sqrt{2\pi}\, 2^{m/2}\, e^{m/3}\, 3^{(m+1)/2} \cdot m! \sum_{c=2}^{m} \binom{m}{c} E|g_c^n| \prod_{s=1}^{c} |g_1^n(X_s)|^{2/c} \cdot \frac{1}{\sqrt{n}}.
\end{aligned} \tag{49}$$

Note that in (49) $\gamma_n(2) \le \gamma_n(p)$ and $\beta_n^{1/2}\gamma_n^{1/2}(2) \le (\beta_n + \gamma_n(2))/2$. Then, letting $\varepsilon = (2-p)/2$ in (49) and simplifying some constants on the right-hand side of this inequality, we get the proof of Theorem 1.

In what follows we shall apply the inequality (3).

Proof of Corollary 1. First we estimate the value of β_n. Consider

$$\begin{aligned}
\sqrt{n}E|g_c|I(|g_c| > n^{(2c-1)/2}) &= n^c P(|g_c| > n^{(2c-1)/2}) + \sqrt{n}\int_{n^{(2c-1)/2}}^{\infty} P(|g_c| > x)dx \\
&\le B_c(n^{(2c-1)/2}, \gamma_c) + \sqrt{n}B_c(n^{(2c-1)/2}, \gamma_c)\int_{n^{(2c-1)/2}}^{\infty} x^{-\gamma_c}dx \\
&= 2cB_c(n^{(2c-1)/2}, \gamma_c).
\end{aligned} \tag{50}$$

Therefore,

$$\beta_n \le 2\sum_{c=2}^{m}\binom{m}{c} cB_c(n^{(2c-1)/2},\gamma_c) \to 0,$$

when $n \to \infty$. Further, the value $\gamma_n(p)$ has to be estimated. Let $4/3 < p \le 2$. Denote

$$b_n(c,p) := n^{(2c-p(2c-1))/2} E|g_c|^p I(|g_c| \le n^{(2c-1)/2}) \tag{51}$$

for $2 \le c \le m$.

Estimate $b_n(c,p)$**.** For this purpose consider a constant α_c, $0 < \alpha_c < (2c-1)/2$, and

$$\int_0^{n^{(2c-1)/2}} x^p dP(|g_c| \le x) = -\int_0^{n^{(2c-1)/2}} x^p dP(|g_c| > x)$$

$$\begin{aligned}
&= -n^{p(2c-1)/2} P(|g_c| > n^{(2c-1)/2}) + \int_0^{n^{(2c-1)/2}} x^{p-1} P(|g_c| > x)dx \\
&\le p\int_0^{n^{(2c-1)/2}} x^{p-1} P(|g_c| > x)dx \\
&= p\int_0^{n^{\alpha_c}} x^{p-1} P(|g_c| > x)dx + p\int_{n^{\alpha_c}}^{n^{(2c-1)/2}} x^{p-\gamma_c-1}(x^{\gamma_c} P(|g_c| > x))dx \\
&\le B_c(0,\gamma_c)p\int_0^{n^{\alpha_c}} x^{p-\gamma_c-1}dx + B_c(n^{\alpha_c},\gamma_c)p\int_{n^{\alpha_c}}^{n^{(2c-1)/2}} x^{p-\gamma_c-1}dx \\
&\le \frac{p}{p-\gamma_c} B_c(0,\gamma_c) n^{\alpha_c(p-\gamma_c)} + \frac{p}{p-\gamma_c} B_c(n^{\alpha_c},\gamma_c) n^{\frac{(2c-1)}{2}(p-\gamma_c)}.
\end{aligned}$$

Therefore, for $\alpha_c = (2c-1)/4$ the following inequality is true

$$b_n(c,p) \le \frac{p}{p-\gamma_c} B_c(0,\gamma_c) \cdot n^{-\frac{2c-1}{4}(p-\gamma_c)} + \frac{p}{p-\gamma_c} B_c(n^{(2c-1)/4},\gamma_c), \tag{52}$$

where $2 \le c \le m$, $4/3 < p \le 2$. Consequently,

$$\gamma_n(p) \to 0, \quad n \to \infty.$$

This proves Corollary 1.

Proof of Theorem 2. Under the conditions of the theorem we estimate the expression on the right-hand side of (3). First note that

$$\begin{aligned}
Eg_1^2 I(|g_1| > \sqrt{n}) &\le \frac{3}{\sqrt{n}} B_1(\sqrt{n},\delta_1) \\
&\le \frac{3}{\sqrt{n}} B_1(0,\delta_1).
\end{aligned} \tag{53}$$

Analogously to (50) it can be deduced that

$$\begin{aligned}\sqrt{n}E|g_c|I(|g_c| > n^{(2c-1)/2}) &\le \frac{2c+1}{2}B_c(n^{(2c-1)/2},\delta_c)\frac{1}{\sqrt{n}} \\ &\le \frac{2c+1}{2}B_c(0,\delta_c)\frac{1}{\sqrt{n}}. \qquad (54)\end{aligned}$$

Moreover

$$\beta_n \le \frac{2m+1}{2}\sum_{c=2}^{m}\binom{m}{c}B_c(0,\delta_c)\frac{1}{\sqrt{n}}. \qquad (55)$$

Consider the value of $b_n(c,p)$ in the formula (51) when $5/3 < p \le 2$ and $c = 2,\dots,m$. For this purpose it can be written analogously to (52)

$$\begin{aligned}E|g_c|^pI(|g_c| \le n^{(2c-1)/2}) &= -n^{p(2c-1)/2}P(|g_c| > n^{(2c-1)/2}) \\ +p\int_0^{n^{(2c-1)/2}} x^{p-1}P(|g_c|>x)dx &\le p\int_0^{n^{(2c-1)/2}} x^{p-1}P(|g_c|>x)dx \\ = p\int_0^{n^{(2c-1)/2}} x^{p-\delta_c-1}(x^{\delta_c}P(|g_c|>x))dx &\le B_c(0,\delta_c)p\int_0^{n^{(2c-1)/2}} x^{p-\delta_c-1}dx \\ &= \frac{p}{p-\delta_c}B_c(0,\delta_c)n^{\frac{2c-1}{2}(p-\delta_c)} \qquad (56)\end{aligned}$$

for all $p > \delta_c$.

Consequently, for $5/3 < p \le 2$ and $c = 2,3,\dots,m$

$$b_n(c,p) \le \frac{p}{p-5/3}B_c(0,\delta_c)\frac{1}{\sqrt{n}}. \qquad (57)$$

Therefore

$$\gamma_n(p) \le \frac{p}{p-5/3}\sum_{c=2}^{m}\binom{m}{c}B_c(0,\delta_c)\frac{1}{\sqrt{n}}. \qquad (58)$$

Put $p = 15/8$ in (3). Then (4) can be obtained from (53), (55) and (58). Note for $2 \le c \le m$ by Hölder inequality

$$E|g_c|\prod_{s=1}^{c}|g_1(X_s)|^{2/c} \le \left(E|g_c|^{\frac{3c}{3c-2}}\right)^{\frac{3c-2}{3c}}\left(E|g_1|^3\right)^{\frac{2}{3}} \qquad (59)$$

where

$$E|g_c|^{\frac{3c}{3c-2}} \le 1 + \frac{3c(2c-1)}{2(c-1)}B_c(0,\delta_c), \quad \delta_c = \frac{3c}{3c-2} + \frac{2(c-1)}{(2c-1)(3c-2)}. \qquad (60)$$

Proof of Corollary 2. In well-known inequality

$$a\cdot b \le p^{-1}a^p + q^{-1}b^q, \quad a \ge 0, \quad b \ge 0, \quad p^{-1}+q^{-1} = 1$$

we take for $c = 1, \dots, m$

$$a = |g_c|, \quad b = \prod_{s=1}^{c} |g_1(X_s)|^{2/c}, \quad p = \delta_c = \frac{2c+1}{2c-1}, \quad q = \frac{\delta_c}{\delta_c - 1} = \frac{2c+1}{2}.$$

Then we have

$$E\,|g_c| \prod_{s=1}^{c} |g_1(X_s)|^{2/c} \le \frac{2c-1}{2c+1}\,E\,|g_c|^{\delta_c} + \frac{2}{2c+1}(E\,|g_c|^{(2c+1)/c})^c.$$

Here by Hölder inequality

$$E\,|g_1|^{(2c+1)/c} = E\,|g_1|^{\frac{3}{c}} \cdot |g_1|^{\frac{2(c-1)}{c}} \le$$

$$\le (E\,|g_1|^3)^{1/c}(E\,|g_1|^2)^{\frac{c-1}{c}} = (E\,|g_1|^3)^{1/c}.$$

Therefore for $c = 1, \dots, m$

$$E\,|g_c| \prod_{s=1}^{c} |g_1(X_s)|^{2/c} \le \frac{2c-1}{2c+1} E\,|g_c|^{\delta_c} + \frac{2}{2c+1} \cdot E\,|g_1|^3.$$

This proves Corollary 2.

Proof of Theorem 3. According to (54)

$$\beta_n \le \frac{2m+1}{2} \sum_{c=2}^{m} \binom{m}{c} B_c(n^{(2c-1)/2}, \delta_c) \frac{1}{\sqrt{n}}. \tag{61}$$

Estimate $\gamma_n(p)$ for $5/3 < p \le 2$. Analogously to the deduction of (52) and (57) the following inequality can be obtained

$$b_n(c,p) \le \frac{p}{p-\delta_c} B_c(0,\delta_c) n^{\alpha_c(p-\delta_c)+(2c-p(2c-1))/2}$$

$$+ \frac{p}{p-\delta_c} B_c(n^{\alpha_c}, \delta_c) \frac{1}{\sqrt{n}},$$

which is true for all $5/3 < p \le 2$, $c = 2, \dots, m$, and every α_c such that $0 < \alpha_c < (2c-1)/2$. Assume $\alpha_c = (2c-1)/4$. Taking into account that $1 < \delta_c \le 5/3$ and $B_c(n^{\alpha_c}, \delta_c) \le B_c(\sqrt{n}, \delta_c)$ the following value can be estimated

$$b_n(c,p) \le \frac{p}{p-5/3}\{B_c(0,\delta_c) n^{-\frac{1}{4}(3p-5)} + B_c(\sqrt{n}, \delta_c)\} \frac{1}{\sqrt{n}}, \tag{62}$$

where $5/3 < p \le 2$ and $c = 2, \dots, m$.

Therefore, for all $5/3 < p \le 2$

$$\gamma_n(p) \le \frac{p}{p-5/3} \sum_{c=2}^{m} \binom{m}{c} \{B_c(0,\delta_c) n^{-\frac{1}{4}(3p-5)} + B_c(\sqrt{n}, \delta_c)\} \frac{1}{\sqrt{n}}. \tag{63}$$

In (3) we can put $p = 35/18$. Note that for every $x > 0$

$$B_c(0, \delta_c) \le x^{\delta_c} + B_c(x, \delta_c). \tag{64}$$

Assume $x = n^{1/8}$. Then from (63) and (64) it can be deduced

$$\begin{aligned}\gamma_n(35/18) &\le 7\sum_{c=2}^{m}\binom{m}{c}\{(n^{\delta_c/8}+B_c(n^{1/8},\delta_c))n^{-5/32}+B_c(\sqrt{n},\delta_c)\}\frac{1}{\sqrt{n}}\\ &\le 7\cdot 2^m E|g_1|^3\frac{1}{\sqrt{n}}+14\sum_{c=2}^{m}\binom{m}{c}B_c(n^{1/8},\delta_c)\frac{1}{\sqrt{n}}, \qquad (65)\end{aligned}$$

because $1 \le E|g_1|^3$ and $B_c(\sqrt{n},\delta_c) \le B_c(n^{1/8},\delta_c)$. As a result, from (3), (53), (61) and (65) the inequality (5) can be obtained.

Proof of Theorem 4. To refine the estimate for $\zeta_4(t)$ given by (43). According to (36) the following representation of $\zeta_4(t)$ can be written

$$\zeta_4(t) = \frac{(m-1)\sqrt{n}}{2\sigma_n}\cdot\varphi^{n-2}(\frac{t}{\sigma_n\sqrt{n}})E\{g_{n2}(X_1,X_2)\prod_{s=1}^{2}(e^{it\xi_s}-1)\}+R_{n6}(t), \qquad (66)$$

where

$$R_{n6}(t) = \frac{\sqrt{n}}{\sigma_n m}\sum_{c=3}^{m}\binom{m}{c}\varphi^{n-c}\left(\frac{t}{\sigma_n\sqrt{n}}\right)E\{g_{nc}(X_1,\dots,X_c)\prod_{s=1}^{c}(e^{it\xi_s}-1)\}.$$

First $R_{n6}(t)$ has to be estimated. According to the estimate $|e^{ix}-1| \le |x|$, $x \in R$, it can be written

$$|E\{g_{nc}(X_1,\dots,X_c)\prod_{s=1}^{c}(e^{it\xi_s}-1)\}| \le |t|^c\sigma_n^{-c}n^{-c/2}E|g_{nc}|\prod_{s=1}^{c}|Z_s|, \qquad (67)$$

where

$$Z_s = g_{n1}(X_s)+\sum_{c=2}^{m}\binom{m}{c}\frac{c}{m}h_{nc}(X_s).$$

According to Hölder's inequality it can be written as

$$E|g_{nc}|\prod_{s=1}^{c}|Z_s| \le \left(E|g_{nc}|^{3/2}\right)^{2/3}\left(E|Z_1|^3\right)^{c/3}.$$

Reconciling (17) with (61)

$$\begin{aligned}E|Z_1|^3 &\le 8E|g_1^n|^3+2^{2m+6}\sqrt{n}\beta_n\\ &\le 8E|g_1|^3+(2m+1)2^{2m+5}\sum_{c=2}^{m}\binom{m}{c}B_c(n^{(2c-1)/2},\delta_c),\end{aligned}$$

and, then, according to (26) can be obtained:

$$E|g_{nc}|^{3/2} \le ((c+1)!)^{1/2}\cdot 2^c\cdot E|g_c|^{3/2}I(|g_c| \le n^{(2c-1)/2}).$$

If $3 \le c \le m$, then $\delta_c < 3/2$. Therefore (56) can be applied assuming $p = 3/2$. Hence

$$E|g_{nc}|^{3/2} \le ((c+1)!)^{1/2}\frac{2^c\cdot 3(2c-1)}{2c-5}B_c(0,\delta_c)n^{(2c-5)/4}.$$

Consequently,

$$|R_{n6}(t)| \le \{\sum_{c=3}^{m}\binom{m}{c} c!\, 2^{c+1}(|t|^c \cdot |\varphi(\frac{t}{\sigma_n\sqrt{n}})|)\sigma_n^{-c-1}B_c^{2/3}(0,\delta_c)$$

$$\times(8E|g_1|^3 + (2m+1)2^{2m+5}\sum_{d=2}^{m}\binom{m}{d}B_d(n^{(2d-1)/2},\delta_d))^{c/3}n^{-\frac{c-1}{6}}\}\frac{1}{\sqrt{n}}. \qquad (68)$$

To make an estimate for the first term on the right-hand side of (66) we assume $c = 2$ in (37) and (38). Hence,

$$|E\{g_{n2}(X_1,X_2)\prod_{s=1}^{2}(e^{it\xi_s}-1)\}|$$

$$\le |E\{g_2^n(X_1,X_2)\prod_{s=1}^{2}(e^{\frac{it}{\sigma_n\sqrt{n}}(g_1^n(X_s)+\bar{h}^n(X_s))}-1)\}| + 4E|g_2|I(|g_2|>n^{\frac{3}{2}}).$$

According to (39) when $c = 2$

$$|E\{g_2^n(X_1,X_2)\prod_{s=1}^{2}(e^{\frac{it}{\sigma_n\sqrt{n}}(g_1^n(X_s)+\bar{h}^n(X_s))}-1)\}|$$

$$\le t^2n^{-1}\sigma_n^{-2}E|g_2^n||g_1^n(X_1)||g_1^n(X_2)| + 2t^2n^{-1}\sigma_n^{-2}E|g_2^n||\bar{h}^n(X_1)||g_1^n(X_2)|$$

$$+t^2n^{-1}\sigma_n^{-2}E|g_2^n||\bar{h}^n(X_1)||\bar{h}^n(X_2)|.$$

Applying (43), Hölder's inequality and the equality $Eg_1^2 = 1$ we can obtain

$$E|g_2^n||\bar{h}^n(X_1)||g_1^n(X_2)| \le \left(E|g_2^n|^2\right)^{1/2} 2^{m/2}\beta_n^{1/2},$$

$$E|g_2^n||\bar{h}^n(X_1)||h^n(X_2)| \le \left(E|g_2^n|^2\right)^{1/2} 2^{m}\beta_n.$$

Combining these inequalities with (66) for all $t \in R$ it appears that

$$|\zeta_4(t)| \le \frac{(m-1)}{2\sqrt{n}}\sigma_n^{-3}t^2\left|\varphi\left(\frac{t}{\sigma_n\sqrt{n}}\right)\right|^{n-2} E|g_2^n||g_1^n(X_1)||g_1^n(X_2)|$$

$$+\frac{(m-1)2^{m/2}}{\sqrt{n}}\sigma_n^{-3}t^2\left|\varphi\left(\frac{t}{\sigma_n\sqrt{n}}\right)\right|^{n-2}\left(E|g_2^n|^2\right)^{1/2}\beta_n^{1/2}$$

$$+\frac{(m-1)2^{m-1}}{\sqrt{n}}\sigma_n^{-3}t^2\left|\varphi\left(\frac{t}{\sigma_n\sqrt{n}}\right)\right|^{n-2}\left(E|g_2^n|^2\right)^{1/2}\beta_n$$

$$+\frac{2(m-1)}{\sigma_n}\left|\varphi\left(\frac{t}{\sigma_n\sqrt{n}}\right)\right|^{n-2}\{\sqrt{n}E|g_2|I(|g_2|>n^{3/2})\} + R_{n6}(t), \qquad (69)$$

where $R_{n6}(t)$ satisfies (68).

Referring to (59) when $c = 2$

$$E|g_2||g_1(X_1)||g_1(X_2)| \le (1 + 9B_2(0, 5/3))^{2/3} \cdot (E|g_1|^3)^{2/3}. \tag{70}$$

The inequality (6) is obtained as a result of the application of (69) to (45).

Proof of Theorem 5. We denote

$$\Phi_n(x) = F(x) + \frac{\kappa_3}{6} \cdot \frac{1}{\sqrt{2\pi n}}(1 - x^2)e^{-\frac{x^2}{2}}, \quad x \in R.$$

It can be verified that

$$\phi_n(t) = \int_{-\infty}^{\infty} e^{itx} d\Phi_n(x) = e^{-\frac{t^2}{2}}(1 + \frac{\kappa_3}{6}(it)^3 \frac{1}{\sqrt{n}}), \quad t \in R.$$

Employing Esseen's inequality

$$\sup_x |F_n(x) - \Phi_n(x)| \le \int_{-\sqrt{n}\ln n}^{\sqrt{n}\ln n} |\varphi_n(t) - \phi_n(t)||t|^{-1} dt + c(1 + |\kappa_3|)(\sqrt{n}\ln n)^{-1}, \tag{71}$$

where $c > 0$ is an absolute constant.

Estimate the integral in formula (71) applying (18) with $p = 15/8$ and $\varepsilon = 1/16$. First represent this integral as a sum

$$\int_{-\sqrt{n}\ln n}^{\sqrt{n}\ln n} |\varphi_n(t) - \phi_n(t)||t|^{-1} dt = I_n + J_n, \tag{72}$$

where

$$I_n = \int_{-\sqrt{n}/a}^{\sqrt{n}/a} |\varphi_n(t) - \phi_n(t)||t|^{-1} dt,$$
$$J_n = \int_{\frac{\sqrt{n}}{a} \le |t| \le \sqrt{n}\ln n} |\varphi_n(t) - \phi_n(t)||t|^{-1} dt.$$

Estimate of J_n. According to the definition $a = E|Z_1|^3 \sigma_n^{-3/2}$. Further, consider

$$E|g_{n1}|^3 - E|Z_1|^3 = E(|g_{n1}| - |Z_1|)(|g_{n1}|^2 + |g_{n1}||Z_1| + |Z_1|^2).$$

Taking into account the inequalities

$$|g_{n1}| \le 2\sqrt{n}, \quad |Z_1| \le 2\sqrt{n} \sum_{c=1}^{m} \binom{m}{c} \frac{c}{m},$$

and (54) we get

$$\begin{aligned} |E|g_{n1}|^3 - E|Z_1|^3| &\le 4n \sum_{c=2}^{m} \binom{m}{c} \frac{c}{m} E|h_{nc}(X_1)|(1 + \sum_{c=1}^{m} \binom{m}{c} \frac{c}{m})^2 \\ &\le 2^{2m+3} \sum_{c=2}^{m} \binom{m}{c} \{nE|g_c| I(|g_c| > n^{(2c-1)/2})\} \\ &\le 2^{2m+2} \sum_{c=2}^{m} \binom{m}{c} (2c+1) B_c(n^{(2c-1)/2}, \delta_c). \end{aligned} \tag{73}$$

Using analogous reasoning

$$|E|g_{n1}|^3 - E|g_1^n|^3| \leq 7E|g_1|^3 I(|g_1| > \sqrt{n}) \tag{74}$$

¿From (73), (74) and the formula

$$E|g_1^n|^3 = E|g_1|^3 - E|g_1|^3 I(|g_1| > \sqrt{n})$$

it follows that

$$-b_n + E|g_1|^3 \leq E|Z_1|^3 \leq E|g_1|^3 + b_n, \tag{75}$$

where

$$b_n = 2^{2m+2} \sum_{c=1}^{m} \binom{m}{c} (2c+1) B_c(n^{(2c-1)/2}, \delta_c).$$

Taking into account that $b_n \to 0$ as $n \to \infty$ the following inequality

$$b_n \leq \frac{1}{2} E|g_1|^3, \tag{76}$$

holds for sufficiently large n. Therefore, from (75) it can be obtained

$$\frac{1}{2} E|g_1|^3 \leq E|Z_1|^3 \leq E|g_1|^3 \tag{77}$$

for $n \to \infty$. Analogously to the deduction of (47) and using (76)

$$\sigma_n^2 \geq 1 - \frac{2}{\sqrt{n}} b_n \geq 1 - \frac{1}{\sqrt{n}} E|g_1|^3.$$

Consequently,

$$\sigma_n^2 \geq \frac{1}{2} \tag{78}$$

for sufficiently large n: $\sqrt{n} \geq 2E|g_1|^3$.

Further, consider $\varphi(\frac{t}{\sigma_n \sqrt{n}})$ in the domain $\frac{\sqrt{n}}{a} \leq |t| \leq \sqrt{n} \ln n$. The following representation can be written

$$\varphi(\frac{t}{\sigma_n \sqrt{n}}) = E \exp\left(\frac{itg_1(X_1)}{\sigma_n \sqrt{n}}\right)$$

$$+E \exp\left(\frac{itg_{n1}(X_1)}{\sigma_n \sqrt{n}}\right) \left\{ \exp\left(\frac{it}{\sigma_n \sqrt{n}} \sum_{c=2}^{m} \binom{m}{c} \frac{c}{m} h_{nc}(X_1)\right) - \exp\left(\frac{it\bar{g}_{n1}(X_1)}{\sigma_n \sqrt{n}}\right) \right\}$$

where $\bar{g}_{n1}(X_1) = g_1(X_1) I(|g_1(X_1)| > \sqrt{n}) - E g_1(X_1) I(|g_1(X_1)| > \sqrt{n})$. With the help of the inequality $|e^{ix} - 1| \leq |x|$, $x \in R$, it can be obtained

$$\begin{aligned} |\varphi(\frac{t}{\sigma_n \sqrt{n}})| &\leq |\chi(\frac{t}{\sigma_n \sqrt{n}})| + \frac{2|t|}{\sigma_n \sqrt{n}} E|g_1| I(|g_1| > \sqrt{n}) \\ &\quad + \frac{2|t|}{\sigma_n \sqrt{n}} \sum_{c=2}^{m} \binom{m}{c} \frac{c}{m} E|g_c| I(|g_c| > n^{(2c-1)/2}) \\ &\leq |\chi(\frac{t}{\sigma_n \sqrt{n}})| + \frac{2|t|}{\sigma_n \sqrt{n}} \frac{b_n}{n}, \end{aligned} \tag{79}$$

where $\chi(t) = E\exp(itg_1(X_1))$, $t \in R$. ¿From (76)–(79) it can be concluded that

$$\sup_{\frac{\sqrt{n}}{a} \le |t| \le \sqrt{n}\ln n} |\varphi(\frac{t}{\sigma_n\sqrt{n}})| \le \sup_{|t| \ge \frac{1}{2E|g_1|^3}} |\chi(t)| + \frac{\ln n}{\sqrt{n}}. \tag{80}$$

Further due to $p = 15/8$ and $\varepsilon = 1/16$ in (18) $k \ge n^{1/34}$ for $n \to \infty$. According to the Cramer condition of this theorem $\sup_{|t| \ge \frac{1}{2E|g_1|^3}} |\chi(t)| < 1$. Therefore for every $s \ge 1$

$$(\sup_{\frac{\sqrt{n}}{a} \le |t| \le \sqrt{n}\ln n} |\varphi(\frac{t}{\sigma_n\sqrt{n}})|)^k = O(n^{-s}) \tag{81}$$

according to (80) as $n \to \infty$. Furthermore, it is clear that

$$\begin{aligned} J_n &\le \int_{\frac{\sqrt{n}}{2E|g_1|^3} \le |t| \le \sqrt{n}\ln n} |\varphi_n(t) - \phi_n(t)||t|^{-1}dt \\ &\le \int_{\frac{\sqrt{n}}{2E|g_1|^3} \le |t| \le \sqrt{n}\ln n} |\varphi_n(t)||t|^{-1}dt + \int_{\frac{\sqrt{n}}{2E|g_1|^3} \le |t| \le \sqrt{n}\ln n} |\phi_n(t)||t|^{-1}dt. \end{aligned}$$

Due to (44) and (81)

$$\int_{\frac{\sqrt{n}}{2E|g_1|^3} \le |t| \le \sqrt{n}\ln n} |\varphi_n(t)||t|^{-1}dt = o(n^{-1/2}).$$

According to the property of function $\phi_n(t)$

$$\int_{\frac{\sqrt{n}}{2E|g_1|^3} \le |t| \le \sqrt{n}\ln n} |\phi_n(t)||t|^{-1}dt = o(n^{-1/2}).$$

Consequently,

$$J_n = o(n^{-1/2}). \tag{82}$$

Estimate of I_n. The function $\phi_n(t)$ can be represented as a sum $\phi_n(t) = \phi_{n1}(t) + \phi_{n2}(t)$, where

$$\phi_{n1}(t) = e^{-t^2/2}(1 + (it)^3\frac{1}{6} \cdot E\, g_1^3 \cdot \frac{1}{\sqrt{n}}),$$

$$\phi_{n2}(t) = e^{-t^2/2}(it)^3\frac{1}{2}(m-1)E(g_2(X_1, X_2)g_1(X_1)g_1(X_2)) \cdot \frac{1}{\sqrt{n}}.$$

Therefore, according to (44)

$$\varphi_n(t) - \phi_n(t) = \left(\varphi^n(\frac{t}{\sigma_n\sqrt{n}}) - \phi_{n1}(t)\right) + (it\zeta_4(t) - \phi_{n2}(t)) + R_{n7}(t)$$

in the domain $|t| \le \sqrt{n}/a$ where

$$R_{n7}(t) = R_{n1}(t) + R_{n2}(t) + it(\zeta_1(t) + \zeta_2(t)) + it\varphi^k(\frac{t}{\sigma_n\sqrt{n}})(R_{n3}(t) + R_{n4}(t)).$$

Therefore the following inequality is true

$$I_n \leq I_{n1} + I_{n2} + I_{n3} + I_{n4} \tag{83}$$

where

$$I_{n1} = \int_{-\frac{\sqrt{n}}{2\sqrt{b}}}^{\frac{\sqrt{n}}{2\sqrt{b}}} |\varphi^n(\frac{t}{\sigma_n\sqrt{n}}) - \phi_{n1}(t)||t|^{-1}dt, \quad I_{n2} = \int_{\frac{\sqrt{n}}{2\sqrt{b}} \leq |t| \leq \frac{\sqrt{n}}{a}} |\varphi^n(\frac{t}{\sigma_n\sqrt{n}}) - \phi_{n1}(t)||t|^{-1}dt,$$

$$I_{n3} = \int_{-\frac{\sqrt{n}}{a}}^{\frac{\sqrt{n}}{a}} |it\zeta_4 - \phi_{n2}(t)||t|^{-1}dt, \quad I_{n4} = \int_{-\frac{\sqrt{n}}{a}}^{\frac{\sqrt{n}}{a}} |R_{n7}(t)||t|^{-1}dt.$$

Here

$$b = \sigma_n^{-2}E|Z_1|^4.$$

Note that $a \leq \sqrt{b}$ is always true. Consider every summand on the right-hand side of (83). First have a look at I_{n1}. ¿From Bhattacharya and Ranga Rao (1986) it is known that the following inequality is true

$$\left|\varphi^n(\frac{t}{\sigma_n\sqrt{n}}) - \exp(-\frac{1}{2}t^2)\left(1 + \frac{(it)^3}{\sigma\sqrt{n}}EZ_1^3\right)\right| \leq \frac{b}{n}t^4(1+t^2)\exp\left(-\frac{t^2}{4}\right). \tag{84}$$

Here

$$\begin{aligned} E|Z_1|^4 &\leq 8\left(E|g_{n1}|^4 + E|\sum_{c=2}^{m}\binom{m}{c}\frac{c}{m}h_n(X_1)|^4\right) \\ &\leq 8(8E|g_1|^4 I(|g_1| \leq \sqrt{n})) + (m-1)^3\sum_{c=2}^{m}\binom{m}{c}^4 Eh_{nc}^4(X_1) \\ &\leq 64E|g_1|^4 I(|g_1| \leq \sqrt{n}) + 64(m-1)^3\sum_{c=2}^{2}\binom{m}{n}^4\{n^{3/2}E|g_c|I(|g_c| > n^{(2c-1)/2})\}, \end{aligned}$$

that is

$$\frac{1}{n}E|Z_1|^4 \leq 64n^{-1}E|g_1|^4 I(|g_1| \leq \sqrt{n}) + 64m^{4m}\beta_n.$$

Substituting (84) into I_{n1} and taking into account (75) and (78), it can be observed that $I_{n1} = o(n^{-1/2})$ as $n \to \infty$. To estimate I_{n2} it should be noted that

$$\frac{\sqrt{n}}{2\sqrt{b}} \geq \frac{n^{1/4}}{2\sqrt{3}\,2^m E|g_1|^3}.$$

According to (15) $I_{n2} = o(n^{-1/2})$, for $n \to \infty$. To estimate the I_{n3}, formula (66) is applied. Using (66) it can be written: $I_{n3} \leq I_{n3}^1 + I_{n3}^2$, where

$$I_{n3}^1 = \int_{-\frac{\sqrt{n}}{a}}^{\frac{\sqrt{n}}{a}} |itf_n(t) - \phi_{n2}(t)||t|^{-1}dt, \quad I_{n3}^2 = \int_{-\frac{\sqrt{n}}{a}}^{\frac{\sqrt{n}}{a}} |R_{n6}(t)|dt,$$

and

$$f_n(t) = \frac{(m-1)\sqrt{n}}{2\sigma_n}\varphi^{n-2}(\frac{t}{\sigma_n\sqrt{n}})E\{g_{n2}(X_1, X_2)\prod_{s=1}^{2}(e^{it\xi_s} - 1)\}.$$

Consequently, using (68) and taking (14) $I_{n3}^2 = o(n^{-1/2})$.

According to (15) integral I_{n3}^1 can be estimated in the following way

$$I_{n3}^1 \le \frac{(m-1)}{2} \int_{-\frac{\sqrt{n}}{a}}^{\frac{\sqrt{n}}{a}} |\delta_n(t)| \exp\left(-\frac{t^2}{2}\right) dt + o(n^{-1/2}),$$

where

$$\delta_n(t) = \frac{\sqrt{n}}{\sigma_n} E\left\{g_{n2}(X_1, X_2) \prod_{s=1}^{2} \left(e^{\frac{itZ_s}{\sigma_n\sqrt{n}}} - 1\right)\right\} - E\left(g_2(X_1, X_2)g_1(X_1)g_1(X_2)\right) \frac{(it)^2}{\sqrt{n}}.$$

To analyze $\delta_n(t)$, the formula

$$e^{ix} - 1 = ix(1 + \int_0^1 (e^{ixu} - 1)du)$$

for $x = tZ_s/(\sigma_n\sqrt{n})$, $s = 1, 2$, is applied. Therefore,

$$E\left\{g_{n2}(X_1, X_2) \prod_{s=1}^{2} \left(e^{\frac{itZ_s}{\sigma_n\sqrt{n}}} - 1\right)\right\} = \frac{(it)^2}{\sigma_n^2 n} E(g_{n2}(X_1, X_2)Z_1 \cdot Z_2)$$

$$+\frac{2(it)^2}{\sigma_n^2 n} \int_0^1 du E\left(g_{n2}(X_1, X_2)Z_1 \cdot Z_2 \left(e^{\frac{itZ_1}{\sigma_n\sqrt{n}}u} - 1\right)\right)$$

$$+\frac{(it)^2}{\sigma_n^2 n} \int_0^1 du_1 \int_0^1 du_2 E\left(g_{n2}(X_1, X_2)Z_1 \cdot Z_2 \left(e^{\frac{itZ_1}{\sigma_n\sqrt{n}}u_1} - 1\right)\left(e^{\frac{itZ_2}{\sigma_n\sqrt{n}}u_2} - 1\right)\right).$$

Substituting this formula into $\delta_n(t)$ and applying the inequality $|e^{ix} - 1| \le 2|x|^q$, $0 < q < 1$, we obtain

$$|\delta_n(t)| \le \frac{t^2}{\sqrt{n}} \left\{\left|\frac{1}{\sigma_n^3} E\left(g_{n2}(X_1, X_2)Z_1 \cdot Z_2\right) - E\left(g_2(X_1, X_2)g_1(X_1) \cdot g_1(X_2)\right)\right|\right.$$

$$+4|t|^q \sigma_n^{-2-q} n^{-q/2} E\left(|g_{n2}(X_1, X_2)||Z_1|^{1+q} \cdot |Z_2|\right)$$

$$\left.+4|t|^{2q} \sigma_n^{-2-2q} n^{-q} E\left(|g_{n2}(X_1, X_2)||Z_1 \cdot Z_2|^{1+q}\right)\right\}. \tag{85}$$

Assuming $q = 1/2$ in (85) and taking into account (78) the following relations can be obtained

$$\left|\frac{1}{\sigma_n^3} E\left(g_{n2}(X_1, X_2)Z_1 \cdot Z_2\right) - E\left(g_2(X_1, X_2)g_1(X_1) \cdot g_1(X_2)\right)\right| \to 0, \tag{86}$$

$$n^{-1/4} E\left(|g_{n2}(X_1, X_2)||Z_1|^{3/2} \cdot |Z_2|\right) \to 0, \tag{87}$$

$$n^{-1/2} E\left(|g_{n2}(X_1, X_2)||Z_1 \cdot Z_2|^{3/2}\right) \to 0. \tag{88}$$

Indeed, relation (86) follows from the representations

$$E\left(g_{n2}(X_1, X_2)Z_1 \cdot Z_2\right) = Eg_2^n(X_1, X_2)g_{n1}(X_1)g_{n1}(X_2)$$

$$+2g_2^n(X_1, X_2)g_{n1}(X_1)\bar{h}^n(X_1) + Eg_2^n(X_1, X_2)\bar{h}^n(X_1)\bar{h}^n(X_2),$$

$$\frac{1}{\sigma_n^3} = 1 + (1-\sigma_n^2)\frac{1+\sigma_n+\sigma_n^2}{\sigma_n^3(1+\sigma_n)},$$

the estimates (47), (48) and the Hölder inequality.

Relations (87) and (88) can be deduced from the following estimates which are obtained with the help of the Hölder inequality

$$\begin{aligned} n^{-1/4}E|g_{n2}(X_1,X_2)||Z_1|^{3/2}|Z_2| &\le n^{-1/4}\left(E|g_{n2}(X_1,X_2)|^2\right)^{1/2}\left(E|Z_1|^3\cdot E|Z_2|^2\right)^{1/2} \\ &\le \left(\frac{1}{\sqrt{n}}E|g_2^n|^2\right)^{1/2}\left(E|Z_1|^3\right)^{5/6} \to 0, \end{aligned}$$

$$\begin{aligned} n^{-1/2}E|g_{n2}(X_1,X_2)||Z_1|^{3/2}|Z_2|^{3/2} &\le n^{-1/2}\left(E|g_{n2}(X_1,X_2)|^2\right)^{1/2}E|Z_1|^3 \\ &\le n^{-1/4}\left(\frac{1}{\sqrt{n}}E|g_2^n|^2\right)^{1/2}E|Z_1|^3 \to 0. \end{aligned}$$

Consequently, $I_{n3}^1 = o(n^{-1/2})$ for $n\to\infty$. Estimate $I_{n3}^2 = o(n^{-1/2})$ for $n\to\infty$ follows from (14) and (68). Therefore $I_{n3} = o(n^{-1/2})$. In (45) it was proved that $I_{n4} = o(n^{-1/2})$. Hence,

$$I_n = o(n^{-1/2}) \quad \text{as} \quad n\to\infty. \tag{89}$$

From (71), (72), (82) and (89) the relation (7) can be obtained. Therefore, Theorem 5 is proved.

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