Large Deviations of U-Statistics

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Large deviation results with explicit order terms and Cramér's series are developed for non-degenerate U-statistics of degree m under Cramér type conditions on the kernel. The method of the proof is based on the contraction technique of Keener, Robinson and Weber (1998), which is the natural generalization of the classical method of Cramér (1938). Other techniques used in the proofs include truncation, decoupling inequalities, Borell's inequality for Rademacher chaos and a partitioning method to bound the degenerate remainder term.

1 Introduction and Results.

Let $X, X_1, \ldots, X_n$ be independent and identically distributed random variables with the distribution F taking values in the measurable space $(\mathbf{X}, \mathcal{X})$. Consider a U-statistic $U_n = U_n(X_1, \ldots, X_n)$ of the form

$$U_n = \frac{(m-1)\cdots 2 \cdot 1}{(n-1)\cdots(n-m+1)} \sum_{1 \le i_1 < \cdots < i_m \le n} h(X_{i_1}, \ldots, X_{i_m})$$

where the kernel $h : \mathbf{X}^m \to R$ is an integrable, symmetric function. According to the Hoeffding representation

$$U_n = S_n - \frac{n(m-1)}{m}\mu + v_n,$$

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where

$$\mu = Eh(X_1,\dots,X_m), \qquad S_n = \sum_{i=1}^{n} g(X_i),$$
$$g(x) = E(h(X_1,\dots,X_{m-1},X_m)|X_m = x), \quad \text{and}$$

$$\begin{aligned} v_n &= \frac{m-1}{n-1}\sum_{1\le i_1<i_2\le n} g_2(X_{i_1},X_{i_2}) + \cdots \\ &+ \frac{(m-1)\cdots 2\cdot 1}{(n-1)\cdots(n-m+1)}\sum_{1\le i_1<\cdots<i_m\le n} g_m(X_{i_1},\dots,X_{i_m}), \end{aligned}$$

where $g_c : \mathbf{X}^c \to R,\ c = 2,\dots,m$ are canonical functions which are defined for a given kernel h by the operator formula

$$g_c(x_1,\dots,x_c) = (\delta_{x_1} - F)\cdots(\delta_{x_c} - F)F^{m-c}h,$$

where $Q_1\cdots Q_m h = \int\cdots\int h(x_1,\dots,x_m)dQ_1(x_1)\cdots dQ_m(x_m)$. The following useful representation holds:

$$g_c(x_1,\dots,x_c) = \sum_{d=0}^{c}(-1)^{c-d}\sum_{1\le i_1<\cdots<i_d\le c} E\,h(x_{i_1},\dots,x_{i_d},X_{d+1},\dots,X_m).$$

The functions g_c possess the degeneracy property $Eg_c(x_1,\dots,x_{c-1},X_c) = 0$ *a.s.*, $c = 2,\dots,m$.

Let $\sigma^2 = E(g(X)-\mu)^2$. Assume that $0 < \sigma^2 < \infty$ and without loss of generality, take $\mu = 0$. By the central limit theorem for non-degenerate U-statistics,

$$P\left(U_n/(\sigma\sqrt{n}) \le x\right) \to \Phi(x),$$

uniformly in x as $n\to\infty$, provided $t^{2c/(2c-1)}P(|g_c(X_1,\dots,X_c)| > t) \to 0$ as $t\to\infty$ for each $c = 2,\dots,m$ and $\Phi(x)$ denotes the standard normal distribution function.

We are interested in the relative error

$$\frac{P\left(U_n/(\sigma\sqrt{n}) > x\right)}{(1-\Phi(x))} = (1+o(1))$$

as x tends to infinity together with n.

This result was proved initially for $x = O((\ln n)^{1/2})$ under different conditions on the kernel by Ghosh(1974), Rubin and Sethuraman(1965), Slastnikov(1978), Malevich and Abdalimov(1979). For non-degenerate kernels h such that

$$E|h(X_1,\dots,X_m)|^k \le C^k k^{\gamma k},\ k = 1,2,\dots$$

for some constants $C > 0$ and $\gamma \geq 0$ Malevich and Abdalimov(1979) have shown that Cramér type large deviations are valid when $x = o(n^{1/(10+4\gamma)})$. Vandemaele(1982) refined this theorem to obtain the result for $x = o(n^{1/(6+4\gamma)})$. If $\gamma = 0$, which corresponds to the kernel being bounded, then this result gives the natural range $0 \leq x \leq o(n^{1/6})$. For this range results have also been obtained by Inglot, Kallenberg and Ledwina(1992), Inglot and Ledwina(1993) and Wang Qiying(1996). With $\gamma \geq 1$ Aleshkevichene(1990) established this large deviation theorem for $x = O(n^{1/(4\gamma+2)})$ and for $\gamma \geq 2$ Aleshkevichene(1992) proved that the error was $o(1) = O((1 + x^3 + x \ln n)/\sqrt{n})$ for x in the range $0 \leq x \leq O(n^{1/(4\gamma-2)})$.

Keener, Robinson and Weber(1998) proposed a contraction technique for establishing Cramér type large deviation theorems for U-statistics of degree 2. Their approach is the natural generalization of the classical method of Cramér (1938). Borovskikh and Weber(2000) developed this contraction technique for U-statistics with bounded kernels to establish a Cramér type large deviation theorem with convergent Cramér series for $x = o\left(\sqrt{n}\right)$. In this paper we extend the contraction technique to establish Cramér's series for large deviation probabilities for U-statistics of any degree m when the kernel h satisfies Cramér's condition or Linnik's condition. Following Borovskikh and Weber(2000) but under Cramér type conditions we find the explicit solution of the analogue of the contraction equation in the form of a convergent power series in t with coefficients depending on $x \in \mathbf{X}$. This enables the construction of an explicit Cramér series. Aleshkevichiene (1990) established Cramér type large deviations, including a truncated Cramér series and bounds on the rate of convergence, for U-statistics of degree 2 with bounded kernels via the cumulant method.

Borovskikh and Weber(2001) established large deviation results with Cramér's series for U-statistics with the kernel $h(x_1, \ldots, x_m) = x_1 \cdots x_m$ under Cramér's condition $E \exp\{a|X|\} < \infty$, for some $a > 0$.

Assume $E \exp\{t \sum_{s=1}^m g(X_s)\}|h(X_1, \ldots, X_m)| < \infty$ and $E \exp\{t|g(X)|\} < \infty$ for $t \in [0, \alpha]$, for some $\alpha > 0$. Define the operator $\mathcal{H}_t^{m-1}$, by

$$\mathcal{H}_t^{m-1} g = \frac{E\{\prod_{s=1}^{m-1} e^{tg(X_s)} h(X_1, \ldots, X_{m-1}, \cdot)\}}{(Ee^{tg(X)})^{m-1}}, \ t \in [0, \alpha],$$

for any fixed integer $m = 2, 3, \ldots$.

In what follows we will show that the equation $g = \mathcal{H}_t^{m-1} g$ has an essentially unique solution g_t , i.e.

$$g_t = \mathcal{H}_t^{m-1} g_t. \tag{1}$$

The function g_t is the key to defining the Cramér series. As in Borovskikh and Weber(2000) we give the solution of (1), $g_t(x)$, in the form of an analytic function in t, but here we only assume that a Cramér type condition holds. To formulate this result we need some notation which we develop in line with Borovskikh and Weber(2000). For $x \in \mathbf{X}$ and $k = 1, 2, \ldots$ let

$$c_k(x) = \sum_{l=0}^{k-1} a_l(x) \ b_{k-1-l}, \tag{2}$$

where,

$$\begin{aligned} a_0(x) &= g(x), \\ a_l(x) &= \sum_{\nu_1+2\nu_2+\cdots+l\nu_l=l} Eh(X_1, \ldots, X_{m-1}, x) \prod_{s=1}^{l} \frac{1}{\nu_s!} \left(\sum_{j=1}^{m-1} c_s(X_j) \right)^{\nu_s}, \quad l \geq 1 \end{aligned}$$

and

$$\begin{aligned} b_0 &= 1, \\ b_l &= \sum_{\nu_1+2\nu_2+\cdots+l\nu_l=l} (\nu_1 + \cdots + \nu_l)! \prod_{s=1}^{l} \frac{(-1)^{\nu_s}}{\nu_s!} \beta_s^{\nu_s}, \quad l \geq 1, \end{aligned}$$

where

$$\beta_l = \sum_{\nu_1+2\nu_2+\cdots+l\nu_l=l} E \prod_{s=1}^{l} \frac{1}{\nu_s!} \left(\sum_{j=1}^{m-1} c_s(X_j) \right)^{\nu_s}, \quad l \geq 1.$$

The summation in $a_l(x)$, b_l and β_l is carried out over all integer-valued vectors $(\nu_1, \ldots, \nu_l)$ such that $\nu_1 + 2\nu_2 + \cdots + l\nu_l = l$, $\nu_j \geq 0$, $j = 1, \ldots, l$. Thus, for example,

$$\begin{aligned} a_1(x) &= (m-1) Eh\left(X_1, \ldots, X_{m-1}, x\right) c_1\left(X_1\right), \\ a_2(x) &= (m-1) Eh\left(X_1, \ldots, X_{m-1}, x\right) c_2\left(X_1\right) + \\ &+ \frac{1}{2}(m-1) Eh\left(X_1, \ldots, X_{m-1}, x\right) c_1^2\left(X_1\right) + \\ &+ \binom{m-1}{2} Eh\left(X_1, \ldots, X_{m-1}, x\right) c_1\left(X_1\right) c_1\left(X_2\right), \end{aligned}$$

and

$$b_1 = -\beta_1, \quad b_2 = \beta_1^2 - \beta_2,$$

with

$$\begin{aligned}
\beta_1 &= (m-1)Ec_1(X),\\
\beta_2 &= (m-1)Ec_2(X) + \frac{1}{2}(m-1)Ec_1^2(X) + \binom{m-1}{2}\left(Ec_1(X)\right)^2.
\end{aligned}$$

Here, by definition, $\binom{n}{k} = n(n-1)\cdots(n-k+1)/k!$, if $n \geq k$ and $\binom{n}{k} = 0$, if $n < k$. The expression on the right in (2) depends only on $c_1(x), \ldots, c_{k-1}(x)$. Hence (2) is a recurrence relation defining $c_k(x),\ k = 1, 2, \ldots$. For example,

$$\begin{aligned}
c_1(x) &= g(x),\\
c_2(x) &= (m-1)Eh(X_1, \ldots, X_{m-1}, x)g(X_1),\\
c_3(x) &= \binom{m-1}{2}Eh(X_1, \ldots, X_{m-1}, x)g(X_1)g(X_2)\\
&\quad +(m-1)^2Eh(X_1, \ldots, X_{m-1}, x)h(X_1, X_m \ldots, X_{2m-2})g(X_m)\\
&\quad +\frac{1}{2}(m-1)Eh(X_1, \ldots, X_{m-1}, x)g^2(X_1)\\
&\quad -\left((m-1)^2 + \frac{1}{2}(m-1)\right) g(x)Eg^2(X_1).
\end{aligned} \tag{3}$$

In what follows we will need the notation

$$\begin{aligned}
c(x) &= \left(Eh^2(X_1, \ldots, X_{m-1}, x)\right)^{1/2}, x \in \mathbf{X}\\
\| c \| &= \left(Ec^2(X)\right)^{1/2}
\end{aligned}$$

and under Cramér's condition

$$E\exp\{a \mid h(X_1, \ldots, X_m) \mid\} < \infty \tag{4}$$

for some $a > 0$, we denote

$$b = E\exp\{a \mid h(X_1, \ldots, X_m) \mid\} \quad \text{and} \quad d = \left(E\exp\{ac(X)\}\right)^{1/2}.$$

Note that

$$E\exp\{ac(X)\} \leq (1+\sqrt{2})b. \tag{5}$$

In fact,

$$E\exp\{ac(X)\} = 1 + aEc(X) + \sum_{k=2}^{\infty}\frac{a^k}{k!}Ec^k(X).$$

Since $x^2 \le 2e^x, x \ge 0$ and using Hölder's inequality

$$\begin{aligned} aEc(X) &\le \left(E\{a^2h^2(X_1,\dots,X_m)\}\right)^{1/2} \\ &\le \sqrt{2}\,(E\exp\{a|h(X_1,\dots,X_m)|\})^{1/2} \\ &\le \sqrt{2}b. \end{aligned}$$

Again by Hölder's inequality

$$Ec^k(X) \le E \mid h(X_1,\dots,X_m) \mid^k, \quad k \ge 2.$$

This proves (5). The inequality (5) indicates that the condition $d < \infty$ on the kernel h is weaker than the Cramér condition (4).

Theorem 1. *Assume $d < \infty$. Then for all $x \in \mathbf{X}$ and $k = 1, 2, \dots$*

$$\mid c_k(x) \mid \le c(x)\,\frac{1}{8}\,(8d(1+d))^k \left(\frac{m-1}{a}\right)^{k-1} \tag{6}$$

and the function

$$g_t(x) = \sum_{k=1}^{\infty} c_k(x) t^{k-1},\ x \in \mathbf{X} \tag{7}$$

is the unique analytical solution of equation (1) in the region $0 \le t < a/\left(8d(1+d)(m-1)\right)$. *Further,*

$$\mid g_t(x) \mid \le 2dc(x),$$

$$\mid g_t(x) - g(x) \mid \le c(x)\,t\,\frac{m-1}{a}\,(8d(d+1))^2 \,/\, (1 - t8d(1+d)(m-1)/a)\,, \tag{8}$$

and if $0 \le t \le a/\left((m-1)8d(1+d)(8d(1+d)+1)\right)$ *then*

$$E\exp\left\{\frac{a}{2} \mid g_t(X) \mid \right\} \le 2b. \tag{9}$$

Note that the function $g_t(x)$ in (7) can be written as

$$g_t(x) = g(x) + \sum_{k=1}^{\infty} c_{k+1}(x) t^k.$$

Hence, the estimate (8) is true by virtue of (6), and (9) follows from (8). Moreover, throughout we assume that $m \ge 2$ since if $m = 1$ we have $g_t(x) = g(x)$ for all $t > 0$.

In the following theorem we suppose that $x = o(\sqrt{n})$, $n \to \infty$ and we use the iterated logarithmic function $\ln_l n = \ln(\ln_{l-1} n)$, $\quad l = 1, 2, \ldots$, setting $\ln_0 n = n$.

Theorem 2. *Let $\sigma^2 = Eg^2(X) > 0$ and assume the kernel h satisfies Cramér's condition (4). Then for $x = O(\sqrt{n}/\ln_l n)$ with any fixed integer $l \geq 1$*

$$(10) \quad P\left(U_n > x\sigma\sqrt{n}\right) / (1 - \Phi(x)) = \exp\left\{\frac{x^3}{\sqrt{n}}\, \lambda_m\left(\frac{x}{\sqrt{n}}\right)\right\}\left(1 + O\left(\frac{1+x}{\sqrt{n}}\right)\right)$$

with Cramér series

$$(11) \qquad \lambda_m(u) = \sum_{k=0}^{\infty} \lambda_{mk} u^k,$$

where

$$|\lambda_{mk}| \leq \lambda^{k+3}, \quad k = 0, 1, \ldots$$

with

$$\lambda = \left(\frac{152md(1+d)^2}{a\sigma}\right)\left(1 + \frac{8d^2 \parallel c \parallel}{a\sigma^2}\right).$$

Equation (10) is also valid in the region $O(\sqrt{n}/\ln_l n) \leq x \leq o(\sqrt{n})$ if the function h^2 satisfies Cramér's condition, i.e. $E\exp(a_0 h^2) < \infty$ for some constant $a_0 > 0$.

Explicit formulae for the evaluation of the coefficients λ_{mk} for any $k = 0, 1, \ldots$ are given at (29). The power series in (11) is convergent for $|u| < \lambda^{-1}$.

Let $0 < \alpha < 1/2$ and $s = [4\alpha/(1-2\alpha)]$ denote the integer part of $4\alpha/(1-2\alpha)$. Introduce the truncated Cramér series

$$\lambda_m^{[s]}(u) = \lambda_{m0} + \lambda_{m1}u + \cdots + \lambda_{ms}u^s.$$

Theorem 3. *Let $x = o(n^\alpha)$, $n \to \infty$. Suppose that Linnik's condition*

$$(12) \qquad E\exp\left(a_\alpha |h|^{4\alpha/(2\alpha+1)}\right) < \infty$$

holds for some $a_\alpha > 0$. Then

$$(13) \quad P\left(U_n > x\sigma\sqrt{n}\right) / (1 - \Phi(x)) = \exp\left\{\frac{x^3}{\sqrt{n}}\, \lambda_m^{[s]}\left(\frac{x}{\sqrt{n}}\right)\right\}\left(1 + O\left(\frac{1+x}{\sqrt{n}}\right)\right).$$

2 Proofs.

Proof of Theorem 1. The following lemma provides the inequalities needed to establish (6).

Lemma 1. *If the kernel h satisfies the condition $d < \infty$, then for all $k = 1, 2, \ldots$*

$$|c_k(x)| \le c(x)\gamma_k, \quad x \in \mathbf{X} \tag{14}$$

where

$$\gamma_k = \binom{2k}{k} \frac{(2d(1+d))^k}{4(2k-1)(1+d)} \left(\frac{m-1}{a}\right)^{k-1}.$$

Proof. We will use induction on k. By the definition (3) and Hölder's inequality

$$\begin{aligned} \mid c_1(x) \mid &\le c(x) \\ \mid c_2(x) \mid &\le (m-1)c(x)\left(Ec^2(X)\right)^{1/2}. \end{aligned}$$

Next according to the definition of $\{\gamma_k\}$

$$\gamma_1 = d, \quad \text{and} \quad \gamma_2 = 2d^2(1+d)\,\frac{m-1}{a}.$$

Here $d \ge 1$ and moreover

$$d = (E\exp\{ac(X)\})^{1/2} = \left(1 + aEc(X) + \frac{1}{2}a^2Ec^2(X) + \cdots\right)^{1/2} \ge \frac{a}{\sqrt{2}}\left(Ec^2(X)\right)^{1/2}$$

and

$$\gamma_2 \ge 2d\,\frac{m-1}{a} \ge (m-1)\sqrt{2}\left(Ec^2(X)\right)^{1/2}.$$

Therefore

$$\mid c_1(x) \mid \le c(x)\gamma_1, \quad \text{and} \quad \mid c_2(x) \mid \le c(x)\gamma_2.$$

Now suppose that (14) is valid for $s = 1, \ldots, k-1$, that is,

$$\mid c_s(x) \mid \le c(x)\,\gamma_s, \quad s = 1, \ldots, k-1.$$

From (2)

$$\mid c_k(x) \mid \le \sum_{l=0}^{k-1} \mid a_l(x) \mid \mid b_{k-l-1} \mid,$$

where by the induction hypothesis

$$\begin{aligned}
|a_l(x)| &\leq \sum_{\nu_1+2\nu_2+\cdots+l\nu_l=l} E \mid h\,(X_1,\ldots,X_{m-1},x) \mid \prod_{s=1}^{l} \frac{1}{\nu_s!}\Big\{\Big(\sum_{j=1}^{m-1} c(X_j)\Big)\,\gamma_s\Big\}^{\nu_s} \\
&= \sum_{\nu_1+2\nu_2+\cdots+l\nu_l=l} E \mid h\,(X_1,\ldots,X_{m-1},x) \mid \Big(\sum_{j=1}^{m-1} c(X_j)\Big)^{\nu_1+\cdots+\nu_l} \prod_{s=1}^{l} \frac{1}{\nu_s!}\gamma_s^{\nu_s}.
\end{aligned}$$

Since $c^{\nu}(x) \leq \frac{\nu!}{a^{\nu}} \exp\{ac(x)\}, \quad a>0, \ \nu=\nu_1+\cdots+\nu_l$ and using Hölder's inequality

$$\begin{aligned}
&|a_l(x)| \\
&\leq E \mid h\,(X_1,\ldots,X_{m-1},x) \mid \ \exp\Big\{\frac{a}{2(m-1)}\sum_{j=1}^{m-1} c(X_j)\Big\} \\
&\quad \times \sum_{\nu_1+2\nu_2+\cdots+l\nu_l=l} (\nu_1+\cdots+\nu_l)! \prod_{s=1}^{l}\frac{1}{\nu_s!}\left(\frac{2(m-1)}{a}\gamma_s\right)^{\nu_s} \\
&\leq c(x)\left(E\exp\left(\frac{a}{m-1}c(X)\right)\right)^{\frac{m-1}{2}} \sum_{\nu_1+2\nu_2+\cdots+l\nu_l=l} (\nu_1+\cdots+\nu_l)! \prod_{s=1}^{l}\frac{1}{\nu_s!}\left(\frac{2(m-1)}{a}\gamma_s\right)^{\nu_s} \\
&\leq c(x)\,(E\exp\{ac(X)\})^{1/2}\,\bar{a}_l \\
&= c(x)\,d\,\bar{a}_l.
\end{aligned}$$

Also

$$\begin{aligned}
|\beta_l| &\leq \sum_{\nu_1+2\nu_2+\cdots+l\nu_l=l} E \prod_{s=1}^{l}\frac{1}{\nu_s!}\Big\{\Big(\sum_{j=1}^{m-1} c(X_j)\Big)\,\gamma_s\Big\}^{\nu_s} \\
&= \sum_{\nu_1+2\nu_2+\cdots+l\nu_l=l} E\Big(\sum_{j=1}^{m-1} c(X_j)\Big)^{\nu_1+\cdots+\nu_l} \prod_{s=1}^{l}\frac{1}{\nu_s!}\gamma_s^{\nu_s} \\
&\leq E\exp\Big\{\frac{a}{2(m-1)}\sum_{j=1}^{m-1} c(X_j)\Big\} \\
&\quad \times \sum_{\nu_1+2\nu_2+\cdots+l\nu_l=l} (\nu_1+\cdots+\nu_l)! \prod_{s=1}^{l}\frac{1}{\nu_s!}\left(\frac{2(m-1)}{a}\gamma_s\right)^{\nu_s} \\
&\leq (E\exp\{ac(X)\})^{1/2}\,\bar{a}_l \\
&= d\,\bar{a}_l
\end{aligned}$$

and

$$|b_l| \leq \sum_{\nu_1+2\nu_2+\cdots+l\nu_l=l} (\nu_1+\cdots+\nu_l)! \prod_{s=1}^{l}\frac{1}{\nu_s!}\,(d\,\bar{a}_s)^{\nu_s} = \bar{b}_l,$$

setting $\bar{a}_0=1$ and $\bar{b}_0=1$.

Hence,

$$\mid c_k(x) \mid \leq c(x)\,d\sum_{l=0}^{k-1}\bar{a}_l\ \bar{b}_{k-l-1}. \tag{15}$$

It is sufficient to show that in (15)

$$\gamma_k = d\sum_{l=0}^{k-1} \bar{a}_l\, \bar{b}_{k-l-1}. \tag{16}$$

Let

$$\gamma(z) \;=\; \sum_{k=1}^{\infty} \gamma_k\, z^k$$

be the generating function of the sequence $\{\gamma_k\}$. If (16) holds then

$$\begin{aligned}
\gamma(z) &= d\sum_{k=1}^{\infty}\left(\sum_{l=0}^{k-1} \bar{a}_l\, \bar{b}_{k-l-1}\right) z^k = d\, z \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k} \bar{a}_l\, \bar{b}_{k-l}\right) z^k \\
&= d\, z \left(\sum_{k=0}^{\infty} \bar{a}_k\, z^k\right)\left(\sum_{k=0}^{\infty} \bar{b}_k\, z^k\right),
\end{aligned}$$

where

$$\begin{aligned}
\sum_{k=0}^{\infty} \bar{a}_k\, z^k &= 1 + \sum_{k=1}^{\infty}\left(\sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} (\nu_1+\cdots+\nu_k)! \prod_{s=1}^{k} \frac{1}{\nu_s!}\left(\frac{2(m-1)}{a}\gamma_s\right)^{\nu_s}\right) z^k \\
&= \left(1 - \frac{2(m-1)}{a}\sum_{k=1}^{\infty} \gamma_k\, z^k\right)^{-1}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} \bar{b}_k\, z^k &= 1 + \sum_{k=1}^{\infty}\left(\sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} (\nu_1+\cdots+\nu_k)! \prod_{s=1}^{k} \frac{1}{\nu_s!}(d\,\bar{a}_s)^{\nu_s}\right) z^k \\
&= \left(1 - d\sum_{k=1}^{\infty} \bar{a}_k\, z^k\right)^{-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\gamma(z) &= d\, z\left(1 - \frac{2(m-1)}{a}\gamma(z)\right)^{-1}\left(1 + d - \frac{d}{1 - \frac{2(m-1)}{a}\gamma(z)}\right)^{-1} \\
&= \frac{d\, z}{(1+d)\left(1 - \frac{2(m-1)}{a}\gamma(z)\right) - d}.
\end{aligned}$$

Therefore from (16) we obtain the quadratic equation for the generating function

$$\gamma^2(z) - \frac{a}{2(m-1)(1+d)}\gamma(z) + \frac{a\, d\, z}{2(m-1)(1+d)} = 0, \quad \gamma(0) = 0.$$

From here for $|z| \le a/(8d(1+d)(m-1))$ we find the solution

$$\gamma(z) = \frac{a}{4(m-1)(1+d)} - \frac{a}{4(m-1)(1+d)}\left(1 - 8d(1+d)\frac{(m-1)z}{a}\right)^{1/2}. \tag{17}$$

Using the well-known expansion

$$(1-4z)^{1/2} = 1 - \sum_{k=1}^{\infty} \binom{2k}{k} \frac{1}{2k-1} z^k, \quad |z| \le \frac{1}{4}$$

we have from (17)

$$\gamma(z) = \sum_{k=1}^{\infty} \gamma_k \, z^k$$

with coefficients

$$\gamma_k = \binom{2k}{k} \frac{(2d(1+d))^k}{4(2k-1)(1+d)} \left(\frac{m-1}{a}\right)^{k-1}.$$

This proves Lemma 1.

From (17) it follows that if $|z| \le a/(8d(1+d)(m-1))$ then

$$|\gamma(z)| \le 2d\,|z|.$$

The coefficients $\{\gamma_k\}$ satisfy the inequality

$$\gamma_k \le \frac{1}{8}\,(8d(1+d))^k \left(\frac{m-1}{a}\right)^{k-1}, \quad k = 1, 2, \ldots \tag{18}$$

since $\binom{2k}{k} \le 2^{2k}$ and $d \ge 1$.

From (14) and (18) we get the estimate (6).

Lemma 2. *If $d < \infty$ and $0 \le t < a/(8d(1+d)(m-1))$ then the function $g_t(x)$ defined in (7) is the solution of equation (1).*

Proof. Denote $z_t(x) = t\, g_t(x)$ and rewrite equation (1) in the form

$$z_t(x) = t\, \frac{E \prod_{s=1}^{m-1} e^{z_t(X_s)} h\,(X_1, \ldots, X_{m-1}, x)}{\left(Ee^{z_t(X)}\right)^{m-1}}. \tag{19}$$

We must prove that under the condition $d < \infty$ the analytical function

$$z_t(x) = \sum_{k=1}^{\infty} c_k(x) t^k \tag{20}$$

satisfies equation (19). In (20) the power series converges in the region $0 \le t < a/(8d(1+d)(m-1))$ due to (6). The numerator in (19) has the convergent expansion

$$\begin{aligned}
& E \prod_{s=1}^{m-1} e^{z_t(X_s)} h\,(X_1, \ldots, X_{m-1}, x) \\
&= E \exp\Big\{ \sum_{k=1}^{\infty} \left(\sum_{j=1}^{m-1} c_k(X_j) \right) t^k \Big\} h\,(X_1, \ldots, X_{m-1}, x) \\
&= \sum_{k=0}^{\infty} \Big\{ \sum_{\nu_1 + 2\nu_2 + \cdots + k\nu_k = k} Eh\,(X_1, \ldots, X_{m-1}, x) \prod_{s=1}^{k} \frac{1}{\nu_s!} \left(\sum_{j=1}^{m-1} c_s(X_j) \right)^{\nu_s} \Big\} t^k \\
&= \sum_{k=0}^{\infty} a_k(x)\, t^k, \quad a_0(x) = g(x).
\end{aligned}$$

Similarly the denominator in (19) has the convergent expansion

$$
\begin{aligned}
&\Big(Ee^{z_t(X)}\Big)^{-(m-1)} \\
&= \left(E\exp\Big\{\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m-1} c_k(X_j)\right)t^k\Big\}\right)^{-1} \\
&= \left(1+\sum_{k=1}^{\infty}\Big\{\sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} E\prod_{s=1}^{k}\frac{1}{\nu_s!}\left(\sum_{j=1}^{m-1} c_s(X_j)\right)^{\nu_s}\Big\}t^k\right)^{-1} \\
&= (1+\sum_{k=1}^{\infty}\beta_k t^k)^{-1} \\
&= 1+\sum_{k=1}^{\infty}\Big\{\sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k}(\nu_1+\cdots+\nu_k)!\prod_{s=1}^{k}\frac{(-1)^{\nu_s}}{\nu_s!}\beta_s^{\nu_s}\Big\}t^k \\
&= \sum_{k=0}^{\infty} b_k\, t^k, \quad b_0 = 1.
\end{aligned}
$$

Hence

$$\frac{E\prod_{s=1}^{m-1} e^{z_t(X_s)} h\,(X_1,\ldots,X_{m-1},x)}{\big(Ee^{z_t(X)}\big)^{m-1}} = \sum_{k=0}^{\infty}\Big\{\sum_{l=0}^{k} a_l(x)\, b_{k-l}\Big\}t^k$$

and thus

$$t\,\frac{E\prod_{s=1}^{m-1} e^{z_t(X_s)} h\,(X_1,\ldots,X_{m-1},x)}{\big(Ee^{z_t(X)}\big)^{m-1}} = \sum_{k=1}^{\infty}\Big\{\sum_{l=0}^{k-1} a_l(x)\, b_{k-l-1}\Big\}t^k, \tag{21}$$

where due to (6) the series converges in the region $0 \le t < a/(8d(1+d)(m-1))$.

Comparing (19) - (21) we see that for all $k = 1, 2, \ldots$

$$c_k(x) \quad = \quad \sum_{l=0}^{k-1} a_l(x)\, b_{k-1-l}, \quad x \in \mathbf{X}.$$

This formula coincides with (2). As a result we obtain the proof of Lemma 2.

Theorem 1 follows from Lemma 1 and Lemma 2.

Proof of Theorem 2. Define F_t by

$$dF_t = e^{tg_t-\psi(t)}dF,$$

where

$$\psi(t) = \ln E\exp\{tg_t(X)\}.$$

Let E_t denote the expectation under P_t, where P_t is a probability measure under which $X_1, X_2, \ldots$ are i.i.d. from F_t.

We will specify t via the saddle-point equation

$$\mu_t = \frac{m\sigma x}{\sqrt{n}}, \tag{22}$$

where $\mu_t = E_t g_t(X)$ and $xn^{-1/2} \to 0$.

Lemma 3. *Let $t = t\left(\frac{x}{\sigma\sqrt{n}}\right)$ be the solution of equation (22). Then*

$$t = \sum_{k=1}^{\infty} \pi_k \left(\frac{x}{\sigma\sqrt{n}}\right)^k \tag{23}$$

is a convergent power series with $\pi_1 = 1$ and coefficients $\pi_k,\ k \geq 2$ specified at (26), which satisfy the inequality

$$|\pi_k| \leq \frac{2}{k}\left(\frac{128md(1+d)^2}{a}\right)^{k-1}\left(1 + \frac{8d^2 \parallel c \parallel}{a\sigma^2}\right)^{k-1}, \quad k \geq 1.$$

Proof. By definition

$$\mu_t = \frac{E\exp\left\{t\ \sum_{j=1}^m g_t(X_j)\right\} h\left(X_1, \ldots, X_m\right)}{E\exp\left\{t\ \sum_{j=1}^m g_t(X_j)\right\}}.$$

Using (7) we obtain the expansions

$$E\exp\left\{t\sum_{j=1}^m g_t(X_j)\right\} h\left(X_1, \ldots, X_m\right) = \sum_{k=1}^{\infty} d_k(h) t^k, \quad d_1(h) = m\sigma^2$$

and

$$E\exp\left\{t\sum_{j=1}^m g_t(X_j)\right\} = 1 + \sum_{k=1}^{\infty} d_k(1) t^k,$$

with

$$d_k(h) = \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} Eh\left(X_1, \ldots, X_m\right) \prod_{s=1}^k \frac{1}{\nu_s!}\left\{\sum_{j=1}^m c_s(X_j)\right\}^{\nu_s}, \quad k \geq 1$$

where due to (6)

$$\begin{aligned}
|d_k(h)| &\leq \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} E \mid h\left(X_1, \ldots, X_m\right) \mid \\
&\quad \times \prod_{s=1}^k \frac{1}{\nu_s!}\left\{\left(\sum_{j=1}^m c(X_j)\right)\left(8d(1+d)\right)^s \left(\frac{m}{a}\right)^{s-1} \frac{1}{2}\right\}^{\nu_s} \\
&= \left(\frac{8md(1+d)}{a}\right)^k \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} E \mid h\left(X_1, \ldots, X_m\right) \mid
\end{aligned}$$

$$
\begin{aligned}
& \times \prod_{s=1}^{k} \frac{1}{\nu_s!}\Big\{\Big(\sum_{j=1}^{m} c(X_j)\Big)\frac{a}{2m}\Big\}^{\nu_s} \\
\leq \quad & \left(\frac{8md(1+d)}{a}\right)^k E \mid h(X_1,\ldots,X_m) \mid \exp\Big\{\frac{a}{2m}\sum_{j=1}^{m} c(X_j)\Big\} \\
& \times \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} (\nu_1+\cdots+\nu_k)! \prod_{s=1}^{k}\frac{1}{\nu_s!} \\
\leq \quad & \frac{1}{2}\left(\frac{16md(1+d)}{a}\right)^k \parallel c \parallel \, d, \quad k \geq 1
\end{aligned}
$$

by virtue of Hölder's inequality and the formula

$$
(24) \qquad \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} (\nu_1+\cdots+\nu_k)! \prod_{s=1}^{k}\frac{1}{\nu_s!} y^{\nu_s} = y(1+y)^{k-1}, \quad k \geq 1.
$$

Similarly,

$$
|d_k(1)| \leq \frac{1}{2}\left(\frac{16md(1+d)}{a}\right)^k d, \quad k \geq 1.
$$

Further

$$
\begin{aligned}
\sum_{k=1}^{\infty} d_k(h)t^k &= tm\sigma^2\left(1+\sum_{k=1}^{\infty}\left(\frac{d_{k+1}(h)}{m\sigma^2}\right)t^k\right) \\
&= tm\sigma^2 \exp\Big\{\ln\left(1+\sum_{k=1}^{\infty}\left(\frac{d_{k+1}(h)}{m\sigma^2}\right)t^k\right)\Big\} \\
&= tm\sigma^2 \exp\Big\{\sum_{k=1}^{\infty}\mathcal{A}_k t^k\Big\},
\end{aligned}
$$

where

$$
\mathcal{A}_k = \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} (\nu_1+\cdots+\nu_k-1)! \prod_{s=1}^{k}\frac{(-1)^{\nu_s-1}}{\nu_s!}\left(\frac{d_{s+1}(h)}{m\sigma^2}\right)^{\nu_s},
$$

in addition

$$
\begin{aligned}
|\mathcal{A}_k| \quad \leq \quad & \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} (\nu_1+\cdots+\nu_k)! \prod_{s=1}^{k}\frac{1}{\nu_s!}\Big\{\frac{1}{2m\sigma^2}\left(\frac{16md(1+d)}{a}\right)^{s+1} \parallel c \parallel \, d\Big\}^{\nu_s} \\
= \quad & \left(\frac{16md(1+d)}{a}\right)^k \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} (\nu_1+\cdots+\nu_k)! \times \\
& \times \prod_{s=1}^{k}\frac{1}{\nu_s!}\Big\{\frac{8d(1+d)}{\sigma^2 a} \parallel c \parallel \, d\Big\}^{\nu_s} \\
= \quad & \left(\frac{16md(1+d)}{a}\right)^k \left(\frac{8d^2(1+d) \parallel c \parallel}{a\sigma^2}\right)\left(1+\frac{8d^2(1+d) \parallel c \parallel}{a\sigma^2}\right)^{k-1} \\
\leq \quad & \left(\frac{16md(1+d)}{a}\right)^k \left(1+\frac{8d^2(1+d) \parallel c \parallel}{a\sigma^2}\right)^k, \quad k \geq 1
\end{aligned}
$$

taking account of (24).

Further,

$$\left(1+\sum_{k=1}^{\infty} d_k(1)t^k\right)^{-1} = \exp\Big\{-\ln\left(1+\sum_{k=1}^{\infty} d_k(1)t^k\right)\Big\} = \exp\Big\{-\sum_{k=1}^{\infty} \mathcal{B}_k t^k\Big\},$$

where

$$\mathcal{B}_k = \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} (\nu_1+\cdots+\nu_k-1)! \prod_{s=1}^{k} \frac{(-1)^{\nu_s-1}}{\nu_s!}(d_s(1))^{\nu_s},$$

in addition

$$\begin{aligned}
|\mathcal{B}_k| &\le \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} (\nu_1+\cdots+\nu_k)! \prod_{s=1}^{k} \frac{1}{\nu_s!}\Big\{\frac{d}{2}\left(\frac{16md(1+d)}{a}\right)^{s}\Big\}^{\nu_s} \\
&= \left(\frac{16md(1+d)}{a}\right)^{k} \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} (\nu_1+\cdots+\nu_k)! \prod_{s=1}^{k} \frac{1}{\nu_s!}\left(\frac{d}{2}\right)^{\nu_s} \\
&= \left(\frac{16md(1+d)}{a}\right)^{k} \frac{d}{2}\left(1+\frac{d}{2}\right)^{k-1} \\
&\le \left(\frac{16md(1+d)}{a}\right)^{k} \left(1+\frac{d}{2}\right)^{k}, \quad k \ge 1
\end{aligned}$$

because of (24).

Hence the equation (22) is reduced to the form

$$t = \frac{x}{\sigma\sqrt{n}} \exp\{\mathcal{D}(t)\}, \tag{25}$$

where

$$\mathcal{D}(t) = \sum_{k=1}^{\infty} \mathcal{D}_k t^k,$$

with the coefficients $\mathcal{D}_k = \mathcal{B}_k - \mathcal{A}_k$, and

$$|\mathcal{D}_k| \le 2\left(\frac{16md(1+d)^2}{a}\right)^{k}\left(1+\frac{8d^2 \parallel c \parallel}{a\sigma^2}\right)^{k}, \quad k \ge 1.$$

Using the Lagrange expansion (see, Whittaker and Watson (1927)) we find the solution of equation (25) is

$$t = \sum_{k=1}^{\infty} \pi_k \left(\frac{x}{\sigma\sqrt{n}}\right)^{k},$$

where

$$\pi_k = \frac{1}{k} \sum_{\nu_1+2\nu_2+\cdots+(k-1)\nu_{k-1}=k-1} \prod_{s=1}^{k-1} \frac{1}{\nu_s!}(k\mathcal{D}_s)^{\nu_s}, \quad k \ge 1, \tag{26}$$

setting $\pi_1 = 1$. Note that

$$
\begin{aligned}
|\pi_k| &\le \frac{1}{k} \sum_{\nu_1+2\nu_2+\cdots+(k-1)\nu_{k-1}=k-1} \prod_{s=1}^{k-1} \frac{1}{\nu_s!}\left\{2k\left(\frac{16md(1+d)^2}{a}\right)^s\left(1+\frac{8d^2\parallel c\parallel}{a\sigma^2}\right)^s\right\}^{\nu_s} \\
&= \frac{4^{k-1}}{k}\left(\frac{16md(1+d)^2}{a}\right)^{k-1}\left(1+\frac{8d^2\parallel c\parallel}{a\sigma^2}\right)^{k-1} \\
&\quad\times \sum_{\nu_1+2\nu_2+\cdots+(k-1)\nu_{k-1}=k-1} \prod_{s=1}^{k-1} \frac{1}{\nu_s!}\left\{2k\left(\frac{1}{4}\right)^s\right\}^{\nu_s} \\
&\le \frac{4^{k-1}}{k}\left(\frac{16md(1+d)^2}{a}\right)^{k-1}\left(1+\frac{8d^2\parallel c\parallel}{a\sigma^2}\right)^{k-1} \prod_{s=1}^{k-1}\left\{\sum_{\nu_s=0}^{\infty}\frac{1}{\nu_s!}\left(2k\left(\frac{1}{4}\right)^s\right)^{\nu_s}\right\} \\
&= \frac{4^{k-1}}{k}\left(\frac{16md(1+d)^2}{a}\right)^{k-1}\left(1+\frac{8d^2\parallel c\parallel}{a\sigma^2}\right)^{k-1} \prod_{s=1}^{k-1}\exp\left\{2k\left(\frac{1}{4}\right)^s\right\} \\
&= \frac{4^{k-1}}{k}\left(\frac{16md(1+d)^2}{a}\right)^{k-1}\left(1+\frac{8d^2\parallel c\parallel}{a\sigma^2}\right)^{k-1} \exp\left\{\frac{2k}{3}\left(1-\left(\frac{1}{4}\right)^k\right)\right\} \\
&\le \frac{2}{k}\left(\frac{128md(1+d)^2}{a}\right)^{k-1}\left(1+\frac{8d^2\parallel c\parallel}{a\sigma^2}\right)^{k-1}, \quad k\ge 1.
\end{aligned}
$$

This proves Lemma 3.

Let $U_n(t)$ denote the U-statistic U_n, where $X_1, X_2, \ldots$ are i.i.d. with the distribution F_t. Using the Hoeffding representation under P_t

$$U_n(t) = S_n(t) - \frac{n(m-1)}{m}\mu_t + v_n(t),$$

where

$$
\begin{aligned}
S_n(t) &= \sum_{i=1}^{n} g_t(X_i), \\
g_t(x) &= E_t(h(X_1,\ldots,X_{m-1},X_m)|X_m = x), \\
&\text{and} \\
v_n(t) &= \frac{m-1}{n-1}\sum_{1\le i_1<i_2\le n} g_{2t}(X_{i_1},X_{i_2}) + \cdots \\
&\quad + \frac{(m-1)\cdots 2\cdot 1}{(n-1)\cdots(n-m+1)} \sum_{1\le i_1<\cdots<i_m\le n} g_{mt}(X_{i_1},\ldots,X_{i_m}),
\end{aligned}
$$

where $g_{ct} : \mathbf{X}^c \to R,\ c = 2,\ldots,m$, are canonical functions satisfying the property of complete degeneracy $E_t g_{ct}(x_1,\ldots,x_{c-1},X_c) = 0$, $a.s.$ and having the representation

$$g_{ct}(x_1,\ldots,x_c) = \sum_{d=0}^{c}(-1)^{c-d}\sum_{1\le i_1<\cdots<i_d\le c} E_t\, h(x_{i_1},\ldots,x_{i_d},X_{d+1},\ldots,X_m). \tag{27}$$

In what follows t is the solution of the equation (22), that is, t has the expansion (23). By virtue of (22)

$$U_n(t) - x\sigma\sqrt{n} = S_n(t) - n\mu_t + v_n(t)$$

and hence

$$\begin{aligned}
& P(U_n > x\sigma\sqrt{n}) \\
&= \int\cdots\int I(U_n(y_1,\ldots,y_n) > x\sigma\sqrt{n}) \prod_{i=1}^n dF(y_i) \\
&= \int\cdots\int \exp\{\sum_{i=1}^n(\psi(t) - tg_t(y_i))\}\ I(U_n(y_1,\ldots,y_n) > x\sigma\sqrt{n}) \prod_{i=1}^n dF_t(y_i) \\
&= E_t[e^{n\psi(t)-tS_n(t)} I(U_n(t) > x\sigma\sqrt{n})] \\
&= e^{n\psi(t)-nt\mu_t}\ \mathcal{T}_n(t), \qquad (28)
\end{aligned}$$

where

$$\mathcal{T}_n(t) = E_t[e^{-t(S_n(t)-n\mu_t)} I(S_n(t) - n\mu_t + v_n(t) > 0)].$$

Definition of Cramér's series.

With the help of (7), under the condition $xn^{-1/2} \to 0$ we have

$$\psi(t) = \ln(1 + \sum_{k=2}^\infty \mathcal{G}_k t^k) = \sum_{k=2}^\infty \mathcal{H}_k t^k,$$

where

$$\mathcal{G}_1 = 0, \quad \mathcal{G}_k = \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} E \prod_{s=1}^k \frac{1}{\nu_s!} c_s^{\nu_s}(X), \quad \text{and,}$$

$$\mathcal{H}_0 = \mathcal{H}_1 = 0, \quad \mathcal{H}_k = \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} (\nu_1+\cdots+\nu_k-1)! \prod_{s=1}^k \frac{(-1)^{\nu_s-1}}{\nu_s!} \mathcal{G}_s^{\nu_s},\ k \geq 2.$$

Using (6) for $k \geq 2$

$$\begin{aligned}
|\mathcal{G}_k| &\leq \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} E \prod_{s=1}^k \frac{1}{\nu_s!}\left\{c(X)\,(8d(1+d))^s \left(\frac{m-1}{a}\right)^{s-1}\right\}^{\nu_s} \\
&= (8d(1+d))^k \left(\frac{m-1}{a}\right)^k \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} E\left(\frac{a}{2}\,c(X)\right)^{\nu_1+\cdots+\nu_k} \prod_{s=1}^k \frac{1}{\nu_s!}\left(\frac{2}{m-1}\right)^{\nu_s} \\
&\leq (8d(1+d))^k \left(\frac{m-1}{a}\right)^k d \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} (\nu_1+\cdots+\nu_k)! \prod_{s=1}^k \frac{1}{\nu_s!}\left(\frac{2}{m-1}\right)^{\nu_s} \\
&\leq \left(\frac{24md(1+d)}{a}\right)^k d
\end{aligned}$$

and

$$
\begin{aligned}
|\mathcal{H}_k| &\leq \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} (\nu_1+\cdots+\nu_k)! \prod_{s=1}^{k} \frac{1}{\nu_s!} \Big\{ \left(\frac{24md(1+d)}{a} \right)^s d \Big\}^{\nu_s} \\
&= \left(\frac{24md(1+d)}{a} \right)^k \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} (\nu_1+\cdots+\nu_k)! \prod_{s=1}^{k} \frac{1}{\nu_s!} d^{\nu_s} \\
&= \left(\frac{24md(1+d)}{a} \right)^k d(1+d)^{k-1} \leq \left(\frac{24md(1+d)^2}{a} \right)^k .
\end{aligned}
$$

Since t is the solution of the saddle-point equation (22)

$$
\psi(t) = \sum_{k=2}^{\infty} \psi_k (\frac{x}{\sigma\sqrt{n}})^k
$$

with coefficients

$$
\psi_k = \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} (\nu_1+\cdots+\nu_k)!\ \mathcal{H}_{\nu_1+\cdots+\nu_k} \prod_{s=1}^{k} \frac{1}{\nu_s!} \pi_s^{\nu_s},
$$

and, moreover, due to (6)

$$
\begin{aligned}
|\psi_k| &\leq \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} (\nu_1+\cdots+\nu_k)! \left(\frac{24md(1+d)^2}{a} \right)^{\nu_1+\cdots+\nu_k} \\
&\quad \times \prod_{s=1}^{k} \frac{1}{\nu_s!} \Big\{ \left(\frac{128md(1+d)^2}{a} \right)^{s-1} \left(1 + \frac{8d^2 \parallel c \parallel}{a\sigma^2} \right)^{s-1} \Big\}^{\nu_s} \\
&\leq \left(\frac{128md(1+d)^2}{a} \right)^k \left(1 + \frac{8d^2 \parallel c \parallel}{a\sigma^2} \right)^k \\
&\quad \times \sum_{\nu_1+2\nu_2+\cdots+k\nu_k=k} (\nu_1+\cdots+\nu_k)! \prod_{s=1}^{k} \frac{1}{\nu_s!} \left(\frac{3}{16} \right)^{\nu_s} \\
&\leq \left(\frac{128md(1+d)^2}{a} \right)^k \left(1 + \frac{8d^2 \parallel c \parallel}{a\sigma^2} \right)^k (1+\frac{3}{16})^k, \quad k \geq 1.
\end{aligned}
$$

Since $g_t(x)$ can be written as in (7) and $t = t\left(\frac{x}{\sigma\sqrt{n}}\right)$ is the solution of the saddle-point equation (22) then in (28)

$$
\begin{aligned}
n(\psi(t) - t\mu_t) &= n\psi\left(t\left(\frac{x}{\sigma\sqrt{n}} \right) \right) - mt\left(\frac{x}{\sigma\sqrt{n}} \right) x\sigma\sqrt{n} \\
&= -\frac{x^2}{2} + \frac{x^3}{\sqrt{n}} \sum_{k=0}^{\infty} \lambda_{mk} \left(\frac{x}{\sqrt{n}} \right)^k = -\frac{x^2}{2} + \frac{x^3}{\sqrt{n}} \lambda_m \left(\frac{x}{\sqrt{n}} \right), \qquad (29)
\end{aligned}
$$

defining the Cramér series (11) with the coefficients

$$
\lambda_{mk} = \frac{1}{\sigma^{k+3}} \psi_{k+3} - \frac{m}{\sigma^{k+1}} \pi_{k+2}, \quad k = 0, 1, \ldots .
$$

Since

$$d^2 = Ee^{ac(X)} = 1 + aEc(X) + \frac{1}{2}a^2Ec^2(X) + \cdots \geq \frac{1}{2}a^2Ec^2(X) \geq \frac{1}{2}a^2\sigma^2,$$

then from here we find that λ_{mk} satisfy the inequality

$$|\lambda_{mk}| \leq \left\{ \left(\frac{152md(1+d)^2}{a\sigma} \right) \left(1 + \frac{8d^2 \parallel c \parallel}{a\sigma^2} \right) \right\}^{k+3}, \quad k = 0, 1, \ldots.$$

Estimate of $\mathcal{T}_n(t)$.

Next we need to estimate $\mathcal{T}_n(t)$ in (28). Denote

$$v_{nc}(t) = \frac{(m-1)\cdots(m-c+1)}{(n-1)\cdots(n-c+1)} \sum_{1\leq i_1<\cdots<i_c\leq n} g_{ct}(X_{i_1}, \ldots, X_{i_c}). \tag{30}$$

Then $v_n(t) = v_{n2}(t) + \cdots + v_{nm}(t)$.

Further for $c = 2, \ldots, m$ and $\delta > 0$ let

$$\begin{aligned}
f_c(x_1, \ldots, x_c) &= (\delta_{x_1} - F_t)\cdots(\delta_{x_c} - F_t)\, g_{ct}\; I\left(|g_{ct}| \leq \frac{2^{2c+2}}{a}\delta\sqrt{n}/x \right), \\
\bar{f}_c(x_1, \ldots, x_c) &= (\delta_{x_1} - F_t)\cdots(\delta_{x_c} - F_t)\, g_{ct}\; I\left(|g_{ct}| > \frac{2^{2c+2}}{a}\, e_{l-1}\left(\delta\sqrt{n}/x\right) \right), \\
\varphi_{pc}(x_1, \ldots, x_c) &= (\delta_{x_1} - F_t)\cdots(\delta_{x_c} - F_t) \\
&\quad \times g_{ct} I \left(\frac{2^{2c+2}}{a}\, e_{p-1}\left(\delta\sqrt{n}/x\right) < |g_{ct}| \leq \frac{2^{2c+2}}{a}\, e_p\left(\delta\sqrt{n}/x\right) \right),
\end{aligned}$$

where $e_p(y) = \exp\{\exp_{p-1}(y)\}$ is the iterated exponential function, $p = 1, \ldots, l-1$, setting $\exp_0(y) = y$, and $l \geq 2$ is any fixed integer. Each of these functions can be written in the form (27). Next we define $v^f_{nc}(t)$ and $v^f_n(t)$ by analogy with (30), where instead of g_{ct} we take f_c. Obviously

$$g_{ct} = f_c + \bar{f}_c + \sum_{p=1}^{l-1} \varphi_{pc}, \quad v_{nc}(t) = v^f_{nc}(t) + v^{\bar{f}}_{nc}(t) + \sum_{p=1}^{l-1} v^{\varphi_p}_{nc}(t),$$

and

$$v_n(t) = \sum_{c=2}^{m} v^f_{nc}(t) + \sum_{c=2}^{m} v^{\bar{f}}_{nc}(t) + \sum_{c=2}^{m}\sum_{p=1}^{l-1} v^{\varphi_p}_{nc}(t),$$

$$v_n(t) = v^f_n(t) + v^{\bar{f}}_n(t) + \sum_{p=1}^{l-1} v^{\varphi_p}_n(t). \tag{31}$$

Estimate of exponential moments. Now we will use the decomposition (31), the representation in Hoeffding (1963), the decoupling inequalities due to de la Peña (1992) and the inequality in Borell (1979) for Rademacher chaos.

Hoeffding's formula. Denote

$$\phi_n(x_1, \ldots, x_n) = \frac{1}{k} \sum_{j=1}^{k} \phi(x_{jm-m+1}, \ldots, x_{jm}),$$

for some symmetric function ϕ, where $k = [n/m]$ is the integer part of the number n/m. Then

$$\binom{n}{m}^{-1} \sum_{1 \le i_1 < \cdots < i_m \le n} \phi(X_{i_1}, \ldots, X_{i_m}) = \frac{1}{n!} \sum \phi_n(X_{i_1}, \ldots, X_{i_n}), \tag{32}$$

where the summation on the right is carried out over all the permutations $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$.

de la Peña decoupling inequalities. Let $\phi : \mathbf{X}^m \to B$ be a symmetric degenerate kernel taking values in a Banach space $(B, \| \cdot \|)$ and let $\Psi : R \to R$ be any convex increasing function. If $X_1^j, \ldots, X_n^j, \quad j = 1, \ldots, m$, are independent copies of $X_1, \ldots, X_n$, then the *upper decoupling inequality*

$$E\,\Psi\left(\| \sum_{1 \le i_1 < \cdots < i_m \le n} \phi(X_{i_1}, \ldots, X_{i_m}) \| \right)$$

$$\le E\,\Psi\left(m^m \| \sum_{1 \le i_1 < \cdots < i_m \le n} \phi(X_{i_1}^1, \ldots, X_{i_m}^m) \| \right) \tag{33}$$

and the *lower decoupling inequality*

$$E\,\Psi\left(\| \sum_{1 \le i_1 < \cdots < i_m \le n} \phi(X_{i_1}^1, \ldots, X_{i_m}^m) \| \right)$$

$$\le E\,\Psi\left(2^{2m-2}(m-1)! \, \| \sum_{1 \le i_1 < \cdots < i_m \le n} \phi(X_{i_1}, \ldots, X_{i_m}) \| \right) \tag{34}$$

are valid.

Borell's inequality. Let $\{\varepsilon_j\}, \; \{\varepsilon_j^i\}, \quad i, j \ge 1$, be the independent sequences of i.i.d. random variables with distribution $2^{-1}\delta_{-1} + 2^{-1}\delta_1$ and let $x_{i_1, \ldots, i_m}$ be elements in a Banach space $(B, \| \cdot \|)$. Denote

$$X = \sum_{1 \le i_1 < \cdots < i_m \le n} \varepsilon_{i_1}^1 \cdots \varepsilon_{i_m}^m x_{i_1, \ldots, i_m}.$$

By Borell's inequality for all $p \ge 2$

$$E \, \| X \|^p \le (p-1)^{pm/2} \, s^p, \tag{35}$$

where

$$s^2 = \sum_{1 \le i_1 < \cdots < i_m \le n} x^2_{i_1,\ldots,i_m}.$$

It follows from (35) that

$$(36) \quad E \exp\left(\theta \parallel X \parallel^\alpha\right) \le 2 \exp\left\{\frac{2-\alpha m}{2}(4\alpha m)^{\alpha m/(2-\alpha m)} \theta^{2/(2-\alpha m)} s^{2\alpha/(2-\alpha m)}\right\}$$

for any $\theta > 0$ and $0 < \alpha < 2/m$ (see, Arcones (1995) or Arcones and Giné (1993)).

In the following we will use the geometric-arithmetic mean inequality

$$(37) \ |x_1 \cdots x_m| \le q_1^{-1}|x_1|^{q_1} + \cdots + q_m^{-1}|x_m|^{q_m}, \quad q_1^{-1} + \cdots + q_m^{-1} = 1, \quad x_i \in R.$$

Lemma 4. *If $\delta > 0$ satisfies the condition $(4m)^{2m} 12 e \delta t \sqrt{n}/x < a$ then*

$$E_t \exp\left\{3t|v_n^f(t)|\right\} \le \left(1 - (4m)^{2m} 12 e \frac{\delta}{a}\, t\, \frac{\sqrt{n}}{x}\right)^{-1}.$$

Proof. We have from (37) and (33)

$$\begin{aligned} E_t \exp\left\{3t|v_n^f(t)|\right\} &\le E_t \prod_{c=2}^m \exp\left\{3t|v_{nc}^f(t)|\right\} \\ &\le \frac{1}{m-1} \sum_{c=2}^m E_t \exp\left\{3(m-1)t|v_{nc}^f(t)|\right\} \\ &\le \frac{1}{m-1} \sum_{c=2}^m E_t \exp\left\{3 m^m c^c t n^{-c+1} \mid \sum_{1 \le i_1 < \cdots < i_c \le n} f_c\left(X_{i_1}^1, \ldots, X_{i_c}^c\right) \mid \right\}. \end{aligned}$$

By symmetrization (cf. Ledoux and Talagrand (1991), p.152) and noting $|f_c| \le 2^c (2^{2c+2}/a)\, \delta \sqrt{n}/x$,

$$\begin{aligned} & E_t \exp\left\{3t|v_n^f(t)|\right\} \\ &\le \frac{1}{m-1} \sum_{c=2}^m E \exp\left\{3\, m^m\, c^c\, 2^c\, t\, n^{-c+1} \mid \sum_{1 \le i_1 < \cdots < i_c \le n} \varepsilon_{i_1}^1 \cdots \varepsilon_{i_c}^c f_c\left(X_{i_1}^1, \ldots, X_{i_c}^c\right) \mid \right\} \\ &\le \frac{1}{m-1} \sum_{c=2}^m E \exp\left\{3\, m^m\, c^c\, 2^c\, 2^{(3c+2)(c-2)/c} \left(\frac{\delta}{a}\right)^{(c-2)/c} (t\sqrt{n}/x) \right. \\ &\quad \left. \times \mid n^{-c/2} \sum_{1 \le i_1 < \cdots < i_c \le n} \varepsilon_{i_1}^1 \cdots \varepsilon_{i_c}^c f_c\left(X_{i_1}^1, \ldots, X_{i_c}^c\right) \mid^{2/c} \right\} \\ &= \frac{1}{m-1} \sum_{c=2}^m \sum_{k=0}^c \frac{1}{k!} \left(3\, m^m\, c^c\, 2^c\, 2^{(3c+2)(c-2)/c} \left(\frac{\delta}{a}\right)^{(c-2)/c} (t\sqrt{n}/x)\right)^k \end{aligned}$$

$$\times E\,|n^{-c/2}\sum_{1\le i_1<\cdots<i_c\le n}\varepsilon^1_{i_1}\cdots\varepsilon^c_{i_c}f_c\left(X^1_{i_1},\ldots,X^c_{i_c}\right)|^{2k/c}$$

$$+\frac{1}{m-1}\sum_{c=2}^{m}\sum_{k=c+1}^{\infty}\frac{1}{k!}\left(3\,m^m\,c^c\,2^c\,2^{(3c+2)(c-2)/c}\,\left(\frac{\delta}{a}\right)^{(c-2)/c}\,(t\sqrt{n}/x)\right)^k$$

$$\times E\,|n^{-c/2}\sum_{1\le i_1<\cdots<i_c\le n}\varepsilon^1_{i_1}\cdots\varepsilon^c_{i_c}f_c\left(X^1_{i_1},\ldots,X^c_{i_c}\right)|^{2k/c}$$

$$\le\ \frac{1}{m-1}\sum_{c=2}^{m}\sum_{k=0}^{c}\frac{1}{k!}\left(3\,m^m\,c^c\,2^c\,2^{(3c+2)(c-2)/c}\,\left(\frac{\delta}{a}\right)^{(c-2)/c}\,(t\sqrt{n}/x)\right)^k$$

$$\times E_t\left(n^{-c}\sum_{1\le i_1<\cdots<i_c\le n}f_c^2\left(X^1_{i_1},\ldots,X^c_{i_c}\right)\,\frac{x^2}{n}\right)^{\frac{k}{c}}$$

$$+\frac{1}{m-1}\sum_{c=2}^{m}\sum_{k=c+1}^{\infty}\frac{1}{k!}\left(3\,m^m\,c^c\,2^c\,2^{(3c+2)(c-2)/c}\,\left(\frac{\delta}{a}\right)^{(c-2)/c}\,(t\sqrt{n}/x)\right)^k\,k^k$$

$$\times E_t\left(n^{-c}\sum_{1\le i_1<\cdots<i_c\le n}f_c^2\left(X^1_{i_1},\ldots,X^c_{i_c}\right)\,\frac{x^2}{n}\right)^{\frac{k}{c}}$$

where the last inequality follows from Hölder's inequality and Borell's inequality (35). Since $|f_c|\le 2^c\,(2^{2c+2}/a)\delta\,\sqrt{n}/x$ and $k!\ge e^{-k}\,k^k$ we have

$$\begin{aligned}E_t\exp\left\{3t|v^f_n(t)|\right\}\ &\le\ \frac{1}{m-1}\sum_{c=2}^{m}\sum_{k=0}^{c}\frac{1}{k!}\left(3\,m^m\,c^c\,2^{4c+2}\,\frac{\delta}{a}\,t\,\frac{\sqrt{n}}{x}\right)^k\\&\quad+\frac{1}{m-1}\sum_{c=2}^{m}\sum_{k=c+1}^{\infty}\left(3\,m^m\,c^c\,2^{4c+2}\,e\,\frac{\delta}{a}\,t\,\frac{\sqrt{n}}{x}\right)^k\\&\le\ \sum_{k=0}^{\infty}\left((4m)^{2m}12\,e\,\frac{\delta}{a}\,t\,\frac{\sqrt{n}}{x}\right)^k\\&=\ \left(1-(4m)^{2m}12\,e\,\frac{\delta}{a}\,t\,\frac{\sqrt{n}}{x}\right)^{-1},\end{aligned}$$

since $(4m)^{2m}12\,e\,\delta\,t\,\sqrt{n}/x<a$. This proves Lemma 4.

Lemma 5. *If $m\ge 2$ and $0\le t\le 2^{-2m-8}a/m$ then under Cramér's condition (4)*

$$E_t\,\exp\left\{3\,t\,|v^{\bar f}_n(t)|\right\}\le 2\,\exp\left\{t^2\,n\left(e_l\left(\delta\sqrt{n}/x\right)\right)^{-1}A_t\right\},$$

where

$$A_t=3m2^{4m}\left(E_th^8\right)^{1/4}\,E_t\,\exp\left(\frac{a}{2}|h|\right).$$

Proof. By (32) and the convexity of $e^{|x|}$,

$$E_t\,\exp\left\{3\,t\,|v^{\bar f}_n(t)|\right\}\ \le\ \frac{1}{m-1}\sum_{c=2}^{m}E_t\,\exp\left\{3\,(m-1)\,t\,|v^{\bar f}_{nc}(t)|\right\}$$

$$
\begin{aligned}
&\le \frac{1}{m-1}\sum_{c=2}^{m} E_t \,\exp\Big\{6\,m\,t\,|\sum_{j=1}^{[n/c]} \bar f_c(X_{jc-c+1},\dots,X_{jc})|\Big\} \\
&\le \frac{1}{m-1}\sum_{c=2}^{m} E_t \,\exp\Big\{6\,m\,t\,\sum_{j=1}^{[n/c]} \bar f_c(X_{jc-c+1},\dots,X_{jc})\Big\} \\
&\quad + \frac{1}{m-1}\sum_{c=2}^{m} E_t \,\exp\Big\{-\,6\,m\,t\,\sum_{j=1}^{[n/c]} \bar f_c(X_{jc-c+1},\dots,X_{jc})\Big\}, \\
&\qquad \text{since } e^{|x|}\le e^{x}+e^{-x}, \\
&\le \frac{1}{m-1}\sum_{c=2}^{m}\Big(E_t\,\exp\Big\{6\,m\,t\,\bar f_c(X_1,\dots,X_c)\Big\}\Big)^{[n/c]} \\
&\quad + \frac{1}{m-1}\sum_{c=2}^{m}\Big(E_t\,\exp\Big\{-\,6\,m\,t\,\bar f_c(X_1,\dots,X_c)\Big\}\Big)^{[n/c]}, \qquad (38)
\end{aligned}
$$

using the independence of $X_1,\dots,X_n$. Next apply the formula

$$
e^{x}=\sum_{r=0}^{s-1}\frac{1}{r!}\,x^{r}+\frac{1}{(s-1)!}x^{s}\int_0^1 (1-u)^{s-1}\,e^{ux}\,du \qquad (39)
$$

which is valid for all $x\in R$ and integers $s=1,2,\dots$. Then

$$
\begin{aligned}
E_t\,\exp\Big\{6\,m\,t\,\bar f_c(X_1,\dots,X_c)\Big\} &= 1+(6mt)^2\int_0^1 (1-u)\Big\{E_t \bar f_c^{2}\,e^{6mtu\bar f_c}\Big\}\,du \\
&\le 1+2(3mt)^2\,E_t\,\bar f_c^{2}\,e^{6mt|\bar f_c|} \\
&\le 1+2(3mt)^2\,\Big(E_t\,\bar f_c^{4}\Big)^{1/2}\Big(E_t\,e^{12mt|\bar f_c|}\Big)^{1/2} \\
&\le \exp\Big\{2\,(3m)^2\,t^2\,\Big(E_t\,\bar f_c^{4}\Big)^{1/2}\Big(E_t\,e^{12mt|\bar f_c|}\Big)^{1/2}\Big\}. \qquad (40)
\end{aligned}
$$

Here

$$
\begin{aligned}
E_t\,\bar f_c^{4} &\le 2^{4c}\,E_t\,g_{ct}^{4}\,I\Big(|g_{ct}|>2^{2c+2}a^{-1}\,e_{l-1}\,\big(\delta\,\sqrt{n}/x\big)\Big) \\
&\le 2^{4c}\,e^{-2e_{l-1}\,(\delta\,\sqrt{n}/x)}\,E_t\,g_{ct}^{4}\,e^{2^{-2c-1}a|g_{ct}|} \\
&\le 2^{4c}\,\Big(e_l\,\big(\delta\,\sqrt{n}/x\big)\Big)^{-2}\Big(E_t\,g_{ct}^{8}\Big)^{1/2}\Big(E_t\,e^{2^{-2c-1}a|g_{ct}|}\Big)^{1/2}, \\
E_t\,g_{ct}^{8} &\le 2^{8c}\,E_t\,h^{8}, \quad \text{and} \\
E_t\,e^{2^{-2c-1}a|g_{ct}|} &\le E_t\,\exp\Big\{2^{c+1}\,2^{-2c-1}\,a\,|h|\Big\} \le E_t\,\exp\Big\{\frac{a}{2}\,|h|\Big\}.
\end{aligned}
$$

Since $t\le 2^{-2m-6}\,a/m$ then

$$
E_t\,e^{12mt|\bar f_c|}\le E_t\,\exp\Big\{2^{2c+4}\,3\,m\,t\,|h|\Big\}\le E_t\,\exp\Big\{\frac{a}{2}\,|h|\Big\}.
$$

Lemma 5 now follows from (38) and (40).

Lemma 6. *Suppose that Cramér's condition (4) is satisfied. Let*

$$B_t = l^2\,(8m)^{4m}\left(1 + t\,\frac{\sqrt{n}}{x}\right)^2\left(E_t\,h^8\right)^{1/4}E_t\,\exp\left(\frac{a}{2}\,|h|\right).$$

If

$$l\,(4m)^{4m}\left(t + \frac{x}{\sqrt{n}}\right) \le a$$

and for $c = 2, \ldots, m$

$$B_t\,t^c\left(\frac{x}{\sqrt{n}}\right)^{c-2} \le 1 \quad \textit{and} \quad l\,(4m)^{4m}B_t^{c-1}\,t^{c+1}\left(\frac{x}{\sqrt{n}}\right)^{c-1} \le a^2,$$

then

$$E_t\,\exp\Big\{3\,t\,|\sum_{p=1}^{l-1} v_n^{\varphi_p}(t)|\Big\} \le l\,(2m)^{2m}.$$

Proof. From (37)

$$E_t\,\exp\Big\{3\,t\,|\sum_{c=2}^{m}\sum_{p=1}^{l-1} v_{nc}^{\varphi_p}(t)|\Big\}$$

$$\le \frac{1}{(m-1)(l-1)}\sum_{c=2}^{m}\sum_{p=1}^{l-1} E_t\,\exp\Big\{3\,(m-1)\,(l-1)\,t\,|v_{nc}^{\varphi_p}(t)|\Big\}.$$

Here by the upper decoupling inequality (33) and symmetrization

$$E_t\,\exp\Big\{3\,(m-1)\,(l-1)\,t\,|v_{nc}^{\varphi_p}(t)|\Big\} \le E\,\exp\Big\{3\,m\,l\,(2c)^c\,t\,|v_{nc}^{d\varphi_p}(t)|\Big\}, \tag{41}$$

where $v_{nc}^{d\varphi_p}(t)$ denotes a decoupling symmetrization version $v_{nc}^{\varphi_p}(t)$, that is,

$$v_{nc}^{d\varphi_p}(t) = \frac{(c-1)\cdots 2\cdot 1}{(n-1)\cdots(n-c+1)}\sum_{1\le i_1<\cdots<i_c\le n}\varepsilon_{i_1}^1\cdots\varepsilon_{i_c}^c\,\varphi_{pc}\left(X_{i_1}^1,\ldots,X_{i_c}^c\right)$$

and further

$$\begin{aligned} &E\,\exp\Big\{3\,m\,l\,(2c)^c\,t\,|v_{nc}^{d\varphi_p}(t)|\Big\} \\ &= E\,\exp\Big\{3\,m\,l\,(2c)^c\,t\,|v_{nc}^{d\varphi_p}(t)|\Big\}\,I\left(|v_{nc}^{d\varphi_p}(t)| \le 2B_t x\sqrt{n}/e_p\left(\delta\,\sqrt{n}/x\right)\right) \\ &\quad + E\,\exp\Big\{3\,m\,l\,(2c)^c\,t\,|v_{nc}^{d\varphi_p}(t)|\Big\}\,I\left(|v_{nc}^{d\varphi_p}(t)| > 2B_t x\sqrt{n}/e_p\left(\delta\,\sqrt{n}/x\right)\right). \end{aligned} \tag{42}$$

In (42), by Chebyshev's inequality

$$E\,\exp\Big\{3\,m\,l\,(2c)^c\,t\,|v_{nc}^{d\varphi_p}(t)|\Big\}\,I\left(|v_{nc}^{d\varphi_p}(t)| > 2B_t x\sqrt{n}/e_p\left(\delta\,\sqrt{n}/x\right)\right)$$

$$
\begin{aligned}
&\le \exp\Big\{-2\,x^2\,B_t/e_p\left(\delta\sqrt{n}/x\right)\Big\}E\,\exp\Big\{\left(3\,m\,l\,(2c)^c\,t + x/\sqrt{n}\right)|v_{nc}^{d\varphi_p}(t)|\Big\}\\
&\le \exp\Big\{-2\,x^2\,B_t/e_p\left(\delta\sqrt{n}/x\right)\Big\}E\,\exp\Big\{3\,l\,(2c)^c\,2^{2m-2}\,m!\,\frac{n}{c}\binom{n}{c}^{-1}\left(t+\frac{x}{\sqrt{n}}\right)\\
&\quad\times|\sum_{1\le i_1<\cdots<i_c\le n}\varepsilon_{i_1}\cdots\varepsilon_{i_c}\varphi_{pc}(X_{i_1},\ldots,X_{i_c})|\Big\},\quad \text{by (34).}
\end{aligned}
$$

By Hoeffding's formula (32), the convexity of $e^{|x|}$ and noting $m!\le m^m$

$$
\begin{aligned}
&E\,\exp\Big\{3\,m\,l\,(2c)^c\,t\,|v_{nc}^{d\varphi_p}(t)|\Big\}\,I\left(|v_{nc}^{d\varphi_p}(t)|>2B_t x\sqrt{n}/e_p\left(\delta\sqrt{n}/x\right)\right)\\
&\quad\le \exp\Big\{-2\,x^2\,B_t/e_p\left(\delta\sqrt{n}/x\right)\Big\}E\,\exp\Big\{3\,l\,(4m)^{2m}\left(t+x/\sqrt{n}\right)\\
&\quad\quad\times|\sum_{j=1}^{[n/c]}\varepsilon_{jc-c+1}\cdots\varepsilon_{jc}\varphi_{pc}(X_{jc-c+1},\ldots,X_{jc})|\Big\}\\
&\quad\le 2\,\exp\Big\{-x^2\,B_t/e_p\left(\delta\sqrt{n}/x\right)\Big\},
\end{aligned}
\tag{43}
$$

where the last inequality follows from (38), (40) and noting that $24\,l\,(8m)^{2m}\,(t+x/\sqrt{n})\le a$. Further we will show that in (42)

$$
\begin{aligned}
&E\,\exp\Big\{3\,m\,l\,(2c)^c\,t\,|v_{nc}^{d\varphi_p}(t)|\Big\}\,I\left(|v_{nc}^{d\varphi_p}(t)|\le 2B_t x\sqrt{n}/e_p\left(\delta\sqrt{n}/x\right)\right)\\
&\quad\le 1+6\,m\,l\,(2c)^c.
\end{aligned}
\tag{44}
$$

To do this we set

$$
V_c^{dec}(u)=P\left(|v_{nc}^{d\varphi_p}(t)|\le u\right),\quad u\ge 0
$$

$$
u_t=2\,B_t\,x\,\sqrt{n}/e_p\left(\delta\sqrt{n}/x\right),\quad\text{and}\quad a_c=3\,m\,l\,(2c)^c.
$$

Then

$$
\begin{aligned}
E\,\exp\Big\{a_c\,t\,|v_{nc}^{d\varphi_p}(t)|\Big\}\,I\left(|v_{nc}^{d\varphi_p}(t)|\le u_t\right) &= -\int_0^{u_t} e^{a_c\,t\,u}\,d\left(1-V_c^{dec}(u)\right)\\
&\le 1+a_c\,t\int_0^{u_t} e^{a_c\,t\,u}\left(1-V_c^{dec}(u)\right)du.
\end{aligned}
\tag{45}
$$

Here

$$
1-V_c^{dec}(u)\le e^{-\theta u^\alpha}\,E\,\exp\Big\{\theta|v_{nc}^{d\varphi_p}(t)|^\alpha\Big\}
$$

for any $\theta>0$ and $0<\alpha<1$.

Further by the inequality (36) with $m=c,\ \alpha=\frac{2}{c+1}$ and $\theta=(a_c+1)\,t\,u^{1-\alpha}$

$$
1-V_c^{dec}(u)\ \le\ 2\,e^{-(2a_c+1)\,t\,u}E_t\,\exp\Big\{\frac{2a_c+1}{c+1}\left(\frac{8c}{c+1}\right)^c t^{c+1}\,u^{c-1}
$$

$$
\begin{aligned}
&\times \frac{((c-1)!)^2}{((n-1)\cdots(n-c+1))^2} \sum_{1\le i_1<\cdots<i_c\le n} \varphi_{pc}^2(X_{i_1}^1,\ldots,X_{i_c}^c)\Big\} \\
\le\ & 2\,e^{-(2a_c+1)\,t\,u} E_t \exp\Big\{\frac{2a_c+1}{c+1}\left(\frac{8c}{c+1}\right)^c t^{c+1}\,u^{c-1}\,\frac{2^{2c-2}((c-1)!)^3}{((n-1)\cdots(n-c+1))^2} \\
&\times \sum_{1\le i_1<\cdots<i_c\le n} \varphi_{pc}^2(X_{i_1},\ldots,X_{i_c})\Big\}, \quad \text{using (34)},
\end{aligned}
$$

$$
\le\ 2\,e^{-(2a_c+1)tu}\,E_t \exp\Big\{b_c\,t^{c+1}\,u^{c-1}\,\frac{1}{n^{c-1}}\sum_{j=1}^{[n/c]}\varphi_{pc}^2(X_{jc-c+1},\ldots,X_{jc})\Big\}, \tag{46}
$$

by (32) and convexity, where

$$
b_c = \frac{2a_c+1}{c+1}\left(\frac{8c}{c+1}\right)^c 2^{2c-2}\,c^{c-1}\,(c-1)! \le m\,l\,(4m)^{3m}.
$$

In (46), since $1+x\le e^x\le 1+x\,e^x, \quad x\ge 0,$

$$
\begin{aligned}
&E_t \exp\Big\{b_c\,t^{c+1}\,u^{c-1}\,\frac{1}{n^{c-1}}\sum_{j=1}^{[n/c]}\varphi_{pc}^2(X_{jc-c+1},\ldots,X_{jc})\Big\} \\
&\le \left(E_t \exp\Big\{b_c\,t^{c+1}\,u^{c-1}\,\frac{1}{n^{c-1}}\varphi_{pc}^2(X_1,\ldots,X_c)\Big\}\right)^n \\
&\le \left(1+b_c\,t^{c+1}\,u^{c-1}\,\frac{1}{n^{c-1}}E_t\left(\varphi_{pc}^2(X_1,\ldots,X_c)\,e^{b_c\,t^{c+1}\,u^{c-1}\,n^{1-c}\varphi_{pc}^2(X_1,\ldots,X_c)}\right)\right)^n \\
&\le \exp\Big\{b_c\,t^{c+1}\left(\frac{u}{n}\right)^{c-2} uE_t\left(\varphi_{pc}^2(X_1,\ldots,X_c)\,\exp\left(b_c\,t^{c+1}\left(\frac{u}{n}\right)^{c-1}\varphi_{pc}^2(X_1,\ldots,X_c)\right)\right)\Big\}.
\end{aligned}
$$

By Hölder's inequality

$$
\begin{aligned}
&E_t\left(\varphi_{pc}^2(X_1,\ldots,X_c)\,\exp\left(b_c\,t^{c+1}\left(\frac{u}{n}\right)^{c-1}\varphi_{pc}^2(X_1,\ldots,X_c)\right)\right) \\
&\le \left(E_t\,\varphi_{pc}^4(X_1,\ldots,X_c)\right)^{1/2}\left(E_t\,\exp\left(2b_c\,t^{c+1}\left(\frac{u}{n}\right)^{c-1}\varphi_{pc}^2(X_1,\ldots,X_c)\right)\right)^{1/2}.
\end{aligned}
$$

Taking into account the definition of φ_{pc}, (27) and convexity we have

$$
\begin{aligned}
E_t\,\varphi_{pc}^4(X_1,\ldots,X_c) &\le 2^{8c}E_t\,h^4 \le 2^{8c}\left(E_t\,h^8\right)^{1/2}, \\
|\varphi_{pc}(x_1,\ldots,x_c)| &\le 2^{3c+2}a^{-1}e_p\left(\delta\sqrt{n}/x\right),
\end{aligned}
$$

and since $u\le 2x\sqrt{n}B_t/e_p\,(\delta\,\sqrt{n}/x)$

$$
\begin{aligned}
&E_t\,\exp\left(2b_c\,t^{c+1}\left(\frac{u}{n}\right)^{c-1}\varphi_{pc}^2(X_1,\ldots,X_c)\right) \\
&\le\ E_t\,\exp\left(2^{4c+4}\,b_c\,B_t^{c-1}\,a^{-1}\,t^{c+1}\left(\frac{x}{\sqrt{n}}\right)^{c-1}|\varphi_{pc}|\right) \\
&\le\ E_t\,\exp\left(2^{6c+4}\,b_c\,B_t^{c-1}\,a^{-1}\,t^{c+1}\left(\frac{x}{\sqrt{n}}\right)^{c-1}|h|\right) \\
&\le\ E_t\,\exp\left(\frac{a}{2}|h|\right),
\end{aligned}
$$

as $2^{6c+5}\, b_c\, B_t^{c-1}\, t^{c+1}\, (x/\sqrt{n})^{c-1} \le a^2$ under the conditions of this lemma. Hence in (46),

$$1 - V_c^{dec}(u) \le 2e^{-(2a_c+1)tu} \exp\Big\{ 2^{9c}\, b_c\, \left(E_t\, h^8\right)^{1/4} \left(E_t\, e^{\frac{a}{2}|h|}\right) t^{c+1} \left(\frac{x}{\sqrt{n}}\right)^{c-2} u \Big\}.$$

Since $2^{9c}\, b_c\, (E_t\, h^8)^{1/4} \left(E_t\, e^{\frac{a}{2}|h|}\right) t^c\, (x/\sqrt{n})^{c-2} \le a_c$, under the conditions of this lemma

$$1 - V_c^{dec}(u) \le 2e^{-(a_c+1)tu} \tag{47}$$

Estimate (44) follows from (45) and (47). This combined with (41)-(43) proves Lemma 6.

Lemma 7. *Under the conditions of Theorem 2*

$$E_t\, e^{t\,|v_n(t)|} = O(1).$$

Proof. First we consider the case when Cramér's condition (4) is satisfied and $x \le \delta\sqrt{n}/\ln_l n$. By the inequality (37)

$$E_t\, e^{t\,|v_n(t)|} \le E_t\, e^{3\,t\,|v_n^f(t)|} + E_t\, e^{3\,t\,|v_n^{\bar{f}}(t)|} + E_t\, e^{3\,t\,|\sum_{p=1}^{l-1} v_n^{\varphi_p}(t)|}. \tag{48}$$

Let $\delta > 0$ satisfy the condition of Lemma 4. Then the first expectation on the right side of (48) is $O(1)$. Since $\delta\sqrt{n}/x \ge \ln_l n$ then in Lemma 5 $e_l\,(\delta\sqrt{n}/x) \ge n$ and therefore the second expectation on the right side of (48) is $O(1)$. All of the conditions of Lemma 6 are satisfied, since $x\, n^{-1/2} \to 0$. Hence the third expectation on the right of (48) is also $O(1)$ by Lemma 6.

Next we consider the case where h^2 satisfies Cramér's condition, that is, $E \exp(a_0\, h^2) < \infty$ for some $a_0 > 0$ instead of condition (4), and $x = o(\sqrt{n})$, as $n \to \infty$. For this situation we will develop a bound as in the proof of Lemma 6. Using (37)

$$\begin{aligned} E_t\, e^{t\,|v_n(t)|} &\le \frac{1}{m-1} \sum_{c=2}^{m} E_t\, e^{(m-1)\,t\,|v_{nc}(t)|} \\ &\le \sum_{c=2}^{m} E_t\, e^{m\,t\,|v_{nc}(t)|}\, I\left(|v_{nc}(t)| > n\right) + \sum_{c=2}^{m} E_t\, e^{m\,t\,|v_{nc}(t)|}\, I\left(|v_{nc}(t)| \le n\right). \end{aligned} \tag{49}$$

Here by (32) and convexity

$$E_t\, e^{m\,t\,|v_{nc}(t)|}\, I\left(|v_{nc}(t)| > n\right)$$

$$
\begin{aligned}
&\le\ e^{-x\sqrt{n}}\ E_t \exp\Big\{m\,\Big(t+\frac{x}{\sqrt{n}}\Big)\,|v_{nc}(t)|\Big\}\\
&\le\ e^{-x\sqrt{n}}\ E_t \exp\Big\{2\,m\,\Big(t+\frac{x}{\sqrt{n}}\Big)\ |\sum_{j=1}^{[n/c]} g_{ct}(X_{jc-c+1},\ldots,X_{jc})\ |\ \Big\}\\
&\le\ e^{-x\sqrt{n}}\ \Big(E_t \exp\Big\{2\,m\,\Big(t+\frac{x}{\sqrt{n}}\Big)\,g_{ct}(X_1,\ldots,X_c)\Big\}\Big)^{[n/c]}\\
&\quad +e^{-x\sqrt{n}}\ \Big(E_t \exp\Big\{-2\,m\,\Big(t+\frac{x}{\sqrt{n}}\Big)\,g_{ct}(X_1,\ldots,X_c)\Big\}\Big)^{[n/c]}\\
&\le\ 2e^{-x\sqrt{n}}\ \exp\Big\{m^2\,x^2\,\Big(1+t\,\frac{\sqrt{n}}{x}\Big)^2\,\Big(E_t\,g_{ct}^2\Big)^{1/2}\\
&\quad \times\Big(E_t\ \exp\Big(4\,m\,\Big(t+\frac{x}{\sqrt{n}}\Big)\Big)\,|g_{ct}|\Big)^{1/2}\Big\}
\end{aligned}
$$

$$
(50)\qquad = o(1),\quad \text{as } xn^{-1/2}\to 0,\quad x\to\infty,\quad n\to\infty.
$$

Further in (49) we have

$$
(51)\qquad E_t\,e^{m\,t\,|v_{nc}(t)|}\,I\left(|v_{nc}(t)|\le n\right)\le 1+m\,t\int_0^n e^{mtu}\ (1-V_c(u))\ du,
$$

where $1-V_c(u)=P\left(|v_{nc}(t)|>u\right)$.

By Chebyshev's inequality

$$
(52)\qquad 1-V_c(u)\le e^{-(m+1)tu}\,E_t\ \exp\Big\{(m+1)\,t\,u^{\frac{c-1}{c+1}}\,|v_{nc}(t)|^{2/(c+1)}\Big\}.
$$

Here

$$
\begin{aligned}
&E_t\ \exp\Big\{(m+1)\,t\,u^{\frac{c-1}{c+1}}\,|v_{nc}(t)|^{2/(c+1)}\Big\}\\
&=\ \sum_{k=0}^{\infty}\frac{1}{k!}\Big((m+1)\,t\,u^{\frac{c-1}{c+1}}\Big)^k\ E_t|v_{nc}(t)|^{\frac{2k}{c+1}}\\
&=\ \sum_{k=0}^{[(c+1)/2]}\frac{1}{k!}\Big((m+1)\,t\,u^{\frac{c-1}{c+1}}\Big)^k\ E_t|v_{nc}(t)|^{\frac{2k}{c+1}}\\
(53)\qquad &\quad +\sum_{[(c+1)/2]+1}^{\infty}\frac{1}{k!}\Big((m+1)\,t\,u^{\frac{c-1}{c+1}}\Big)^k\ E_t|v_{nc}(t)|^{\frac{2k}{c+1}},
\end{aligned}
$$

where by Hölder's inequality

$$
E_t|v_{nc}(t)|^{\frac{2k}{c+1}}\le\Big(E_t\,v_{nc}^2(t)\Big)^{\frac{k}{c+1}}\le n^{-\frac{c-2}{c+1}k}\,\Big(E\,g_{ct}^2\Big)^{\frac{k}{c+1}},\quad 0\le k\le[(c+1)/2],
$$

and for $k\ge[(c+1)/2]+1$ by the upper decoupling inequality (34) and symmetrization

$$
E_t|v_{nc}(t)|^{\frac{2k}{c+1}}\ \le\ E_t\left|\frac{(c-1)!\,c^c}{(n-1)\cdots(n-c+1)}\ \sum_{1\le i_1<\cdots<i_c\le n} g_{ct}(X^1_{i_1},\ldots,X^c_{i_c})\right|^{\frac{2k}{c+1}}
$$

$$\leq \; E_t \left| \frac{(c-1)!\, c^c\, 2^c}{(n-1)\cdots(n-c+1)} \sum_{1\leq i_1<\cdots<i_c\leq n} \varepsilon^1_{i_1}\cdots\varepsilon^c_{i_c} g_{ct}(X^1_{i_1},\ldots,X^c_{i_c})\right|^{\frac{2k}{c+1}}.$$

Hence in (53)

$$E_t \,\exp\Big\{(m+1)\, t\, u^{\frac{c-1}{c+1}}\, |v_{nc}(t)|^{2/(c+1)}\Big\} \tag{54}$$

$$\leq \sum_{k=0}^{[(c+1)/2]} \frac{1}{k!}\left((m+1)\, t\, u^{\frac{1}{c+1}}\right)^k \left(\frac{u}{n}\right)^{\frac{c-2}{c+1}k} \left(E_t\, g^2_{ct}\right)^{\frac{k}{c+1}}$$

$$+E\,\exp\Big\{(m+1)\, t\, u^{\frac{c-1}{c+1}} \left|\frac{(c-1)!\, c^c\, 2^c}{(n-1)\cdots(n-c+1)} \sum_{1\leq i_1<\cdots<i_c\leq n} \varepsilon^1_{i_1}\cdots\varepsilon^c_{i_c} g_{ct}(X^1_{i_1},\ldots,X^c_{i_c})\right|^{\frac{2}{c+1}}\Big\}.$$

Here

$$\sum_{k=0}^{[(c+1)/2]} \frac{1}{k!}\left((m+1)\, t\, u^{\frac{1}{c+1}}\right)^k \left(\frac{u}{n}\right)^{\frac{c-2}{c+1}k} \left(E_t\, g^2_{ct}\right)^{\frac{k}{c+1}}$$

$$\leq \exp\Big\{(m+1)\, t\, u^{\frac{1}{c+1}} \left(E_t\, g^2_{ct}\right)^{\frac{1}{c+1}}\Big\}. \tag{55}$$

Using inequality (36) with $m=c,\ \alpha = 2/(c+1)$

$$E\,\exp\Big\{(m+1)\, t\, u^{\frac{c-1}{c+1}} \left|\frac{(c-1)!\, c^c\, 2^c}{(n-1)\cdots(n-c+1)} \sum_{1\leq i_1<\cdots<i_c\leq n} \varepsilon^1_{i_1}\cdots\varepsilon^c_{i_c} g_{ct}(X^1_{i_1},\ldots,X^c_{i_c})\right|^{\frac{2}{c+1}}\Big\}$$

$$\leq \; 2E_t\,\exp\Big\{\frac{(m+1)^{c-1}}{c+1}\left(\frac{8c}{c+1}\right)^c t^{c+1}\, u^{c-1} \left(\frac{(c-1)!\, c^c\, 2^c}{(n-1)\cdots(n-c+1)}\right)^2$$

$$\times \sum_{1\leq i_1<\cdots<i_c\leq n} g^2_{ct}(X^1_{i_1},\ldots,X^c_{i_c})\Big\}$$

$$\leq \; 2E_t\,\exp\Big\{\frac{(m+1)^{c-1}}{c+1}\left(\frac{8c}{c+1}\right)^c 2^{2(c-1)}\,(c-1)!\, t^{c+1}\, u^{c-1} \left(\frac{(c-1)!\, c^c\, 2^c}{(n-1)\cdots(n-c+1)}\right)^2$$

$$\times \sum_{1\leq i_1<\cdots<i_c\leq n} g^2_{ct}(X_{i_1},\ldots,X_{i_c})\Big\},\ \text{using (34)},$$

$$\leq \; 2E_t\,\exp\Big\{(4m)^{5m}\, t^{c+1} \left(\frac{u}{n}\right)^{c-1} \sum_{j=1}^{[n/c]} g^2_{ct}(X_{jc-c+1},\ldots,X_{jc})\Big\},$$

using (32) and convexity,

$$\leq \; 2\left(E_t\,\exp\Big\{(4m)^{5m}\, t^{c+1}\left(\frac{u}{n}\right)^{c-1} g^2_{ct}(X_1,\ldots,X_c)\Big\}\right)^n$$

$$= \; 2\,\exp\Big\{(4m)^{5m}\, t^{c+1}\left(\frac{u}{n}\right)^{c-2} u\, Eg^2_{ct}\Big\}$$

$$\times\left(E_t\,\exp\Big\{(4m)^{5m}\, t^{c+1}\left(\frac{u}{n}\right)^{c-1}\left(g^2_{ct}(X_1,\ldots,X_c) - E\, g^2_{ct}\right)\Big\}\right)^n$$

$$\leq \; 2\,\exp\Big\{(4m)^{5m}\, t^{c+1}\left(\frac{u}{n}\right)^{c-2} u\, Eg^2_{ct}\Big\}\,\exp\Big\{(4m)^{10m}\, t^{2(c+1)}\left(\frac{u}{n}\right)^{2c-2}$$

$$\times n\left(E_t\left(g_{ct}^2 - E\,g_{ct}^2\right)^4\right)^{1/2}\left(E_t \exp\left(2\,(4m)^{5m}\,t^{c+1}\,\left(\frac{u}{n}\right)^{c-1}\,|g_{ct}^2 - E\,g_{ct}^2|\right)\right)^{1/2}\Bigg\}$$

$$\leq\ 2\exp\Big\{(4m)^{5m}\,t^{c+1}\,u\,Eg_{ct}^2\Big\}$$

$$\times\exp\Big\{4\,(4m)^{10m}\,t^{2(c+1)}\,u\left(E_t\,g_{ct}^4\right)^{1/2}\ E_t\exp\left(2\,(4m)^{5m}\,t^{c+1}\,g_{ct}^2\right)\Big\}. \tag{56}$$

Since $xn^{-1/2} \to 0$ and $E\ \exp\{a_0 h^2\} < \infty$, then from (51)-(56) we obtain the estimate

$$E_t\,e^{mt|v_{nc}(t)|}\,I\left(|v_{nc}(t)| \leq n\right) = O(1).$$

This combined with (49)-(50) proves Lemma 7.

Now we will apply Lemmas 4-7 to the estimation of $\mathcal{T}_n(t)$. We introduce the following notation where the dependence on t has been suppressed. Let

$$\tilde{S} = \frac{1}{\sigma_t\sqrt{n}}\sum_{i=1}^n (g_t(X_i) - \mu_t),$$

$$\tilde{v}_n = \frac{1}{\sigma_t\sqrt{n}}\,v_n(t), \qquad \Phi_n(x) = P_t(\tilde{S} + \tilde{v}_n \leq x),\ x \in R,$$

$\sigma_t^2 = E_t(g_t(X_1) - \mu_t)^2$ and $x_t = \sqrt{n}t\sigma_t$. Then $\mathcal{T}_n(t)$ can be written as

$$\mathcal{T}_n(t) = E_t e^{-x_t\tilde{S}}\ I(\tilde{S} + \tilde{v}_n > 0).$$

With the help of (39) we have the following representation

$$\begin{aligned}
\mathcal{T}_n(t) &= E_t e^{x_t\tilde{v}_n}\ \Big\{e^{-x_t(\tilde{S}+\tilde{v}_n)}\ I(\tilde{S}+\tilde{v}_n > 0)\Big\} \\
&= E_t\Big\{e^{-x_t(\tilde{S}+\tilde{v}_n)}\ I(\tilde{S}+\tilde{v}_n > 0)\Big\} \\
&\quad + x_t\int_0^1 E_t\tilde{v}_n e^{ux_t\tilde{v}_n}\ \Big\{e^{-x_t(\tilde{S}+\tilde{v}_n)}\ I(\tilde{S}+\tilde{v}_n > 0)\Big\}\,du \\
&= I_n(t) + t\int_0^1 I_n(u,t)du, \qquad (57)
\end{aligned}$$

where

$$I_n(t) = E_t\Big\{e^{-x_t(\tilde{S}+\tilde{v}_n)}\ I(\tilde{S}+\tilde{v}_n > 0)\Big\},$$

$$I_n(u,t) = E_t v_n(t) e^{utv_n(t)}\ \Big\{e^{-x_t(\tilde{S}+\tilde{v}_n)}\ I(\tilde{S}+\tilde{v}_n > 0)\Big\}.$$

Estimate $\mathbf{I_n(t)}$.

By definition

$$
\begin{aligned}
I_n(t) &= \int e^{-x_t y}\ I(y>0)\ d\Phi_n(y) \\
&= \int_0^\infty e^{-x_t y} d\Phi_n(y) \\
&= \int_0^\infty e^{-x_t y} d\Phi(y) + \int_0^\infty e^{-x_t y} d(\Phi_n(y) - \Phi(y)). \qquad (58)
\end{aligned}
$$

Here by (7), (23) and (39)

$$
\begin{aligned}
\int_0^\infty e^{-x_t y} d\Phi(y) &= \int_0^\infty e^{-x(1+t\cdot O(1))y} d\Phi(y) \\
&= \int_0^\infty e^{-xy} d\Phi(y) \\
&\quad - \int_0^1 \Big\{ xt\cdot O(1) \int_0^\infty y e^{-xy-xtyu\cdot O(1)} d\Phi(y) \Big\} du.
\end{aligned}
$$

Further,

$$\int_0^\infty e^{-xy} d\Phi(y) = e^{x^2/2}\ (1-\Phi(x))$$

and since $t = \frac{x}{\sigma\sqrt{n}}\left(1 + O\left(\frac{x}{\sqrt{n}}\right)\right)$ and $e^{-y^2/2} \le 1$

$$xt \int_0^\infty y e^{-xy-xtyu\cdot O(1)} d\Phi(y) \le \frac{x^2}{\sigma\sqrt{n}}(1+O(1)) \int_0^\infty y e^{-(x+xtuO(1))y} dy.$$

After the substitution $z = (x + xtu\cdot O(1))y$

$$
\begin{aligned}
& xt \int_0^\infty y e^{-xy-xtyu\cdot O(1)} d\Phi(y) \\
&\le \frac{x^2}{\sigma\sqrt{n}}(1+O(1))(x+xtu\cdot O(1))^{-2} \int_0^\infty e^{-z} z dz \\
&= e^{x^2/2}(1-\Phi(x))\cdot O\left(\frac{1+x}{\sqrt{n}}\right).
\end{aligned}
$$

Consequently,

$$\int_0^\infty e^{-x_t y} d\Phi(y) = e^{x^2/2}(1-\Phi(x))\left(1 + O\left(\frac{1+x}{\sqrt{n}}\right)\right). \qquad (59)$$

Then we have in (58)

$$\int_0^\infty e^{-x_t y} d(\Phi_n(y) - \Phi(y)) = -(\Phi_n(0) - \Phi(0)) + x_t \int_0^\infty (\Phi_n(y) - \Phi(y)) e^{-x_t y} dy.$$

By the normal approximation theorem for U-statistics (see Borovskikh (1996), (1998))

$$\sup_y |\Phi_n(y) - \Phi(y)| \le A_m(a_t + b_t)\frac{1}{\sqrt{n}}, \qquad (60)$$

where A_m is a positive constant depending only on m, and

$$\begin{aligned} a_t &= \sigma_t^{-3}\ E_t|g_t(X_1)-\mu_t|^3, \\ b_t &= \sum_{c=2}^m \sigma_t^{-\delta_c}\ E_t|g_{ct}(X_1,\ldots,X_c)|^{\delta_c} \end{aligned}$$

with $\delta_c = (2c+1)/(2c-1)$.

By applying (6)-(8) and (23) and we have in (60)

$$\begin{aligned} a_t &= \sigma^{-3}\ E|g(X)|^3\ (1+O(\frac{x}{\sqrt{n}})), \\ b_t &= \sum_{c=2}^m \sigma^{-\delta_c}\ E|g_c(X_1,\ldots,X_c)|^{\delta_c}\ (1+O(\frac{x}{\sqrt{n}})), \end{aligned}$$

as $xn^{-1/2} \to 0$. Hence we get the following estimate

$$|\int_0^\infty e^{-x_t y} d(\Phi_n(y)-\Phi(y))| = e^{x^2/2}(1-\Phi(x))\ O\left(\frac{1+x}{\sqrt{n}}\right). \tag{61}$$

From (58), (59) and (61) we see that

$$I_n(t) = e^{x^2/2}(1-\Phi(x))\ \left(1+O\left(\frac{1+x}{\sqrt{n}}\right)\right). \tag{62}$$

Estimate $\mathbf{I_n(u,t)}$.

First we will prove that for any $m \geq 2$

$$|I_n(u,t)| \leq c_1\left(\mathcal{R}(t)+\bar{\mathcal{R}}(t)\right) + c_2 n^{-1/2}, \tag{63}$$

where

$$\begin{aligned} \mathcal{R}(t) &= E_t\{e^{2t|v_{n2}(t)|}[e^{-x_t(\tilde{S}+\tilde{v}_{n2})} I(\tilde{S}+\tilde{v}_{n2}+\varepsilon>0)]\}, \\ \bar{\mathcal{R}}(t) &= E_t\Big\{v_{n2}^2(t)\left[e^{-x_t(\tilde{S}+\tilde{v}_{n2})}\ I\left(\tilde{S}+\tilde{v}_{n2}+\varepsilon>0\right)\right]\Big\} \end{aligned}$$

for some positive constants c_i and $\varepsilon = n^{-\frac{1}{2}}$. For $0 \leq u \leq 1$ we have

$$|I_n(u,t)| \leq E_t\{(|v_{n2}(t)|+|r_n|)e^{t(|v_{n2}(t)|+|r_n|)}\ [e^{-x_t(\tilde{S}+\tilde{v}_{n2})} I(\tilde{S}+\tilde{v}_{n2}+\tilde{r}_n>0)]\}$$

where $r_n = v_{n3}(t)+\cdots+v_{nm}(t)$ and $\tilde{r}_n = r_n/(\sigma_t\sqrt{n})$.

Applying the inequalities $e^x - 1 \leq xe^x,\ x \geq 0$ with $x = t|r_n|$ and $\tilde{r}_n > -(\tilde{S}+\tilde{v}_{n2})$ we get

$$\begin{aligned} |I_n(u,t)| \leq\ & E_t\{|v_{n2}(t)|\,e^{t|v_{n2}(t)|}[e^{-x_t(\tilde{S}+\tilde{v}_{n2})} I(\tilde{S}+\tilde{v}_{n2}+\tilde{r}_n>0)]\} \\ & +E_t\{|r_n|(1+t|v_{n2}(t)|)\exp(2t(|v_{n2}(t)|+|r_n|))\}. \end{aligned}$$

By Hölder's inequality and Lemma 7

$$E_t\{|r_n|(1+t|v_{n2}(t)|)\exp(2t(|v_{n2}(t)|+|r_n|)\} \le c_3\left(E_t r_n^2\right)^{1/2}.$$

Here

$$\begin{aligned} E_t r_n^2 &\le (m-2)m^{m-2}n^{-1}\sum_{c=3}^{m} E_t|n^{-c/2}\sum_{1\le i_1<\cdots<i_c\le n} g_{ct}(X_{i_1},\ldots,X_{i_c})|^2 \\ &= (m-2)m^{m-2}n^{-1}\sum_{c=3}^{m}\binom{n}{c} n^{-c}E_t g_{ct}^2(X_1,\ldots,X_c). \end{aligned}$$

Further write

$$\begin{aligned} & E_t\{|v_{n2}(t)|e^{t|v_{n2}(t)|}[e^{-x_t(\tilde S+\tilde v_{n2})}I(\tilde S+\tilde v_{n2}+\tilde r_n>0)]\} \\ &= E_t\{|v_{n2}(t)|e^{t|v_{n2}(t)|}[e^{-x_t(\tilde S+\tilde v_{n2})}I(\tilde S+\tilde v_{n2}+\tilde r_n>0)I(|\tilde r_n|\le\varepsilon)]\} \\ &\quad +E_t\{|v_{n2}(t)|e^{t|v_{n2}(t)|}[e^{-x_t(\tilde S+\tilde v_{n2})}I(\tilde S+\tilde v_{n2}+\tilde r_n>0)I(|\tilde r_n|>\varepsilon)]\} \\ &\le \mathcal{R}(t)+\bar{\mathcal{R}}(t)+E_t\{|v_{n2}(t)|e^{t(|v_{n2}(t)|+|r_n|)}\,I(|\tilde r_n|>\varepsilon)\} \\ &\le \mathcal{R}(t)+\bar{\mathcal{R}}(t)+\sigma_t\,E_t\{|v_{n2}(t)|\cdot|r_n|\,e^{t(|v_{n2}(t)|+|r_n|)}\}. \end{aligned}$$

The inequality (63) follows by applying Hölder's inequality and Lemma 7.

Partitioning estimates. In what follows we shall estimate $\mathcal{R}(t)$ and $\bar{\mathcal{R}}(t)$. We do this by partitioning $v_{n2}(t)$ into several terms. We will give detailed estimates for $\mathcal{R}(t)$ only. The bounds for $\bar{\mathcal{R}}(t)$ can be obtained by repeating the same method step by step. Now we will prove that

$$\mathcal{R}(t)\le c_4 x^{-1}. \tag{64}$$

For $n\to\infty$ let $k=[n/2],\ k_1=[k/2], k_2=k-k_1,\ n_1=[(n-k)/2]$ and $n_2=n-k-n_1$, where $[z]$ denotes the integer part of z. Then

$$v_{n2}(t)=v_k^1+v_{n-k}^2+v_{k_1,n-k}^3+v_{k_2,n-k}^4, \tag{65}$$

where

$$\begin{aligned} v_k^1 &= \frac{m-1}{n-1}\sum_{1\le i_1<i_2\le k} g_{2t}(X_{i_1},X_{i_2}), \\ v_{n-k}^2 &= \frac{m-1}{n-1}\sum_{k+1\le i_1<i_2\le n} g_{2t}(X_{i_1},X_{i_2}), \end{aligned}$$

$$
\begin{aligned}
v^3_{k_1,n-k} &= \frac{m-1}{n-1}\sum_{i_1=1}^{k_1}\sum_{i_2=k+1}^{n} g_{2t}(X_{i_1},X_{i_2}),\\
v^4_{k_2,n-k} &= \frac{m-1}{n-1}\sum_{i_1=k_1+1}^{k}\sum_{i_2=k+1}^{n} g_{2t}(X_{i_1},X_{i_2}).
\end{aligned}
$$

Further let

$$
v^3_{k_1,n-k} = v^{31}_{k_1n_1} + v^{32}_{k_1n_2}, \tag{66}
$$

where

$$
\begin{aligned}
v^{31}_{k_1n_1} &= \frac{m-1}{n-1}\sum_{i_1=1}^{k_1}\sum_{i_2=k+1}^{n_1} g_{2t}(X_{i_1},X_{i_2}),\\
v^{32}_{k_1n_2} &= \frac{m-1}{n-1}\sum_{i_1=1}^{k_1}\sum_{i_2=n_1+1}^{n} g_{2t}(X_{i_1},X_{i_2}),
\end{aligned}
$$

and

$$
v^4_{k_2,n-k} = v^{41}_{k_2n_1} + v^{42}_{k_2n_2}, \tag{67}
$$

where

$$
\begin{aligned}
v^{41}_{k_2n_1} &= \frac{m-1}{n-1}\sum_{i_1=k_1+1}^{k}\sum_{i_2=k+1}^{n_1} g_{2t}(X_{i_1},X_{i_2}),\\
v^{42}_{k_2n_2} &= \frac{m-1}{n-1}\sum_{i_1=k_1+1}^{k}\sum_{i_2=n_1+1}^{n} g_{2t}(X_{i_1},X_{i_2}).
\end{aligned}
$$

We also define the corresponding partitioned terms for the normalized quantity $\tilde{v}_{n2}$. For any real a, b and c we have the following inequality

$$
I(a+b>c) \le I(2a>c) + I(2b>c). \tag{68}
$$

Hence by (65) and four applications of the inequality (68)

$$
\begin{aligned}
I(\tilde{S}+\tilde{v}_{n2}+\varepsilon>0) &\le I(\tilde{S}+2\tilde{v}^1_k>0)\\
&\quad +I(\tilde{S}+2\tilde{v}^2_{n-k}+2\tilde{v}^3_{k_1,n-k}+2\tilde{v}^4_{k_2,n-k}+2\varepsilon>0)\\
&\le I(\tilde{S}+2\tilde{v}^1_k>0)+I(\tilde{S}+4\tilde{v}^2_{n-k}>0)\\
&\quad +I(\tilde{S}+4\tilde{v}^3_{k_1,n-k}+4\tilde{v}^4_{k_2,n-k}+4\varepsilon>0)\\
&\le I(\tilde{S}+2\tilde{v}^1_k>0)+I(\tilde{S}+4\tilde{v}^2_{n-k}>0)\\
&\quad +I(\tilde{S}+8\tilde{v}^3_{k_1,n-k}>0)+I(\tilde{S}+16\tilde{v}^4_{k_2,n-k}>0)\\
&\quad +I(\tilde{S}+16\varepsilon>0).
\end{aligned} \tag{69}
$$

Using (69) we can bound $\mathcal{R}(t)$ as follows

$$\begin{aligned}\mathcal{R}(t) \;\le\; & E_t\{e^{2t|v_{n2}(t)|}[e^{-x_t(\tilde S+\tilde v_{n2})}I(\tilde S+2\tilde v_k^1>0)]\}\\ &+E_t\{e^{2t|v_{n2}(t)|}[e^{-x_t(\tilde S+\tilde v_{n2})}I(\tilde S+4\tilde v_{n-k}^2>0)]\}\\ &+E_t\{e^{2t|v_{n2}(t)|}[e^{-x_t(\tilde S+\tilde v_{n2})}I(\tilde S+8\tilde v_{k_1,n-k}^3>0)]\}\\ &+E_t\{e^{2t|v_{n2}(t)|}[e^{-x_t(\tilde S+\tilde v_{n2})}I(\tilde S+16\tilde v_{k_2,n-k}^4>0)]\}\\ &+E_t\{e^{2t|v_{n2}(t)|}[e^{-x_t(\tilde S+\tilde v_{n2})}I(\tilde S+16\varepsilon>0)]\}.\end{aligned} \tag{70}$$

In (70) each expectation has the same structure. Consider the first expectation in (70). By (37) and (65)

$$\begin{aligned}e^{2t|v_{n2}(t)|}[e^{-x_t(\tilde S+\tilde v_{n2})}] &= e^{2t|v_{n2}(t)|+tv_k^1-tv_{n-k}^2-tv_{k_1,n-k}^3-tv_{k_2,n-k}^4}e^{-x_t(\tilde S+2\tilde v_k^1)}\\ &\le \sum_{i=1}^4 q_{i1}^{-1}e^{\delta_{i1}|v^i|}[e^{-x_t(\tilde S+2\tilde v^1)}],\end{aligned}$$

where for simplicity we write $v^1=v_k^1$, $v^2=v_{n-k}^2$ and so forth. Hence from (70) we obtain

$$\mathcal{R}(t)\le\sum_{i=1}^4\sum_{j=1}^4 q_{ij}^{-1}Q_{ij}+\sum_{i=1}^4 q_i^{-1}Q_i, \tag{71}$$

where

$$\begin{aligned}Q_{ij} &= E_t\{e^{\delta_{ij}t|v^i|}[e^{-x_t(\tilde S+2^j\tilde v^j)}I(\tilde S+2^j\tilde v^j>0)]\},\\ Q_i &= E_t\{e^{\delta_i t|v^i|}[e^{-x_t\tilde S}I(\tilde S+16\varepsilon>0)]\},\end{aligned}$$

δ_i and δ_{ij} are positive constants, and $\varepsilon=n^{-1/2}$.

We shall use (66) and (67) to further partition the expectations when i or j takes the value 3 or 4. Thus using (37), for some constants δ_j^i,

$$\begin{aligned}Q_{31} \;\le\; & E_t\{e^{\delta_1^1 t|v^{31}|}[e^{-x_t(\tilde S+2\tilde v^1)}I(\tilde S+2\tilde v^1>0)]\}\\ &+E_t\{e^{\delta_1^2 t|v^{32}|}[e^{-x_t(\tilde S+2\tilde v^1)}I(\tilde S+2\tilde v^1>0)]\},\end{aligned} \tag{72}$$

with a similar expression for Q_{41}. For terms like Q_{13} and Q_{14} we have

$$\begin{aligned}Q_{13} \;\le\; & E_t\{e^{\delta_2^1 t|v^1|}[e^{-x_t(\tilde S+16\tilde v^{31})}I(\tilde S+16\tilde v^{31}>0)]\}\\ &+E_t\{e^{\delta_2^2 t|v^1|}[e^{-x_t(\tilde S+16\tilde v^{32})}I(\tilde S+16\tilde v^{32}>0)]\}\\ &+E_t\{e^{\delta_2^3 t|v^{31}|}[e^{-x_t(\tilde S+16\tilde v^{31})}I(\tilde S+16\tilde v^{31}>0)]\}\end{aligned}$$

$$+E_t\{e^{\delta_2^4 t|v^{31}|}[e^{-x_t(\tilde{S}+16\tilde{v}^{32})}I(\tilde{S}+16\tilde{v}^{32}>0)]\}$$

$$+E_t\{e^{\delta_2^5 t|v^{32}|}[e^{-x_t(\tilde{S}+16\tilde{v}^{31})}I(\tilde{S}+16\tilde{v}^{31}>0)]\}$$

$$+E_t\{e^{\delta_2^6 t|v^{32}|}[e^{-x_t(\tilde{S}+16\tilde{v}^{32})}I(\tilde{S}+16\tilde{v}^{32}>0)]\}, \tag{73}$$

and for terms like Q_{34} and Q_{43} we have

$$\begin{aligned}
Q_{43} \leq\ & E_t\{e^{\delta_3^1 t|v^{41}|}[e^{-x_t(\tilde{S}+16\tilde{v}^{31})}I(\tilde{S}+16\tilde{v}^{31}>0)]\} \\
& +E_t\{e^{\delta_3^2 t|v^{41}|}[e^{-x_t(\tilde{S}+16\tilde{v}^{32})}I(\tilde{S}+16\tilde{v}^{32}>0)]\} \\
& +E_t\{e^{\delta_3^3 t|v^{42}|}[e^{-x_t(\tilde{S}+16\tilde{v}^{31})}I(\tilde{S}+16\tilde{v}^{31}>0)]\} \\
& +E_t\{e^{\delta_3^4 t|v^{42}|}[e^{-x_t(\tilde{S}+16\tilde{v}^{32})}I(\tilde{S}+16\tilde{v}^{32}>0)]\} \\
& +E_t\{e^{\delta_3^5 t|v^{31}|}[e^{-x_t(\tilde{S}+16\tilde{v}^{31})}I(\tilde{S}+16\tilde{v}^{31}>0)]\} \\
& +E_t\{e^{\delta_3^6 t|v^{31}|}[e^{-x_t(\tilde{S}+16\tilde{v}^{32})}I(\tilde{S}+16\tilde{v}^{32}>0)]\} \\
& +E_t\{e^{\delta_3^7 t|v^{32}|}[e^{-x_t(\tilde{S}+16\tilde{v}^{31})}I(\tilde{S}+16\tilde{v}^{31}>0)]\} \\
& +E_t\{e^{\delta_3^8 t|v^{32}|}[e^{-x_t(\tilde{S}+16\tilde{v}^{32})}I(\tilde{S}+16\tilde{v}^{32}>0)]\}.
\end{aligned} \tag{74}$$

All of the terms we need to bound to obtain (64) have similar structure. Let

$$Q = E_t\{e^{\delta t|\Delta_n|}[e^{-x_t(\xi_n+\eta_n)}I(\xi_n+\eta_n>0)]\}, \tag{75}$$

where

$$\xi_n = \frac{1}{\sigma_t\sqrt{n}}\sum_{j\in J} g_t(X_j) + \tilde{\Delta}_{n1},$$

$$\eta_n = \frac{1}{\sigma_t\sqrt{n}}\sum_{j\in \bar{J}} g_t(X_j) + \tilde{\Delta}_{n2},$$

Δ_n is part of the sum v_n, $\tilde{\Delta}_{n1}$ and $\tilde{\Delta}_{n2}$ are parts of the normalized sum $\tilde{v}_n$, and $\bar{J} = \{1,\ldots,n\}\backslash J$. The set of indices J has cardinality $|J| \sim \lambda n$, $0<\lambda<1$ as $n\to\infty$. Here the random variable ξ_n is independent of Δ_n and η_n.

The expectations Q_i in (71) have a simpler form than (75). All expectations (72) - (74) have the structure (75). The following lemma provides a bound for Q in (75).

Lemma 8. *Under the conditions of Theorem 2*

$$Q \leq c_5 x^{-1} + c_6 n^{-1/2}$$

for some positive constants c_5 and c_6.

Proof. Let $\lambda_n = (|J|/n)^{1/2}$ and

$$\tilde{\xi}_n = \frac{1}{\sigma_t\sqrt{|J|}}\sum_{j\in J} g_t(X_j) + \lambda_n^{-1}\tilde{\Delta}_{n1}$$

and $\tilde{\Phi}_n(u) = P(\tilde{\xi}_n \le u),\ u \in R$. Further, in (75), since ξ_n is independent of Δ_n and η_n we can write

$$Q = E_t\{e^{\delta t|\Delta_n| - x_t\eta_n} E^{\xi} e^{-x_t\xi_n} I(\xi_n + \eta_n > 0)\}, \tag{76}$$

where $E^{\xi}\{\cdots\}$ denotes the expectation with respect to ξ_n. Here (see the proof of (62)),

$$\begin{aligned}
&E^{\xi} e^{-x_t\xi_n} I(\xi_n + \eta_n > 0) \\
&= \int_{-\infty}^{\infty} e^{-x_t\lambda_n u} I(\lambda_n u + \eta_n > 0) d\tilde{\Phi}_n(u) \\
&= \int_{-\eta_n\lambda_n^{-1}}^{\infty} e^{-x_t\lambda_n u} d(\tilde{\Phi}_n(u) - \Phi(u)) + \int_{-\eta_n\lambda_n^{-1}}^{\infty} e^{-x_t\lambda_n u} d\Phi(u).
\end{aligned} \tag{77}$$

In (77)

$$\begin{aligned}
&\int_{-\eta_n\lambda_n^{-1}}^{\infty} e^{-x_t\lambda_n u} d(\tilde{\Phi}_n(u) - \Phi(u)) \\
&= -e^{x_t\eta_n}(\tilde{\Phi}_n(-\eta_n\lambda_n^{-1}) - \Phi(-\eta_n\lambda_n^{-1})) + x_t\lambda_n \int_{-\eta_n\lambda_n^{-1}}^{\infty} e^{-x_t\lambda_n u}(\tilde{\Phi}_n(u) - \Phi(u))du
\end{aligned}$$

and by the inequality (60)

$$\left|\int_{-\eta_n\lambda_n^{-1}}^{\infty} e^{-x_t\lambda_n u} d(\tilde{\Phi}_n(u) - \Phi(u))\right| \le c_7 e^{x_t\eta_n}\frac{1}{\sqrt{n}}.$$

Further in (77)

$$\begin{aligned}
\int_{-\eta_n\lambda_n^{-1}}^{\infty} e^{-x_t\lambda_n u} d\Phi(u) &= \frac{1}{\sqrt{2\pi}}\int_{-\eta_n\lambda_n^{-1}}^{\infty} e^{-x_t\lambda_n u} e^{-u^2/2} du \\
&\le \int_{-\eta_n\lambda_n^{-1}}^{\infty} e^{-x_t\lambda_n u} du \\
&= (x_t\lambda_n)^{-1} e^{x_t\eta_n},
\end{aligned}$$

where $x_t \sim x,\ \lambda_n \sim \lambda$ as $n \to \infty$. Hence in (76)

$$Q \le (c_7\frac{1}{\sqrt{n}} + c_8\frac{1}{x}) E e^{\delta t|\Delta_n|}. \tag{78}$$

Note that for the expectation of Q_i in (71) we can replace (78) by the inequality

$$Q \leq \left(c_9 \frac{1}{\sqrt{n}} + c_{10} \frac{1}{x} \right) e^{\delta^1 \varepsilon x_t} E e^{\delta t |\Delta_n|}. \tag{79}$$

From (78)-(79) and Lemma 7 we obtain the proof of Lemma 8, and hence the desired estimate (64). Combining (28)-(29), (57) and (62)-(64) completes the proof of Theorem 2.

Proof of Theorem 3. Define the truncated kernel $\bar{h} = h\, I\left(|h| \leq n^{\frac{1}{2}+\alpha}\right)$ and introduce the U-statistic $\bar{U}_n$ with this kernel $\bar{h}$. In what follows $\bar{\mu}$, $\bar{\sigma}$, $\bar{t}$, $\bar{\psi}, \ldots$ denote μ, σ, t, $\psi, \ldots$ respectively, where, instead of the kernel h we substitute the truncated kernel $\bar{h}$. For example,

$$\begin{aligned}
\bar{g}(x) &= E\left(\bar{h}(X_1, \ldots, X_m) | X_m = x\right), \\
\bar{\mu} &= E\, \bar{h}(X_1, \ldots, X_m), \\
\bar{\sigma}^2 &= E\left(\bar{g}(X) - \bar{\mu}\right)^2.
\end{aligned}$$

Lemma 9. *Under Linnik's condition (12) for* $1 \leq x \leq o(n^\alpha)$

$$\begin{aligned}
&P\left(U_n - \mu\, n/m > x\sigma\sqrt{n}\right) - P\left(\bar{U}_n - \bar{\mu}\, n/m > x\bar{\sigma}\sqrt{n}\right) \\
&\quad = (1 - \Phi(x))\, \exp\left\{ \frac{x^3}{\sqrt{n}} \lambda_m^{[s]} \left(\frac{x}{\sqrt{n}} \right) \right\} O\left(\frac{x}{\sqrt{n}} \right).
\end{aligned}$$

Proof. It is not difficult to prove (see, for example, Borovskikh (1998)) that

$$\begin{aligned}
&|P\left(U_n - \mu\, n/m > x\sigma\sqrt{n}\right) - P\left(\bar{U}_n - \bar{\mu}\, n/m > x\bar{\sigma}\sqrt{n}\right)| \\
&\quad \leq \frac{1}{2} n\,(n-1)\, P\left(|h| > n^{\frac{1}{2}+\alpha}\right),
\end{aligned}$$

and

$$P\left(|h| > n^{\frac{1}{2}+\alpha}\right) \leq e^{-a_\alpha n^{2\alpha}}\, E\, \exp\left\{ a_\alpha\, |h|^{\frac{4\alpha}{2\alpha+1}} \right\}.$$

Since $1 \leq x \leq o(n^\alpha)$ then

$$\begin{aligned}
e^{-a_\alpha n^{2\alpha}} &= e^{-a_\alpha n^{2\alpha}/2}\, \exp\left\{ \frac{x^3}{\sqrt{n}} \lambda_m^{[s]} \left(\frac{x}{\sqrt{n}} \right) - \frac{x^2}{2} \right\} \\
&\quad \times \exp\left\{ -n^{2\alpha} \left[\frac{a_\alpha}{2} - \left(\frac{x}{n^\alpha} \right)^2 \left(\frac{1}{2} - \frac{x}{\sqrt{n}} \lambda_m^{[s]} \left(\frac{x}{\sqrt{n}} \right) \right) \right] \right\} \\
&= e^{-a_\alpha n^{2\alpha}/2}\, \exp\left\{ \frac{x^3}{\sqrt{n}} \lambda_m^{[s]} \left(\frac{x}{\sqrt{n}} \right) \right\} (1 - \Phi(x)) \cdot O(1).
\end{aligned}$$

This proves Lemma 9.

Lemma 10. *If* $1 \le x \le o(n^\alpha)$ *and Linnik's condition (12) is satisfied then*

$$P\left(\bar U_n - \bar\mu\, n/m > x\bar\sigma\sqrt{n}\right)$$
$$= (1-\Phi(x))\,\exp\Big\{\frac{x^3}{\sqrt{n}}\,\lambda_m^{[s]}\left(\frac{x}{\sqrt{n}}\right)\Big\}\left(1+O\left(\frac{x}{\sqrt{n}}\right)\right).$$

Proof. We can apply Theorem 2. Under our new conditions we take $a = a_\alpha\, n^{-\frac{1}{2}+\alpha}$. Then the connection between Cramér's condition (4) and Linnik's condition (12) is given by the inequality

$$E\,\exp\Big\{a|\bar h|\Big\} \le E\,\exp\Big\{a_\alpha |h|^{\frac{4\alpha}{2\alpha+1}}\Big\}.$$

This guarantees that the key results of Lemma 1 and Lemma 7 hold. By analogy with (29)

$$n\,\left(\bar\psi(\bar t) - \bar t\,\bar\mu_{\bar t}\right) = -\frac{x^2}{2} + \frac{x^3}{\sqrt{n}}\,\bar\lambda_m\left(\frac{x}{\sqrt{n}}\right).$$

Hence by virtue of Theorem 2 for $1 \le x \le o(n^\alpha)$ under the Linnik's condition (12)

$$P\left(\bar U_n - \bar\mu\, n/m > x\bar\sigma\sqrt{n}\right)$$
$$= (1-\Phi(x))\,\exp\Big\{\frac{x^3}{\sqrt{n}}\,\bar\lambda_m\left(\frac{x}{\sqrt{n}}\right)\Big\}\left(1+O\left(\frac{x}{\sqrt{n}}\right)\right)$$

with the Cramér series

$$\bar\lambda_m(u) = \sum_{k=0}^{\infty} \bar\lambda_{mk}\, u^k$$

which converges for $0 \le u \le \delta\, n^{-\frac{1}{2}+\alpha}$ with sufficiently small $\delta > 0$ and which is obtained from the Cramér series (11) where the coefficients $\bar\lambda_{mk}$, depending on the moments of the truncated kernel $\bar h$, are used instead of the coefficients λ_{mk}. It is straightforward to proceed from here to obtain the statement of Lemma 10. Theorem 3 follows directly from Lemmas 9 and 10.

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