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MULTIVARIATE SOBEL-UPPULURI-GALAMBOS-TYPE BOUNDS

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SUMMARY

The upper and lower bounds in a recent multivariate generalization (Galambos and Xu, 1996) of the univariate Sobel-Uppuluri-Galambos inequalities are shown in the following to be weighted averages of individual multivariate bounds, and hence can be sharpened by optimizing over these individual bounds. Examples are used to illustrate the improvement and to illustrate the difference between bounds of this kind, and the kind of multivariate bounds appearing in Chen and Seneta (1996) and Galambos and Lee(1994). We consider only bounds for the probability of the intersection of exactly k_i events occurring, from the ith event group of n_i events, $i = 1, ..., d$.

Keywords: Degree of Bonferroni-type inequality, Meyer's identity, Sobel-Uppuluri inequalities, optimization, weighted average.

1. INTRODUCTION and MAIN RESULTS

Suppose $A_1 ... A_n$ are arbitrary events in an arbitrary probability space (Ω, B, P) and v_1 be the number of events which occur at a given sample point. For any positive integers u and r, Sobel-Uppuluri bounds refer to

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$$
\begin{aligned}
&\sum_{k=0}^{2u+1}(-1)^k\binom{k+r}{r}S_{k+r}+\frac{2u+2}{n-r}\binom{2u+r}{r}S_{2u+r+2}\\
&\leq P(v_1=r) \qquad (1)\\
&\leq \sum_{k=0}^{2u}(-1)^k\binom{k+r}{r}S_{k+r}-\frac{2u+1}{n-r}\binom{2u+r+1}{r}S_{2u+r+1},
\end{aligned}
$$

where

$$S_i = E(\binom{v_1}{i}) \qquad \text{for} \quad 1 \leq i \leq n.$$

(See for example, Galambos 1978 or Recsei and Seneta 1987).

A bound (upper or lower) is called "a degree s bound" if $\max_{i\in I} i = s$, where I is the index set of all S_i used in the bound. A higher degree bound usually provides more accurate evaluation at the cost of more complicated computation in calculating S_i's, since higher dimensional joint probabilities are involved. The application of (1) is extensive; details can be found in sources such as the book by Galambos and Simonelli (1996).

In multivariate setting, d sets of events are considered, A_{ij}, $i=1,...,n_j$ for $j=1,...,d$ respectively. Let $\mathbf{v}=(v_1,...,v_d)'$ and $\mathbf{r}=(r_1,...,r_d)'$. Bounds extending (1) are sought for $P(\mathbf{v}=\mathbf{r})$ using

$$S_{\mathbf{k}} = E(\prod_{i=1}^{d}\binom{v_i}{k_i}) = E(\binom{\mathbf{v}}{\mathbf{k}}),$$

where $\mathbf{k}=(k_1,...,k_d)'$ is a vector of non-negative integers.

Galambos and Lee (1994) proposed bounds extending (1) to a bivariate setting ($d=2$), which were then uniformly improved by Chen and Seneta (1996) as follows, where u_1 and u_2 are non-negative integers:

$$
\begin{aligned}
&\sum_{k_1=0}^{2u_1+1}\sum_{k_2=0}^{2u_2+1}(-1)^{k_1+k_2}\binom{k_1+r_1}{r_1}\binom{k_2+r_2}{r_2}S_{k_1+r_1,k_2+r_2}+\Delta_1+\Delta_2-\Delta_3\\
&\leq P(v_1=r_1,v_2=r_2) \qquad (2)\\
&\leq \sum_{k_1=0}^{2u_1}\sum_{k_2=0}^{2u_2}(-1)^{k_1+k_2}\binom{k_1+r_1}{r_1}\binom{k_2+r_2}{r_2}S_{k_1+r_1,k_2+r_2}-H_1-H_2+H_3
\end{aligned}
$$

where

$$\Delta_1 = 0 \quad \text{for} \quad n_1 = r_1;$$

$$= \binom{r_1 + 2u_1 + 1}{r_1} \frac{r_1 + 2u_1 + 2}{n_1 - r_1} \sum_{y=0}^{n_2 - r_2} (-1)^y S_{r_1+2u_1+2, r_2+y} \binom{r_2 + y}{r_2} \quad \text{for} \quad n_1 > r_1.$$

$$\Delta_2 = 0 \quad \text{for} \quad n_2 = r_2;$$

$$= \binom{r_2 + 2u_2 + 1}{r_2} \frac{r_2 + 2u_2 + 2}{n_2 - r_2} \sum_{x=0}^{n_1 - r_1} (-1)^x S_{r_1+x, r_2+2u_2+2} \binom{r_1 + x}{r_1} \quad \text{for} \quad n_2 > r_2.$$

$$\Delta_3 = 0 \quad \text{for} \quad n_1 = r_1 \quad \text{or} \quad n_2 = r_2;$$

$$= \binom{r_1 + 2u_1 + 1}{r_1} \binom{r_2 + 2u_2 + 1}{r_2} \frac{r_1 + 2u_1 + 2}{2u_1 + 2} \frac{r_2 + 2u_2 + 2}{2u_2 + 2} S_{r_1+2u_1+2, r_2+2u_2+2}$$

$$\text{for} \quad n_1 > r_1 \quad \text{and} \quad n_2 > r_2.$$

$$H_1 = 0 \quad \text{for} \quad n_1 = r_1;$$

$$= \binom{r_1 + 2u_1}{r_1} \frac{r_1 + 2u_1 + 1}{n_1 - r_1} \sum_{y=0}^{2u_2} (-1)^y S_{r_1+2u_1+1, r_2+y} \binom{2u_2 + y}{2u_2} \quad \text{for} \quad n_1 > r_1.$$

$$H_2 = 0 \quad \text{for} \quad n_2 = r_2;$$

$$= \binom{r_2 + 2u_2}{r_2} \frac{r_2 + 2u_2 + 1}{n_2 - r_2} \sum_{x=0}^{2u_1} (-1)^x S_{r_1+x, r_2+2u_2+1} \binom{2u_1 + x}{2u_1} \quad \text{for} \quad n_2 > r_2.$$

$$H_3 = 0 \quad \text{for} \quad n_1 = r_1 \quad \text{or} \quad n_2 = r_2;$$

$$= \binom{r_1 + 2u_1}{r_1} \binom{r_2 + 2u_2}{r_2} \frac{r_1 + 2u_1 + 1}{n_1 - r_1} \frac{r_2 + 2u_2 + 1}{n_2 - r_2} S_{r_1+2u_1+1, r_2+2u_2+1}$$

$$\text{for} \quad n_1 > r_1 \quad \text{and} \quad n_2 > r_2.$$

The nature of this extension is to use the quantity

$$\sum_{i=0}^{a} \sum_{j=0}^{b} (-1)^{i+j} \binom{i + r_1}{r_1} \binom{j + r_2}{r_2} S_{i+r_1, j+r_2}$$

as the generic term for inclusion and exclusion, with upper bounds (lower bounds) being obtained by setting a and b both odd (even), and then to bound the difference from $P(v_1 = r_1, v_2 = r_2)$. The central idea is the application of Meyer's identity

$$\begin{aligned} S_{kt} &= E(\binom{v_1}{k} \binom{v_2}{t}) \\ &= \sum_{i=k}^{n_1} \sum_{j=t}^{n_2} \binom{v_1}{k} \binom{v_2}{t} P(v_1 = i, v_2 = j), \end{aligned}$$

in conjunction with lemmas on combinatorial coefficients via the method of indicators. This direction of generalization in theory separately involves all the degrees for event sets $\{A_{11}, ..., A_{n_1 1}\}$ and $\{A_{12}, ..., A_{n_2 2}\}$ from 0 to $a + r_1$ and 0 to $b + r_2$ respectively.

This involvement of individual bivariate degrees differs from that in multivariate inequalities where degree p is that figuring in the quantity

$$d_p = \sum_{|\mathbf{k}|=p} \binom{\mathbf{k}}{\mathbf{r}} S_{\mathbf{k}},$$

with $|\mathbf{k}| = \sum_{i=1}^{d} k_i$, $\mathbf{k} = (k_1, ..., k_d)'$ and

$$\binom{\mathbf{k}}{\mathbf{r}} = \prod_{i=1}^{d} \binom{k_i}{r_i}, \qquad S_{\mathbf{k}} = S_{k_1,...,k_d},$$

in the sense that p-dimensional joint probabilities are calculated. For example, the multivariate identity stated in Meyer (1969) reads

$$P(\mathbf{v} = \mathbf{r}) = \sum_{i=|\mathbf{r}|}^{|\mathbf{n}|} (-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}}{\mathbf{r}} S_{\mathbf{k}} \tag{3}$$

and its inverse form

$$S_{\mathbf{k}} = \sum_{p=|\mathbf{k}|}^{|\mathbf{n}|} \sum_{|\mathbf{s}|=p} \binom{\mathbf{s}}{\mathbf{k}} P(\mathbf{v} = \mathbf{s}) \tag{4}$$

and so (3) is of degree $|\mathbf{n}|$. Thus, bounds generalizing (1) by using multivariate degree p quantities d_p is a perhaps more natural way of extension, although each direction of generalization has its own advantage under different circumstances. We shall illustrate by constructing numerical examples in Section 3.

The manner of generalizing univariate inequalities using d_p's first appeared in Meyer (1969) where univariate Bonferroni bounds were extended to multivariate. Recently Galambos and Xu (1996) put forward, by using the method of polynomials, the bounds:

$$\begin{aligned}
&\sum_{i=0}^{2t+1} (-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} + \frac{2t+2}{|\mathbf{n}|-|\mathbf{r}|} \sum_{|\mathbf{k}|=2t} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} \\
&\le P(\mathbf{v} = \mathbf{r}) \\
&\le \sum_{i=0}^{2t} (-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} - \frac{2t+1}{|\mathbf{n}|-|\mathbf{r}|} \sum_{|\mathbf{k}|=2t+1} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}},
\end{aligned} \tag{5}$$

aiming at extension of univariate Sobel-Uppuluri-Galambos bounds. They assert that their result is analogous to the way Galambos (1978) extended Bonferroni's classical univariate bounds; but that univariate extension in fact did not come from the method of polynomials. The dissonance between the method used by Galambos and Xu (1996) and the result achieved prompted our investigation, and we obtained by extending Galambos'(1978) original methodology, the following two theorems:

Theorem 1: For any integer $t \geq 0$ with $2t+1 \leq |\mathbf{n}| - |\mathbf{r}|$,

$$\begin{aligned}
&\sum_{i=0}^{2t+1} (-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} + \max_i A_i(2t+1) \\
&\leq P(\mathbf{v} = \mathbf{r}) \qquad (6) \\
&\leq \sum_{i=0}^{2t} (-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} - \max_i A_i(2t),
\end{aligned}$$

where

$$A_i(x) = \sum_{|\mathbf{k}|=x} S_{k_1+r_1,\ldots,k_i+r_i+1\ldots k_d+r_d} \binom{k_1+r_1}{r_1} \cdots \binom{k_i+r_i+1}{r_i} \cdots \binom{k_d+r_d}{r_d} \frac{k_i+1}{n_i-r_i},$$

$i = 1, \ldots, d$. Note that $A_i(x)$ has degree $|\mathbf{r}| + x + 1$. #

Theorem 2: Inequality (5) can be written

$$\begin{aligned}
&\sum_{i=0}^{2t+1} (-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} + \sum_{i=1}^{d} \frac{n_i - r_i}{|\mathbf{n}| - |\mathbf{r}|} A_i(2t+1) \\
&\leq P(\mathbf{v} = \mathbf{r}) \\
&\leq \sum_{i=0}^{2t} (-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} - \sum_{i=1}^{d} \frac{n_i - r_i}{|\mathbf{n}| - |\mathbf{r}|} A_i(2t),
\end{aligned}$$

so (6) is sharper than (5). #

Remark: Theorem 2 shows that Galambos and Xu's (1996) Theorem 1 can be improved uniformly. The nature of improvement, from averaging to optimization, is similar in sense to the way in which Hunter's Inequality improves certain univariate inequalities (Seneta, 1988).

2. PROOF OF THEOREMS

Lemma 1. For any integer $T \geq 0$ and non-negative integers x_1, ..., x_d,

$$\sum_{i=0}^{T} \sum_{|\mathbf{k}|=i} (-1)^i \binom{x_1}{k_1} \cdots \binom{x_d}{k_d} = (-1)^T \sum_{|\mathbf{k}|=T} \binom{x_1 - 1}{k_1} \binom{x_2}{k_2} \cdots \binom{x_d}{k_d}. \tag{7}$$

Since all (x_i, k_i), $i = 1, ..., d$ are interchangeable on the left, the right hand side can be rewritten in d ways.

Proof of Lemma 1: We use proof by induction.

When $T = 0$, left=1=right.

When $T = 1$, the left becomes $1 - x_1 - x_2 - ... - x_d$ while the right equals $(-1)(x_1 - 1 + x_2 + ... + x_d)$ which is the same as the left.

Supposing that (5) is true for any integer $T \geq 0$, in the case of $T + 1$ we have

$$\begin{aligned}
&\sum_{i=0}^{T+1} \sum_{|\mathbf{k}|=i} (-1)^i \binom{x_1}{k_1} \cdots \binom{x_d}{k_d} \\
&= \sum_{i=0}^{T} \sum_{|\mathbf{k}|=i} (-1)^i \binom{x_1}{k_1} \cdots \binom{x_d}{k_d} + (-1)^{T+1} \sum_{|\mathbf{k}|=T+1} \binom{x_1}{k_1} \cdots \binom{x_d}{k_d} \\
&= (-1)^T \sum_{|\mathbf{k}|=T} \binom{x_1 - 1}{k_1} \binom{x_2}{k_2} \cdots \binom{x_d}{k_d} + (-1)^{T+1} \sum_{|\mathbf{k}|=T+1} \binom{x_1}{k_1} \cdots \binom{x_d}{k_d} \\
&= (-1)^{T+1} \{ - \sum_{|\mathbf{k}|=T} \binom{x_1 - 1}{k_1} \binom{x_2}{k_2} \cdots \binom{x_d}{k_d} + \sum_{|\mathbf{k}|=T+1} \binom{x_1}{k_1} \cdots \binom{x_d}{k_d} \} \\
&= (-1)^{T+1} \{ \sum_{|\mathbf{k}|=T} \binom{x_1}{k_1 + 1} \binom{x_2}{k_2} \cdots \binom{x_d}{k_d} - \sum_{|\mathbf{k}|=T} \binom{x_1 - 1}{k_1} \binom{x_2}{k_2} \cdots \binom{x_d}{k_d} \\
&\quad + \sum_{|\mathbf{k}|=T+1} \binom{x_1}{0} \binom{x_2}{k_2} \cdots \binom{x_d}{k_d} \} \\
&= (-1)^{T+1} \{ \sum_{|\mathbf{k}|=T} \binom{x_1 - 1}{k_1 + 1} \binom{x_2}{k_2} \cdots \binom{x_d}{k_d} + \sum_{|\mathbf{k}|=T+1} \binom{x_1}{0} \binom{x_2}{k_2} \cdots \binom{x_d}{k_d} \},
\end{aligned}$$

by putting the first two summations together and using

$$\binom{x_1}{k_1 + 1} - \binom{x_1 - 1}{k_1} = \binom{x_1 - 1}{k_1 + 1}.$$

Thus
$$\sum_{i=0}^{T+1}\sum_{|\mathbf{k}|=i}(-1)^i\binom{x_1}{k_1}\cdots\binom{x_d}{k_d} = (-1)^{T+1}\sum_{|\mathbf{k}|=T+1}\binom{x_1-1}{k_1}\binom{x_2}{k_2}\cdots\binom{x_d}{k_d},$$
since
$$\sum_{|\mathbf{k}|=T}\binom{x_1-1}{k_1+1}\binom{x_2}{k_2}\cdots\binom{x_d}{k_d}+\sum_{|\mathbf{k}|=T+1}\binom{x_1}{0}\binom{x_2}{k_2}\cdots\binom{x_d}{k_d}$$
$$=\sum_{|\mathbf{k}|=T+1}\binom{x_1-1}{k_1}\binom{x_2}{k_2}\cdots\binom{x_d}{k_d}. \quad \#$$

Proof of Theorem 1: Let a be any non-negative integer and write $P(\mathbf{s}) = P(\mathbf{v}=\mathbf{s}) = P(v_1=s_1,...,v_d=s_d)$. Then
$$\sum_{i=0}^{a}(-1)^i\sum_{|\mathbf{k}|=i}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}}S_{\mathbf{k}+\mathbf{r}}$$
$$=\sum_{i=0}^{a}(-1)^i\sum_{|\mathbf{k}|=i}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}}\sum_{p=i+|\mathbf{r}|}^{|\mathbf{n}|}\sum_{|\mathbf{s}|=p}\binom{\mathbf{s}}{\mathbf{k}+\mathbf{r}}P(\mathbf{s}),$$
by (4), noting that $|k|=i$. Thus
$$\sum_{i=0}^{a}(-1)^i\sum_{|\mathbf{k}|=i}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}}S_{\mathbf{k}+\mathbf{r}}$$
$$=\sum_{p=|\mathbf{r}|}^{|\mathbf{n}|}\sum_{|\mathbf{s}|=p}P(\mathbf{s})\sum_{i=0}^{J}\sum_{|\mathbf{k}|=i}(-1)^i\binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}}\binom{\mathbf{s}}{\mathbf{k}+\mathbf{r}}.$$
where $\quad J=\min(a,|\mathbf{n}|-|\mathbf{r}|)$
$$=\sum_{p=|\mathbf{r}|}^{|\mathbf{n}|}\sum_{|\mathbf{s}|=p}P(\mathbf{s})\binom{\mathbf{s}}{\mathbf{r}}\sum_{i=0}^{J}\sum_{|\mathbf{k}|=i}(-1)^i\binom{\mathbf{s}-\mathbf{r}}{\mathbf{k}},$$
This is because
$$\binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}}\binom{\mathbf{s}}{\mathbf{k}+\mathbf{r}}=\binom{\mathbf{s}}{\mathbf{r}}\binom{\mathbf{s}-\mathbf{r}}{\mathbf{k}}, \tag{8}$$
so
$$\sum_{i=0}^{a}(-1)^i\sum_{|\mathbf{k}|=i}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}}S_{\mathbf{k}+\mathbf{r}}$$
$$=P(\mathbf{r})+\sum_{p=|\mathbf{r}|+1}^{|\mathbf{n}|}\sum_{|\mathbf{s}|=p}P(\mathbf{s})\binom{\mathbf{s}}{\mathbf{r}}\sum_{i=0}^{J}\sum_{|\mathbf{k}|=i}(-1)^i\binom{\mathbf{s}-\mathbf{r}}{\mathbf{k}},$$
$$=P(\mathbf{r})+\sum_{p=|\mathbf{r}|+1}^{|\mathbf{n}|}\sum_{|\mathbf{s}|=p}P(\mathbf{s})\binom{\mathbf{s}}{\mathbf{r}}(-1)^J\sum_{|\mathbf{k}|=J}\binom{s_1-r_1-1}{k_1}\binom{s_2-r_2}{k_2}\cdots\binom{s_d-r_d}{k_d}$$

by Lemma 1. Then

$$\sum_{i=0}^{a}(-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}} S_{\mathbf{k}+\mathbf{r}}$$

$$= P(\mathbf{r}) + \sum_{p=|\mathbf{r}|+1}^{|\mathbf{n}|} \sum_{|\mathbf{s}|=p} P(\mathbf{s}) \binom{\mathbf{s}}{\mathbf{r}} (-1)^a \sum_{|\mathbf{k}|=a} \binom{s_1-r_1-1}{k_1}\binom{s_2-r_2}{k_2}\dots\binom{s_d-r_d}{k_d}$$

for $\quad a \le |\mathbf{n}| - |\mathbf{r}|$;

$$= P(\mathbf{r}) + (-1)^a \sum_{|\mathbf{k}|=a} \sum_{p=|\mathbf{r}|+1}^{|\mathbf{n}|} \sum_{|\mathbf{s}|=p} P(\mathbf{s}) \binom{\mathbf{s}}{\mathbf{r}} \binom{s_1-r_1-1}{k_1}\binom{s_2-r_2}{k_2}\dots\binom{s_d-r_d}{k_d}.$$

Thus we have

$$\sum_{i=0}^{a}(-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}} S_{\mathbf{k}+\mathbf{r}} = P(\mathbf{r}) + (-1)^a B_1(a), \tag{9}$$

where

$$B_1(a) = \sum_{|\mathbf{k}|=a} \sum_{p=|\mathbf{r}|+a+1}^{|\mathbf{n}|} \sum_{|\mathbf{s}|=p} P(\mathbf{s}) \binom{\mathbf{s}}{\mathbf{r}} \binom{s_1-r_1-1}{k_1}\binom{s_2-r_2}{k_2}\dots\binom{s_d-r_d}{k_d},$$

since only when $p \ge |\mathbf{r}| + a + 1$ can we obtain

$$\binom{s_1-r_1-1}{k_1}\binom{s_2-r_2}{k_2}\dots\binom{s_d-r_d}{k_d} \neq 0.$$

With (9), we are ready to prove Theorem 1. To show the upper bounds of the theorem, we set $a = 2t$, which makes (9) become

$$\sum_{i=0}^{2t}(-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}} S_{\mathbf{k}+\mathbf{r}} = P(\mathbf{r}) + B_1(2t).$$

We shall first show $A_1(2t) \le B_1(2t)$, which can be seen by recalling that

$$A_1(2t) = \sum_{|\mathbf{k}|=2t} S_{k_1+r_1+1,k_2+r_2,\dots,k_d+r_d} \binom{k_1+r_1+1}{r_1}\binom{k_2+r_2}{r_2}\dots\binom{k_d+r_d}{r_d}\frac{k_1+1}{n_1-r_1}$$

$$= \sum_{|\mathbf{k}|=2t} \sum_{p=|\mathbf{k}|+|\mathbf{r}|+1}^{|\mathbf{n}|} \sum_{|\mathbf{s}|=p} P(\mathbf{s}) \binom{s_1}{k_1+r_1+1}\binom{s_2}{k_2+r_2}\dots\binom{s_d}{k_d+r_d}$$
$$\binom{k_1+r_1+1}{r_1}\binom{k_2+r_2}{r_2}\dots\binom{k_d+r_d}{r_d}\frac{k_1+1}{n_1-r_1}.$$

This is because by (4)

$$S_{k_1+r_1+1,k_2+r_2,\dots,k_d+r_d} = \sum_{p=|\mathbf{k}|+|\mathbf{r}|+1}^{|\mathbf{n}|} \sum_{|\mathbf{s}|=p} P(\mathbf{s}) \binom{s_1}{k_1+r_1+1} \binom{s_2}{k_2+r_2} \cdots \binom{s_d}{k_d+r_d}.$$

On the other hand,

$$B_1(2t) = \sum_{|\mathbf{k}|=2t} \sum_{p=|\mathbf{r}|+2t+1}^{|\mathbf{n}|} \sum_{|\mathbf{s}|=p} P(\mathbf{s}) \binom{s_1}{r_1} \cdots \binom{s_d}{r_d} \binom{s_1-r_1-1}{k_1} \binom{s_2-r_2}{k_2} \cdots \binom{s_d-r_d}{k_d}.$$

Comparing

$$\binom{s_1}{k_1+r_1+1}\binom{k_1+r_1+1}{r_1} \quad \text{vs} \quad \binom{s_1}{r_1}\binom{s_1-r_1-1}{k_1}\frac{n_1-r_1}{k_1+1}$$

we find by expanding the left-hand expression that

$$\binom{s_1}{k_1+r_1+1}\binom{k_1+r_1+1}{r_1} \le \frac{n_1-r_1}{k_1+1}\binom{s_1}{r_1}\binom{s_1-r_1-1}{k_1}. \tag{10}$$

Further, as in (8):

$$\binom{s_i}{k_i+r_i}\binom{k_i+r_i}{r_i} = \binom{s_i}{r_i}\binom{s_i-r_i}{k_i} \quad \text{for} \quad i=2,\dots,d. \tag{11}$$

Thus, for each $P(\mathbf{s})$ in $A_1(2t)$ and $B_1(2t)$, we can compare pairs of associated coefficients in $A_1(2t)$ and $B_1(2t)$ individually, by (10) and (11), and the immediate conclusion is that $A_1(2t) \le B_1(2t)$. Noting that x_1 can be replaced by any one of x_2, ..., x_d establishes the upper bounds in Theorem 1, for $2t \le |\mathbf{n}| - |\mathbf{r}|$.

Taking $a = 2t+1$ in (9) yields

$$\sum_{i=0}^{2t+1} (-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}} S_{\mathbf{k}+\mathbf{r}} = P(\mathbf{r}) - B_1(2t+1).$$

To establish the lower bounds we need first to show

$$\sum_{i=0}^{2t+1} (-1)^i \sum_{|\mathbf{k}|=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{k}} S_{\mathbf{k}+\mathbf{r}} \le P(\mathbf{r}) - A_1(2t+1).$$

This can be achieved if $A_1(2t+1) \leq B_1(2t+1)$, which follows from applying (10) and (11) to the comparison between $A_1(2t+1)$ and $B_1(2t+1)$ term by pairwise term. Considering the fact that x_1 can be replaced by any one of x_2, ..., x_d completes the proof of Theorem 1. #

The following well-known result is necessary to establish Theorem 2.

Lemma 2. For positive numbers a_1, ..., a_n with positive weights c_1, ..., c_n respectively, such that $\sum_{i=1}^n c_i = 1$,

$$\max_{1\leq i\leq n} a_i \leq c_1 a_1 + ... + c_n a_n,$$

and equality can be attained only when a_i=constant for all $i = 1, ..., n$.

Proof of Theorem 2. For notational convenience, we state the proof of Theorem 2 in the bivariate setting. The proof for the multivariate setting follows analogously.

In the bivariate case, Theorem 1 yields

$$\begin{aligned} &\sum_{i=0}^{2t+1}(-1)^i \sum_{k_1+k_2=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} + A_1(2t+1) \\ &\leq P(\mathbf{v}=\mathbf{r}) \\ &\leq \sum_{i=0}^{2t}(-1)^i \sum_{k_1+k_2=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} - A_1(2t) \end{aligned} \tag{12}$$

and

$$\begin{aligned} &\sum_{i=0}^{2t+1}(-1)^i \sum_{k_1+k_2=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} + A_2(2t+1) \\ &\leq P(\mathbf{v}=\mathbf{r}) \\ &\leq \sum_{i=0}^{2t}(-1)^i \sum_{k_1+k_2=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} - A_2(2t) \end{aligned} \tag{13}$$

where

$$A_1(2t) = \sum_{k_1+k_2=2t} S_{k_1+r_1+1,k_2+r_2} \binom{k_1+r_1+1}{r_1} \binom{k_2+r_2}{r_2} \frac{k_1+1}{n_1-r_1}$$

and

$$A_2(2t) = \sum_{k_1+k_2=2t} S_{k_1+r_1,k_2+r_2+1} \binom{k_1+r_1}{r_1} \binom{k_2+r_2+1}{r_2} \frac{k_2+1}{n_2-r_2}.$$

Multiplying (12) by $\frac{n_1-r_1}{|\mathbf{n}|-|\mathbf{r}|}$, (13) by $\frac{n_2-r_2}{|\mathbf{n}|-|\mathbf{r}|}$ respectively and adding them yields

$$\begin{aligned}
&\sum_{i=0}^{2t+1}(-1)^i \sum_{k_1+k_2=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} + A_1(2t+1)\frac{n_1-r_1}{|\mathbf{n}|-|\mathbf{r}|} + A_2(2t+1)\frac{n_2-r_2}{|\mathbf{n}|-|\mathbf{r}|} \\
&\leq P(\mathbf{v}=\mathbf{r}) \qquad (14)\\
&\leq \sum_{i=0}^{2t}(-1)^i \sum_{k_1+k_2=i} \binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}} - A_1(2t)\frac{n_1-r_1}{|\mathbf{n}|-|\mathbf{r}|} - A_2(2t)\frac{n_2-r_2}{|\mathbf{n}|-|\mathbf{r}|}.
\end{aligned}$$

Comparing (14) with (5), if we prove

$$A_1(2t)\frac{n_1-r_1}{|\mathbf{n}|-|\mathbf{r}|} + A_2(2t)\frac{n_2-r_2}{|\mathbf{n}|-|\mathbf{r}|} = \frac{2t+1}{|\mathbf{n}|-|\mathbf{r}|}\sum_{|\mathbf{k}|=2t+1}\binom{\mathbf{k}+\mathbf{r}}{\mathbf{r}} S_{\mathbf{k}+\mathbf{r}}, \qquad (15)$$

we can then claim that the upper bound given in (5) is a weighted average of d upper bounds in Theorem 1, by Lemma 2, we can have the upper bounds of Theorem 2. The proof for the sharperness of lower bounds in Theorem 2 follows similarily, thus the key here is to prove equation (15). To see (15), note that

$$\begin{aligned}
&A_1(2t)\frac{n_1-r_1}{|\mathbf{n}|-|\mathbf{r}|} + A_2(2t)\frac{n_2-r_2}{|\mathbf{n}|-|\mathbf{r}|} \\
&= \sum_{k_1+k_2=2t} S_{k_1+r_1+1,k_2+r_2}\binom{k_1+r_1+1}{r_1}\binom{k_2+r_2}{r_2}\frac{k_1+1}{|\mathbf{n}|-|\mathbf{r}|} + \\
&\quad \sum_{k_1+k_2=2t} S_{k_1+r_1,k_2+r_2+1}\binom{k_1+r_1}{r_1}\binom{k_2+r_2+1}{r_2}\frac{k_2+1}{|\mathbf{n}|-|\mathbf{r}|} \\
&= \sum_{k_1=0}^{2t-1} S_{k_1+r_1+1,2t-k_1+r_2}\binom{k_1+r_1+1}{r_1}\binom{2t-k_1+r_2}{r_2}\frac{k_1+1}{|\mathbf{n}|-|\mathbf{r}|} + \\
&\quad S_{r_1+2t+1,r_2}\binom{2t+r_1+1}{r_1}\frac{2t+1}{|\mathbf{n}|-|\mathbf{r}|} + S_{r_1,2t+r_2+1}\binom{2t+r_2+1}{r_2}\frac{2t+1}{|\mathbf{n}|-|\mathbf{r}|} + \\
&\quad \sum_{k_1=1}^{2t} S_{k_1+r_1,2t-k_1+r_2+1}\binom{k_1+r_1}{r_1}\binom{2t-k_1+r_2+1}{r_2}\frac{2t-k_1+1}{|\mathbf{n}|-|\mathbf{r}|}.
\end{aligned}$$

Setting $k_1 + 1 = j$ in the first summation yields

$$
\begin{aligned}
&A_1(2t)\frac{n_1 - r_1}{|\mathbf{n}| - |\mathbf{r}|} + A_2(2t)\frac{n_2 - r_2}{|\mathbf{n}| - |\mathbf{r}|} \\
&= \sum_{j=1}^{2t} S_{j+r_1,2t-j+r_2+1}\binom{j + r_1}{r_1}\binom{2t - j + r_2 + 1}{r_2}\frac{j}{|\mathbf{n}| - |\mathbf{r}|} + \\
&\quad S_{r_1+2t+1,r_2}\binom{2t + r_1 + 1}{r_1}\frac{2t+1}{|\mathbf{n}| - |\mathbf{r}|} + S_{r_1,2t+r_2+1}\binom{2t + r_2 + 1}{r_2}\frac{2t+1}{|\mathbf{n}| - |\mathbf{r}|} + \\
&\quad \sum_{k_1=1}^{2t} S_{k_1+r_1,2t-k_1+r_2+1}\binom{k_1 + r_1}{r_1}\binom{2t - k_1 + r_2 + 1}{r_2}\frac{2t - k_1 + 1}{|\mathbf{n}| - |\mathbf{r}|} \\
&= S_{r_1+2t+1,r_2}\binom{2t + r_1 + 1}{r_1}\frac{2t+1}{|\mathbf{n}| - |\mathbf{r}|} + S_{r_1,2t+r_2+1}\binom{2t + r_2 + 1}{r_2}\frac{2t+1}{|\mathbf{n}| - |\mathbf{r}|} + \\
&\quad \sum_{k_1=1}^{2t} S_{k_1+r_1,2t-k_1+r_2+1}\binom{k_1 + r_1}{r_1}\binom{2t - k_1 + r_2 + 1}{r_2}\frac{2t+1}{|\mathbf{n}| - |\mathbf{r}|} \\
&= \frac{2t+1}{|\mathbf{n}| - |\mathbf{r}|}\sum_{|\mathbf{k}|=2t+1}\binom{\mathbf{k} + \mathbf{r}}{\mathbf{r}}S_{\mathbf{k}+\mathbf{r}}. \qquad \#
\end{aligned}
$$

3. NUMERICAL EXAMPLES

In this section, we shall present two examples. Example 1 shows that in some cases (5) does not give a useful result while (6) can still be very effective. Example 2, in conjunction with Example 1, clarifies the relationship between the two directions of generalization as mentioned in Section 1. We see from the examples that in some cases, bounds (2) based on the direction of Galambos and Lee (1994) and then Chen and Seneta (1996) work better, while in some other cases, the bounds from (6) are sharper.

EXAMPLE 1. Consider A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 with non-zero probabilities of elementary conjunctions assigned as follows

$$P(A_1A_2A_3^cA_4B_1^cB_2^cB_3^cB_4) = 0.18 \qquad P(A_1A_2A_3A_4^cB_1^cB_2B_3^cB_4^c) = 0.12$$

$$P(A_1A_2A_3A_4B_1B_2B_3B_4^c) = 0.31 \qquad P(A_1A_2A_3A_4B_1B_2^cB_3B_4) = 0.19$$

$$P(A_1A_2A_3A_4^cB_1B_2B_3B_4) = 0.01 \qquad P(A_1^cA_2^cA_3^cA_4^cB_1^cB_2^cB_3^cB_4^c) = 0.19$$

Hence we have the corresponding non-zero probabilities of the number of exactly $A's$ and $B's$ which occur as

$$P(v_A = 3, v_B = 1) = 0.3 \quad P(v_A = 3, v_B = 4) = 0.01$$

$$P(v_A = 4, v_B = 3) = 0.50 \quad P(v_A = 0, v_B = 0) = 0.19$$

and then the associated Bonferroni summations

$$S_{31} = 6.34 \quad S_{32} = 6.06 \quad S_{33} = 2.04 \quad S_{34} = 0.01$$

$$S_{41} = 1.5 \quad S_{42} = 1.5 \quad S_{43} = 0.5 \quad S_{44} = 0. \qquad \#$$

EXAMPLE 2. Consider A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 with non-zero probabilities of elementary conjunctions specified as follows

$$P(A_1A_2A_3A_4^cB_1B_2B_3^cB_4^c) = 0.1 \qquad P(A_1^cA_2A_3A_4B_1^cB_2^cB_3B_4) = 0.1$$

$$P(A_1A_2^cA_3A_4B_1^cB_2B_3B_4^c) = 0.1 \qquad P(A_1A_2A_3^cA_4B_1B_2B_3^cB_4^c) = 0.1$$

$$P(A_1A_2A_3^cA_4B_1B_2^cB_3^cB_4^c) = 0.2 \qquad P(A_1^cA_2^cA_3^cA_4^cB_1^cB_2^cB_3^cB_4^c) = 0.1$$

$$P(A_1A_2A_3A_4^cB_1^cB_2^cB_3^cB_4) = 0.3$$

Such an assignment gives the corresponding non-zero probabilities of the number of exactly $A's$ and $B's$ which occur as

$$P(v_A = 3, v_B = 2) = 0.4 \quad P(v_A = 3, v_B = 1) = 0.5 \quad P(v_A = 0, v_B = 0) = 0.1,$$

and then corresponding Bonferroni summations can be calculated as

$$S_{31} = 1.3 \quad S_{32} = 0.4 \quad S_{33} = 0 \quad S_{34} = 0$$

$$S_{41} = 0 \quad S_{42} = 0 \quad S_{43} = 0 \quad S_{44} = 0. \qquad \#$$

We first compare the upper bounds in (5) and (6).

In Example 1, $d=2, n_1=n_2=4$ and setting $r_1=3, r_2=1$ and $t=1$ in Theorem 1 yields as the upper bounds entering into (6), using $A_i(.)$ at each of $i=1$ and 2 respectively:

$$P(v_A=3, v_B=1) \leq S_{31} - (2S_{32}+4S_{41}) + (3S_{33}+8S_{42}) - 12S_{43} = 0.34$$

and

$$P(v_A=3, v_B=1) \leq S_{31} - (2S_{32}+4S_{41}) + (3S_{33}+8S_{42}) - (4S_{34}+8S_{43}) = 2.3.$$

Thus, the upper bound from Theorem 1 reads

$$P(v_A=3, v_B=1) \leq \min(0.34, 2.3) = 0.34$$

while the upper bound of (5) reads

$$P(v_A=3, v_B=1) \leq S_{31} - (2S_{32}+4S_{41}) + (3S_{33}+8S_{42}) - (3S_{34}+9S_{43}) = 1.81.$$

Thus (5) gives a trivial upper bound (greater than 1). The actual value is $P(v_A=3, v_B=1)=0.30$. (15) in this example reads:

$$0.34 \times \frac{4-3}{4+4-3-1} + 2.3 \times \frac{4-1}{4+4-3-1} = 1.81.$$

Both (5) and (6) give the lower bound 0.3 #

Next we shall use Example 1 to show that (2) can be sharper than (6), and then use Example 2 to show that (6) can be sharper than (2).

In Example 1, we know that (6) gives a upper bound of 0.34 which is very effective in estimating the exact value of 0.3. However, (2) does even better. Letting $u_1=1$, $u_2=0$, $n_1=n_2=4$, $r_1=3$, $r_2=1$ the upper bound in (2) yields

$$\begin{aligned} P(v_A=3, v_B=1) &\leq \sum_{k=0}^{2}\sum_{t=0}^{0}(-1)^{k+t}\binom{k+3}{3}\binom{t+1}{1}S_{k+3,t+1} - H_2 \\ &\leq S_{31} - 4S_{41} - \frac{2}{3}(S_{32} - 4S_{42}). \end{aligned}$$

Applying the values of S_{ij}'s in Example 1 yields

$$P(v_A = 3, v_B = 1) \leq 0.3$$

which exactly hits the value to be evaluated. However, the upper bound in (2) is not always better than (6) as demonsrated by Example 2, where (2) gives the bound

$$P(v_A = 3, v_B = 1) \leq 1.3 - \frac{2}{3} \times 0.4 = 1.03$$

which is a trivial upper bound. On the other hand, (6) provides an upper bound

$$P(v_A = 3, v_B = 1) \leq 1.3 - (2 \times 0.4 + 4 \times 0) = 0.5$$

which is the exact value to be estimated. Both (2) and (6) give the lower bound 0.5. #

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