Sydney University Mathematical Society Problem Competition 2011

This competition is open to undergraduates (including Honours students) at any Australian university or tertiary institution. Entrants may use any source of information except other people. The problems will also be posted on the web page **http://www.maths.usyd.edu.au/u/SUMS/**.

Entrants may submit solutions to as many problems as they wish. Prizes ($60 book vouchers from the Co-op Bookshop) will be awarded for the best correct solution to each of the 10 problems. Students from the University of Sydney are also eligible for the Norbert Quirk Prizes, based on the overall quality of their entry (one for each of 1st, 2nd and 3rd years). Extensions and generalizations of any problem are invited and are taken into account when assessing solutions. If two or more solutions to a problem are essentially equal, preference may be given to students in the earlier year of university; otherwise, prizes may be shared. If a problem receives no correct solutions, its prize-money will be redistributed among the other problems.

Entries must be received by **Friday, August 12, 2011**. They may be posted to Dr Anthony Henderson, School of Mathematics and Statistics, The University of Sydney, NSW 2006, or handed in to Room 805, Carslaw Building. Please mark your entry SUMS Problem Competition 2011, and include your name, university, student number, year of study, and postal address (or email address for University of Sydney students) for the return of your entry and prizes.

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1. Alice and Bess are playing a game with an ordinary six-sided die. Alice's target numbers are $1, 2, 3$, and Bess' target numbers are $4, 5, 6$. They take turns in rolling the die, with Alice going first. If the one whose turn it is rolls a target number which she has not previously rolled, she gets to roll again; if she rolls a target number which she has previously rolled, or a number which is not one of her target numbers, her turn ends. The winner is the first player to have rolled all three of her target numbers (not necessarily all in the one turn). What is the probability that Alice wins?

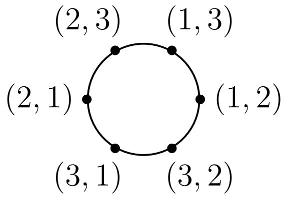
2. Determine all pairs of positive integers a, b such that $4^a + 4^b + 1$ is an integer square.

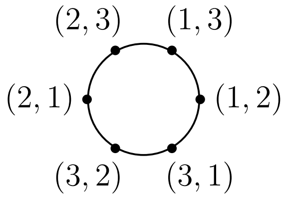
3. Let m and n be positive integers with $m \geq n$. Let A be the $n \times n$ matrix with (i, j)-entry equal to the binomial coefficient $\binom{mj}{i}$. Find the determinant of A.

4. The power series $\displaystyle\sum_{n=0}^{\infty} \cos(\frac{\pi n}{6}) \frac{z^n}{n!}$ converges for all z, to $f(z)$ say. Let $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of $\displaystyle\frac{3}{1+2f(z)}$ about 0. Prove that $a_{3n} = 0$ for all positive integers n.

5. 2011 is a prime number. Let $N = 2^{2011} - 1$, a 606-digit number which can be shown to be composite by computer calculations. Using elementary number theory (and maybe a pocket calculator), prove that N has no prime factors less than $80,000$.

6. Let $n \geq 3$ be an integer. Consider the $n(n-1)$ ordered pairs (i, j), where $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. Show that there is a way to arrange these pairs around a circle, equally spaced, so that for any distinct $i, j, k \in \{1, 2, \cdots, n\}$, the arc from (i, j) to (j, k) which passes through (i, k) is less than half the circumference of the circle. For example, the first of the following pictures for $n = 3$ has this property; the second does not, because (to name one of its failings) the arc from $(1, 3)$ to $(3, 2)$ which passes through $(1, 2)$ is equal to half the circumference.





7. For a positive integer n, let b_n denote the number of binary strings consisting of n zeroes and n ones which have no three consecutive zeroes and no three consecutive ones. Show that

$$b_n = \sum_{k=0}^{n} \left(\binom{k}{n-k} + \binom{k+1}{n-k-1} \right)^2,$$

where the binomial coefficient $\binom{k}{j}$ is defined to be zero if $j < 0$ or $j > k$.

8. Let $(x_1, y_1), \cdots, (x_n, y_n)$ be n distinct points in the plane with $0 < x_i < 1$ and $0 < y_i < 1$ for all i. Let V be the set of all integer translates of these points, i.e. points of the form $(x_i + a, y_i + b)$ for $1 \leq i \leq n$, $a, b \in \mathbb{Z}$. A *periodic hexagon tessellation* with vertex set V consist of a set of continuous curves in the plane called *edges* such that:

- the endpoints of each edge are distinct elements of V, and edges do not intersect except at their endpoints,
- every element of V is the endpoint of exactly three edges,
- every one of the regions between the edges has exactly six edges on its boundary, and
- whenever C is an edge, every integer translate $C + (a, b)$ for $a, b \in \mathbb{Z}$ is also an edge.

Show that a periodic hexagon tessellation with vertex set V exists if and only if n is even.

9. By a *word* in this problem we mean a (possibly empty) string of lowercase letters in the usual alphabet `a`–`z`. If W_1 and W_2 are words then we write W_1W_2 for the concatenation of W_1 and W_2. We say that there is an *elementary transition* between two words W and W' if W has the form $W_1W_2W_3$ and W' equals $W_1W_2W_2W_3$ (in other words, W' is obtained from W by repeating some sub-word), or if W has the form $W_1W_2W_2W_3$ and W' equals $W_1W_2W_3$ (in other words, W' is obtained from W by deleting one copy of a repeated sub-word). We say that two words W and W' are *equivalent* if they are connected by a finite sequence of such elementary transitions. For example, `barbaric` is equivalent to `baariric` because of the following sequence of elementary transitions:

$$\texttt{barbaric} \longleftrightarrow \texttt{baric} \longleftrightarrow \texttt{baaric} \longleftrightarrow \texttt{baariric}$$

Find the number of equivalence classes of words in which every letter of the alphabet appears.

10. Let A be a finite set. Suppose we have a real-valued function f on the set of subsets of A with the property that $\sum_{i \in I} f(I \setminus \{i\}) = 0$ for every $I \subseteq A$. Prove that $f(I) = 0$ whenever $|I| < \frac{|A|}{2}$.