The University of Sydney
School of Mathematics and Statistics

# Solutions to Advanced Methods of Mathematical Physics

MATH440x: Applied Mathematics Honours — Semester 2, 2019

Lecturer: *Nalini Joshi*

It was due 11:59pm Saturday 14 September 2019.

Submit your scanned or typeset answers to `TurnItIn` on Canvas. To ensure compliance with anonymous marking obligations, please do not include your name, only your SID should be present. As per University policy, no late work can be accepted.

**1.** Consider a sequence of monic polynomials $P_n(x)$, $0 \le n \in \mathbb{N}$, with $P_0(x) = 1$, $P_1(x) = x$, which satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} P_n(x) P_m(x)\, e^{-U(x)} dx = h_n \delta_{nm},$$

where $U(x) := x^6 - t\,x^4 - s\,x^2$, and $t$ and $s$ are parameters.

(a) Using the above information, show that these polynomials satisfy a 3-term recurrence relation of the form

$$x P_n(x) = A_n\, P_{n+1}(x) + B_n\, P_n(x) + C_n\, P_{n-1}(x),$$

and find $A_n$, $B_n$, $C_n$ explicitly in terms of $h_n$.

***Solution:***

$$A_n \equiv 1\,,\ B_n \equiv 0\,,\ C_n = \frac{h_n}{h_{n-1}} =: \lambda_n\,.$$

(b) Derive structure relations, i.e., expressions of $\partial P_n/\partial x$, $\partial P_n/\partial t$, $\partial P_n/\partial s$ in terms of the above quantities and $\{P_k\}$, $\{h_k\}$, $k \in \mathbb{N}$.

***Solution:***

$$\begin{aligned}
\frac{\partial P_n}{\partial x} &= n\,P_{n-1} + \left(2\sum_{k=1}^{n-1} \lambda_k - n\lambda_{n-1}\right) P_{n-3} + 6\,\frac{h_n}{h_{n-5}}\,P_{n-5}\,,\\
&= n\,P_{n-1} + \frac{h_n}{h_{n-3}}\left\{6\big(\lambda_{n+1} + \lambda_n + \lambda_{n-1} + \lambda_{n-2} + \lambda_{n-3}\big) - 4t\right\} P_{n-3}\\
&\qquad + \frac{h_0}{h_5}\left\{2\lambda_1\big(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4\big) + \lambda_2\lambda_4\right\} P_{n-5}\,,\\
\frac{\partial P_n}{\partial t} &= -\lambda_n\lambda_{n-1}\big(\lambda_{n+1} + \lambda_n + \lambda_{n-1} + \lambda_{n-2}\big)P_{n-2} - \lambda_n\lambda_{n-1}\lambda_{n-2}\lambda_{n-3}\,P_{n-4}\,,\\
\frac{\partial P_n}{\partial s} &= -\,\lambda_n\lambda_{n-1}\,P_{n-2}\,.
\end{aligned}$$

Note that the alternative solutions given above imply an identity for $\lambda_n$. This identity is equivalent to a difference equation satisfied by $\lambda_n$.

(c) Derive equations for $\partial h_n/\partial t$, $\partial h_n/\partial s$ in terms of the quantities in part (a).

***Solution:***

$$\begin{aligned}
\frac{\partial h_n}{\partial t} &= h_{n+2} + \big(\lambda_{n+1} + \lambda_n\big)^2 h_n + \lambda_n^2\,\lambda_{n-1}^2\,h_{n-2}\,,\\
\frac{\partial h_n}{\partial s} &= h_{n+1} + \lambda_n^2\,h_{n-1}\,.
\end{aligned}$$