The University of Sydney
School of Mathematics and Statistics

Advanced Methods of Mathematical Physics

MATH440x: Applied Mathematics Honours — Semester 2, 2019

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For integer $N \geq 4$, consider the extension of the symmetry generators we considered in lectures to $N-1$ dimensions. That is, we take s_j, $0 \leq j \leq N-1$, and π to be transformations of parameters $\{\alpha_i\}$ and functions $f_i(t)$, $i \in \{0, 1, \ldots, N-1\}$, whose actions on paramaters are defined by

$$s_i(\alpha_j) = \alpha_j - \alpha_i\, A_{ij}, i, j \in \{0, 1, \ldots, N-1\}, \mathrm{mod}\ N\,,$$

where A_{ij} are entries of an $N \times N$ matrix A, given by

$$A_{jj} = 2,\ A_{ij} = \begin{cases} -1,\ j = i \pm 1, \mathrm{mod}\ N\,, \\ \ \ 0,\ j \neq i \pm 1, \mathrm{mod}\ N\,, \end{cases}$$

and

$$\pi(\alpha_i) = \alpha_{i+1}, i \in \{0, 1, \ldots, N-1\}, \mathrm{mod}\ N\,.$$

We also define actions on variables f_0, f_1, …, f_{N-1} by

$$\begin{aligned} s_i(f_j) &= f_j \pm \frac{\alpha_i}{f_i}, j = i \pm 1, \mathrm{mod}\ N\,, \\ s_i(f_j) &= f_j, j \neq i \pm 1, \mathrm{mod}\ N\,, \\ \pi\, f_i &= f_{i+1}, i \in \{0, 1, \ldots, N-1\}, \mathrm{mod}\ N\,. \end{aligned}$$

Below we restrict ourselves to the case $N = 4$, with $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1$, and the assumption that $f_i(t)$, $i \in \{0, 1, 2, 3\}$, satisfy

$$\begin{aligned} f_0' &= f_0(f_1 f_2 - f_2 f_3) + (1/2 - \alpha_2) f_0 + \alpha_0\, f_2\,, \\ f_1' &= f_1(f_2 f_3 - f_3 f_0) + (1/2 - \alpha_3) f_1 + \alpha_1\, f_3\,, \\ f_2' &= f_2(f_3 f_0 - f_0 f_1) + (1/2 - \alpha_0) f_2 + \alpha_2\, f_0\,, \\ f_3' &= f_3(f_0 f_1 - f_1 f_2) + (1/2 - \alpha_1) f_3 + \alpha_3\, f_1\,. \end{aligned} \tag{1}$$

Moreover, we define $T_1 = \pi\, s_3\, s_2\, s_1$.

1. (a) Define $g_i = s_0(f_i)$, $i \in \{0, 1, 2, 3\}$. Show that g_0, g_1, g_2, g_3 satisfy a copy of Equations (1).

(b) Find $T_1(\alpha_j)$, $0 \leq j \leq 3$.

(c) Find $T_1(f_j)$, $0 \leq j \leq 3$.

(d) Assume that $f_0 + f_2 = f_1 + f_3 = t^{1/2}$. Find a system of difference equations satisfied by $u_n = T_1^{(n)}(f_0)$, $v_n = T_1^{(n)}(f_1)$.