The University of Sydney
School of Mathematics and Statistics

Solutions to Advanced Methods of Mathematical Physics

MATH440x: Applied Mathematics Honours — Semester 2, 2019

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For integer $N \geq 4$, consider the extension of the symmetry generators we considered in lectures to $N-1$ dimensions. That is, we take s_j, $0 \leq j \leq N-1$, and π to be transformations of parameters $\{\alpha_i\}$ and functions $f_i(t)$, $i \in \{0, 1, \ldots, N-1\}$, whose actions on paramaters are defined by

$$s_i(\alpha_j) = \alpha_j - \alpha_i\, A_{ij}, i, j \in \{0, 1, \ldots, N-1\}, \mathrm{mod}\ N\,,$$

where A_{ij} are entries of an $N \times N$ matrix A, given by

$$A_{jj} = 2,\ \ A_{ij} = \begin{cases} -1,\ \ j = i \pm 1, \mathrm{mod}\ N\,, \\ \ \ 0,\ \ j \neq i \pm 1, \mathrm{mod}\ N\,, \end{cases}$$

and

$$\pi(\alpha_i) = \alpha_{i+1}, i \in \{0, 1, \ldots, N-1\}, \mathrm{mod}\ N\,.$$

We also define actions on variables f_0, f_1, …, f_{N-1} by

$$\begin{aligned} s_i(f_j) &= f_j \pm \frac{\alpha_i}{f_i}, j = i \pm 1, \mathrm{mod}\ N\,, \\ s_i(f_j) &= f_j, j \neq i \pm 1, \mathrm{mod}\ N\,, \\ \pi\, f_i &= f_{i+1}, i \in \{0, 1, \ldots, N-1\}, \mathrm{mod}\ N\,. \end{aligned}$$

Below we restrict ourselves to the case $N = 4$, with $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1$, and the assumption that $f_i(t)$, $i \in \{0, 1, 2, 3\}$, satisfy

$$\begin{aligned} t f_0' &= f_0(f_1 f_2 - f_2 f_3) + (1/2 - \alpha_2) f_0 + \alpha_0\, f_2\,, \\ t f_1' &= f_1(f_2 f_3 - f_3 f_0) + (1/2 - \alpha_3) f_1 + \alpha_1\, f_3\,, \\ t f_2' &= f_2(f_3 f_0 - f_0 f_1) + (1/2 - \alpha_0) f_2 + \alpha_2\, f_0\,, \\ t f_3' &= f_3(f_0 f_1 - f_1 f_2) + (1/2 - \alpha_1) f_3 + \alpha_3\, f_1\,, \end{aligned} \tag{1}$$

where primes mean differentiation in t. Moreover, we define $T_1 = \pi\, s_3\, s_2\, s_1$.

1. (a) Define $g_i = s_0(f_i)$, $i \in \{0, 1, 2, 3\}$. Show that g_0, g_1, g_2, g_3 satisfy a copy of Equations (1).

Solution: Definitions of s_0 and actions on f_j (along with the condition $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1$) show that

$$\begin{aligned} t g_0' &= g_0(g_1 g_2 - g_2 g_3) + (1/2 - \alpha_2) g_0 - \alpha_0 g_2, \\ t g_1' &= g_1(g_2 g_3 - g_3 g_0) + (1/2 - \alpha_0 - \alpha_3) g_1 + (\alpha_0 + \alpha_1) g_3, \\ t g_2' &= g_2(g_3 g_0 - g_0 g_1) + (1/2 + \alpha_0) g_2 + \alpha_2 g_0, \\ t g_3' &= g_3(g_0 g_1 - g_1 g_2) + (1/2 - \alpha_0 - \alpha_1) g_3 + (\alpha_0 + \alpha_3) g_1. \end{aligned}$$

This is a copy of Equations (1) with new parameter values given by $s_0(\alpha_i)$, $i = 0, 1, 2, 3$.

(b) Find $T_1(\alpha_j)$, $0 \le j \le 3$.

Solution: The results are given by

$$\begin{aligned} T_1(\alpha_0) &= \alpha_0 + 1, \\ T_1(\alpha_1) &= \alpha_1 - 1, \\ T_1(\alpha_2) &= \alpha_2, \\ T_1(\alpha_3) &= \alpha_3. \end{aligned}$$

(c) Find $T_1(f_j)$, $0 \le j \le 3$.

Solution: The results are given by

$$\begin{aligned} T_1(f_0) &= f_1 + \frac{\alpha_0}{f_0} - \frac{\alpha_2 + \alpha_3 + \alpha_0}{f_2 - (\alpha_0 + \alpha_3)/(f_3 - \alpha_0/f_0)}, \\ T_1(f_1) &= f_2 - \frac{\alpha_0 + \alpha_3}{f_3 - \alpha_0/f_0}, \\ T_1(f_2) &= f_3 - \frac{\alpha_0}{f_0} + \frac{\alpha_2 + \alpha_3 + \alpha_0}{f_2 - (\alpha_2 + \alpha_3)/(f_3 - \alpha_0/f_0)}, \\ T_1(f_3) &= f_0 + \frac{\alpha_0 + \alpha_3}{f_3 - \alpha_0/f_0}. \end{aligned}$$

(d) Assume that $f_0 + f_2 = f_1 + f_3 = t^{1/2}$. Find a system of difference equations satisfied by $u_n = T_1^{(n)}(f_0)$, $v_n = T_1^{(n)}(f_1)$.

Solution: Note that from part (c), we have $T_1(f_0 + f_2) = f_1 + f_3$ and $T_1(f_1 + f_3) = f_2 + f_0$. So $T_1(t) = t$. We obtain

$$\begin{aligned} T_1(f_0) &= f_1 + \frac{\alpha_0}{f_0} - \frac{1 - \alpha_1}{T_1(f_1)}, \\ T_1(f_1) &= t^{1/2} - f_0 - \frac{\alpha_0 + \alpha_3}{t^{1/2} - f_1 - \alpha_0/f_0}. \end{aligned}$$

Applying these repeatedly, we obtain

$$\begin{aligned} T_1^{n+1}(f_0) &= T_1^n(f_1) + \frac{T_1^n(\alpha_0)}{f_0} - \frac{1 - T_1^n(\alpha_1)}{T_1^{n+1}(f_1)}, \\ T_1^{n+1}(f_1) &= t^{1/2} - T_1^n(f_0) - \frac{T_1^n(\alpha_0) + T_1^n(\alpha_3)}{t^{1/2} - T_1^n(f_1) - T_1^n(\alpha_0)/T_1^n(f_0)}. \end{aligned}$$

Using $T_1^n(\alpha_0) = n + \alpha_0$, $T_1^n(\alpha_1) = \alpha_1 - n$, $T_1^n(\alpha_3) = \alpha_3$, we obtain

$$\begin{aligned} u_{n+1} &= v_n + \frac{n + \alpha_0}{u_n} - \frac{n + 1 - \alpha_1}{v_{n+1}}, \\ v_{n+1} &= t^{1/2} - u_n - \frac{n + \alpha_3 + \alpha_0}{t^{1/2} - v_n - (n + \alpha_0)/u_n}. \end{aligned}$$