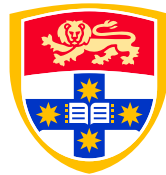


# Fourth Year Pure Mathematics

2022 Handbook  
School of Mathematics and Statistics



THE UNIVERSITY OF  
**SYDNEY**

<http://www.maths.usyd.edu.au/u/UG/HM/>



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Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something completely different.

Goethe, *Maximen und Reflexionen*

# Chapter 1

## The Structure of Pure Mathematics Four

### 1.1 Introduction

In linguistics it is increasingly believed that universal features of language are reflections of the structure of the human brain and its perception of the world around us. In a similar fashion, mathematics is a universal language that has been developed to understand and describe how nature and life work. Mathematics, both in structure and development, is inextricably bound to our attempts to understand the world around us and our perceptions of that world. We see this in the mathematical descriptions and formulations of models in the theoretical and applied sciences: from physics, computer science and information theory on the one hand, to engineering, chemistry, operations research and economics on the other.

Just as remarkable is the way in which esoteric and abstract mathematics finds applications in the applied sciences. Indeed, one of the most exciting developments in science over the past decade has been the re-emergence of a dynamic interaction between pure mathematicians and applied scientists, which is bringing together several decades of the relatively abstract and separate development of pure mathematics and the sciences. Examples include the applications of singularity theory and group theory to symmetry-breaking and bifurcation in engineering; number theory to cryptography; category theory and combinatorics to theoretical and computational computer science; and, most spectacular of all, the developments of general field theories in mathematical physics based on the most profound work in complex analysis and algebraic geometry. Of course, this interaction is not one way. For example, there is the discovery of “exotic” differential structures on  $\mathbb{R}^4$  utilising ideas from Yang-Mills theory.

There are many valid approaches to the study of Pure Mathematics in the final Honours Year. Thus, the course may be regarded as useful in its own right, or may lead on to an M.Sc. or Ph.D. or to a teaching position in University or High School. In another direction, what want a solid base from which to continue with studies in computer science or physics, for example. Finally, you may intend to seek employment with the CSIRO or in the operations research field, or in a financial institution. In the latter circumstances, one well-known advantage of studying mathematics is that mathematics gives training in a

particular way of thinking and an analytic approach to problem solving. Mathematicians are highly adaptable (and employable).

The Fourth Year Honours program in Pure Mathematics caters for the various needs described above by offering a highly flexible and adaptable program, which is both interesting and challenging. We offer a combination of courses, which introduce the major areas of mathematics, together with a smorgasbord of deeper courses that can be arranged to suit your personal requirements.

A description of the various components of the course is given below. For detailed descriptions of the courses, the essay project, and so on, see Chapter 3 and 4.

Overall, the lecture courses offered at the level of PM4 and above are intended to introduce students to the major divisions of modern mathematics and provide a knowledge of some of the main ideas needed for the understanding of much of contemporary mathematics, while still reflecting the research interests within the pure mathematics research groups.

## 1.2 Pure Mathematics Honours/PG units for 2022

The Bachelor of Advanced Studies (new Honours) (Mathematics (Pure)) requires 48 credit points from the following tables including:

- (i) 6 credit points of 4000-level Honours coursework selective units from List 1, and
- (ii) 6 credit points of 4000-level Honours coursework selective units from List 2, and
- (iii) 12 credit points of 4000-level and 5000-level Honours coursework selective units from List 1, List 2, List 3, List 4 or List 5 (listed in the Faculty Handbook).
  - a maximum of 6 credit points of which may be from List 3, and
  - a maximum of 6 credit points of which may be from List 4, and
- (iv) 24 credit points of 4000-level Honours research project units.

See the [Faculty Handbook](#)

### LIST 1

Algebraic Topology (Semester 2)  
Commutative Algebra (Semester 2)  
Representation Theory (Semester 1)

### LIST 2

Functional Analysis (Semester 1)

### LIST 3

4000 or 5000-level units from a different School

### LIST 4

5000-level units from our School.

If you are unsure about the combination of courses you should take, consult with your supervisor or the course coordinator. In any case, you are very welcome to attend all the

lecture courses. Our expectation is that the pure honours students will complete at least 12 credit points of 4000-level Honours coursework selective units from List 1 and List 2.

## **1.3 Pure Mathematics 40xx Units of Study for 2022**

### SEMESTER I

Metric Spaces (MATH4061)

Rings, Fields and Galois Theory (MATH4062)

Dynamical Systems and Applications (MATH4063)

### SEMESTER II

Differential Geometry (MATH4068)

Measure Theory and Fourier Analysis (MATH4069)

## **1.4 AMSI-ACE Courses**

Students are welcomed to check the courses offered in January at the [AMSI Summer School](#) and also courses available through [Advanced Collaborative Environment \(ACE\)](#) during Semester 1 or 2. Enrolment is through the unit AMSI4001 and requires explicit permission from the honours coordinator, Prof [Laurentiu Paunescu](#) (Carlaw 721, phone (02) 9351 2969, email [pm4coord@maths.usyd.edu.au](mailto:pm4coord@maths.usyd.edu.au)).

Students are encouraged to attend the AMSI Summer School even without credit. It is a great way to get to know other students around Australia and to expand their knowledge beyond our coursework program.





I fear that I bore you with these details, but I have to let you see my little difficulties, if you are to understand the situation.

Sherlock Holmes, *A Scandal in Bohemia*

## Chapter 2

# Entry, Administration and Assessment

### 2.1 Entry Requirements for Pure Mathematics 4

Preliminary entrance into the honours program is through the [Faculty of Science application portal](#). The [Faculty requirements](#) which must be met include:

- qualifying for the pass degree with two majors one of which should be cognate to the proposed honours stream, a major which provides a suitable background for the honours stream;
- having a WAM of at least 65;
- securing the agreement of a supervisor.

In addition, the School of Mathematics and Statistics recommends that the students have a total of at least 18CP or 24CP (depending on their major requirements) of relevant 3XXX or 4XXX courses in which :

- the average mark of Advanced level courses is at least 65;
- the average mark of Mainstream level courses is at least 75.

The BSc/BAdvStudies degree requires two majors and hence Honours in this degree requires two majors. Also entry to the standalone BAdvStudies (Hons) requires two majors. However students with a major + minor can complete Honours in the BSc (Hons); this was previously open only to pre-2018 students, but it is now open to students without two majors, including almost all external students, [click here for details](#).

Entry to PM4 is also subject to the approval of the Head of School and judgement of the Honours coordinator.

### 2.2 Structure of Honours

An honours year in Mathematics and Statistics involves four 6CP courses (worth 50% of the final mark) and a project (worth 50%).

### **2.2.1 The honours project (50%)**

The honours project centres around an essay/thesis consisting of about 40-60 pages written on a particular topic from your chosen area. The thesis is due at the end of your final semester, specifically on Monday of week 13. For more information about the essay see Chapter 4.

### **2.2.2 The Talk**

As part of the essay project, students are required to give a talk about their project. The talk is worth 5% of the essay mark. The talk will usually take place about the mid-semester break of student's final semester. The aim of the talk is to explain to a broader audience the purpose and nature of the project. The talk is followed by 5 minutes dedicated to questions from the audience which includes staff members and fellow students.

### **2.2.3 Coursework (50%)**

The honours program specifies a couple of core courses as well as which combination of courses can be taken – please carefully read through the list of constraints! Full-time students will normally attend two 6CP lecture courses each Semester, for a total of four courses.

## 2.3 Important coursework information for all students

### 2.3.1 Selecting your courses

Regardless of whether you are a new or an old curriculum student make sure you **select your courses after consulting the Honours supervisor and the Honours coordinator!** See also Section 1.2 for the offerings.

## 2.4 Actions to be taken

All students intending to take Pure Mathematics 4 should see the *PM4 Course Coordinator*, Prof **Laurentiu Paunescu** (Carslaw 721, phone (02) 9351 2969, email [pm4coord@maths.usyd.edu.au](mailto:pm4coord@maths.usyd.edu.au)) at their earliest opportunity, and in any case well before the beginning of the new teaching year. The Course Coordinator will advise you about choosing a supervisor and a topic for the essay project (see also Section 4.2 below).

## 2.5 Administrative arrangements

The PM4 Course Coordinator is in charge of Pure Mathematics 4 and should be consulted about any organisational problems that may arise.

In particular, students should note that the Course Coordinator's permission should be obtained if you wish to substitute courses from outside, or take a reading course or a postgraduate course. In the first instance, however, you should discuss such matters with your supervisor. Provided you can agree, the Course Coordinator's permission would then normally be a formality.

Please take particular note of the procedure to be followed if you are sick or other circumstances arise that may lead to late submission of your essay (see Section 4.4). Also note that at the end of first semester a progress report must be given to the Course Coordinator (see Chapter 4).

When we know that you are enrolled for PM4 you will be given a computer account. *The usual way in which messages for PM4 students will be distributed will be via e-mail.* Please remember to check your e-mail regularly.

## 2.6 Assessment

Each PM4 lecture course is assessed as advertised in the official unit outline available from [www.sydney.edu.au/units](http://www.sydney.edu.au/units). Usually, a written (sometimes oral) exam is held during the exam period immediately following the course; however, some courses are assessed entirely by assignment.

The essay accounts for 50% of the year's assessment.

As well as assessing the Fourth Year performance, the Department is required to make a recommendation for a grade of Honours. In exceptional cases, the grade of Honours

awarded could differ from the level of performance in the Fourth Year. The possible results for fourth year are First Class Honours, Second Class Honours division 1, Second Class Honours Division 2, Third Class Honours and No Award (Fail), usually abbreviated I, II-1, II-2, III and F.

## 2.7 Honours grades

The Faculty of Science has given the following guidelines for assessment of student performance in fourth year.

- 95–100 Outstanding First Class quality of clear Medal standard, demonstrating independent thought throughout, a flair for the subject, comprehensive knowledge of the subject area and a level of achievement similar to that expected by first rate academic journals. This mark reflects an exceptional achievement with a high degree of initiative and self-reliance, considerable student input into the direction of the study, and critical evaluation of the established work in the area.
- 90–94 Very high standard of work similar to above but overall performance is borderline for award of a Medal. Lower level of performance in certain categories or areas of study above.
- Note:* An honours mark of 90+ and a third year WAM of 80+ are necessary but not sufficient conditions for the award of the Medal. Examiners are referred to the Academic Board Guidelines on the award of Medals found in the general policy pages at the front of the Examiners' Manual.
- 80–89 Clear First Class quality, showing a command of the field both broad and deep, with the presentation of some novel insights. Student will have shown a solid foundation of conceptual thought and a breadth of factual knowledge of the discipline, clear familiarity with and ability to use central methodology and experimental practices of the discipline, and clear evidence of some independence of thought in the subject area. Some student input into the direction of the study or development of techniques, and critical discussion of the outcomes.
- 75–79 Second Class Honours, First Division – student will have shown a command of the theory and practice of the discipline. They will have demonstrated their ability to conduct work at an independent level and complete tasks in a timely manner, and have an adequate understanding of the background factual basis of the subject. Student shows some initiative but is more reliant on other people for ideas and techniques and project is dependent on supervisor's suggestions. Student is dedicated to work and capable of undertaking a higher degree.
- 70–74 Second Class Honours, Second Division – student is proficient in the theory and practice of their discipline but has not developed complete independence of thought, practical mastery or clarity of presentation. Student shows adequate but limited understanding of the topic and has largely followed the direction of the supervisor.
- 65–69 Third Class Honours – performance indicates that the student has successfully completed the work, but at a standard barely meeting honours criteria. The student's understanding of the topic is extremely limited and they have shown little or no independence of thought or performance.

The award of a medal is *not* made just on the basis of a numerical mark or formula. The merits of each eligible candidate are debated by the relevant Board of Examiners.

## 2.8 School Facilities

Pure Mathematics 4 students traditionally enjoy a number of privileges. These include:

- Desk space in the Carslaw Building.
- A computer account with access to email and the internet, as well as  $\text{\TeX}$  and laser printing facilities for the preparation of essays and projects.
- A photocopying account paid by the School for essay/project source material.
- After-hours access to the Carslaw Building. (A deposit is payable.)
- A pigeon-hole in room 728 – please inspect it regularly as lecturers often use it to hand out relevant material.
- Participation in the School's social events.
- Class representative at School meetings.

## 2.9 Scholarships, Prizes and Awards

The following scholarships and prizes may be awarded to Pure Mathematics 4 students of sufficient merit. (Note that unless the conditions of the prize state otherwise, as in the David G.A.Jackson Prize and the A.F.U.W. Prize, these prizes are also open to all Honours students in the School of Mathematics and Statistics.)

### Joye Prize in Mathematics

To the most outstanding student completing fourth year honours in the School of Mathematics and Statistics.

Value: \$5300 plus medal and shield.

### George Allen Scholarship in Pure Mathematics

To a student proceeding to Honours in Pure Mathematics who has shown greatest proficiency in at least 24 credit points of Senior units of study in the School of Mathematics and Statistics.

Value: \$1000.

**Barker Prize**

Awarded at the fourth (Honours) year examiner's meetings for proficiency in Pure Mathematics, Applied Mathematics or Mathematical Statistics.

Value: \$550.

**Ashby Prize**

Offered annually for the best essay, submitted by a student in the Faculty of Science, that forms part of the requirements of Pure Mathematics 4, Applied Mathematics 4 or Mathematical Statistics 4.

Value: \$360.

**Norbert Quirk Prize No IV**

Awarded annually for the best essay on a given mathematical subject by a student enrolled in a fourth year course in mathematics (Pure Mathematics, Applied Mathematics or Mathematical Statistics) provided that the essay is of sufficient merit.

Value: \$250.

**David G.A. Jackson Prize**

Awarded for creativity and originality in any undergraduate Pure Mathematics unit of study.

Value: \$1100.

**Australian Federation of Graduate Women: Prize in Mathematics**

Awarded annually, on the recommendation of the Head of the School of Mathematics and Statistics, to the most distinguished woman candidate for the degree of B.A. or B.Sc. who graduates with first class Honours in Pure Mathematics, Applied Mathematics or Mathematical Statistics.

Value: \$175.

**Rolf Adams Prize**

This annual prize is awarded to the pure mathematics honours student who delivers the best talk.

Value: \$100.

**University Medal**

Awarded to Honours students who perform outstandingly. The award is subject to Faculty rules, which require a Faculty mark of 90 or more in Pure Mathematics 4 and a WAM of 80 or higher in 3rd year. More than one medal may be awarded in any year.

I dislike arguments of any kind. They are always vulgar,  
and often convincing.

Oscar Wilde, *The importance of being earnest*

## Chapter 3

# Course Descriptions

Each course is run at 3 lectures and 1 tutorial per week, and counts as 1 unit. For substitutions by courses not given by Pure Mathematics see Section 1.3. In addition, any 40xx course, not previously examined, is available for credit.

### 3.1 Fourth Year Courses — Semester I

#### Algebraic Topology

Lecturer: TBA

One of the most important aims of algebraic topology is to distinguish or classify topological spaces and maps between them up to homeomorphism. Invariants and obstructions are key to achieve this aim. A familiar invariant is the Euler characteristic of a topological space, which was initially discovered via combinatorial methods and has been rediscovered in many different guises. Modern algebraic topology allows the solution of complicated geometric problems with algebraic methods. Imagine a closed loop of string that looks knotted in space. How would you tell if you can wiggle it about to form an unknotted loop without cutting the string? The space of all deformations of the loop is an intractable set. The key idea is to associate algebraic structures, such as groups or vector spaces, with topological objects such as knots, in such a way that complicated topological questions can be phrased as simpler questions about the algebraic structures. In particular, this turns questions about an intractable set into a conceptual or finite, computational framework that allows us to answer these questions with certainty. In this unit you will learn about fundamental group and covering spaces, homology and cohomology theory. These form the basis for applications in other domains within mathematics and other disciplines, such as physics or biology. At the end of this unit you will have a broad and coherent knowledge of Algebraic Topology, and you will have developed the skills to determine whether seemingly intractable problems can be solved with topological methods.

**Classes:**  $3 \times 1$ hr lecture/week,  $1 \times 1$ hr tutorial/week

**Assessment:** 2x homework assignments (50%), final exam (50%)

**Assumed Knowledge:** Familiarity with abstract algebra and basic topology, e.g., (MATH2922 or MATH2961 or equivalent) and (MATH2923 or equivalent).

## Commutative Algebra

Lecturer: TBA

Commutative Algebra provides the foundation to study modern uses of Algebra in a wide array of settings, from within Mathematics and beyond. The techniques of Commutative Algebra underpin some of the most important advances of mathematics in the last century, most notably in Algebraic Geometry and Algebraic Topology. This unit will teach students the core ideas, theorems, and techniques from Commutative Algebra, and provide examples of their basic applications. Topics covered include affine varieties, Noetherian rings, Hilbert basis theorem, localisation, the Nullstellansatz, ring spectra, homological algebra, and dimension theory. Applications may include topics in scheme theory, intersection theory, and algebraic number theory. On completion of this unit students will be thoroughly prepared to undertake further study in algebraic geometry, algebraic number theory, and other areas of mathematics. Students will also gain facility with important examples of abstract ideas with far-reaching consequences.

**Classes:**  $3 \times 1$ hr lecture/week,  $1 \times 1$ hr tutorial/week

**Assessment:** 2x submitted assignments (20% each), final exam (60%)

**Assumed Knowledge:** Familiarity with abstract algebra, e.g., MATH2922 or equivalent.

## Functional Analysis

Lecturer: TBA

Functional analysis is one of the major areas of modern mathematics. It can be thought of as an infinite-dimensional generalisation of linear algebra and involves the study of various properties of linear continuous transformations on normed infinite-dimensional spaces. Functional analysis plays a fundamental role in the theory of differential equations, particularly partial differential equations, representation theory, and probability. In this unit you will cover topics that include normed vector spaces, completions and Banach spaces; linear operators and operator norms; Hilbert spaces and the Stone-Weierstrass theorem; uniform boundedness and the open mapping theorem; dual spaces and the Hahn-Banach theorem; and spectral theory of compact self-adjoint operators. A thorough mechanistic grounding in these topics will lead to the development of your compositional skills in the formulation of solutions to multifaceted problems. By completing this unit you will become proficient in using a set of standard tools that are foundational in modern mathematics and will be equipped to proceed to research projects in PDEs, applied dynamics, representation theory, probability, and ergodic theory.

**Classes:**  $3 \times 1$ hr lecture/week,  $1 \times 1$ hr tutorial/week

**Assessment:** 3x homework assignments (total 30%), final exam (70%)

**Assumed Knowledge:** Real Analysis (e.g., MATH2X23 or equivalent), and, preferably, knowledge of Metric Spaces.



## **Representation Theory**

Lecturer: TBA

Representation theory is the abstract study of the possible types of symmetry in all dimensions. It is a fundamental area of algebra with applications throughout mathematics and physics: the methods of representation theory lead to conceptual and practical simplification of any problem in linear algebra where symmetry is present. This unit will introduce you to the basic notions of modules over associative algebras and representations of groups, and the ways in which these objects can be classified. You will learn the special properties that distinguish the representation theory of finite groups over the complex numbers, and also the unifying principles which are common to the representation theory of a wider range of algebraic structures. By learning the key concepts of representation theory you will also start to appreciate the power of category-theoretic approaches to mathematics. The mental framework you will acquire from this unit of study will enable you both to solve computational problems in linear algebra and to create new mathematical theory.

**Classes:**  $3 \times 1$ hr lecture/week,  $1 \times 1$ hr tutorial/week

**Assessment:** 2x homework assignments (40%), final exam (60%)

**Assumed Knowledge:** Familiarity with abstract algebra, specifically vector space theory and basic group theory, e.g., MATH2922 or MATH2961 or equivalent.

**Prohibition:** MATH3966



A man ought to read just as the inclination leads him; for what he reads as a task will do him little good.

Samuel Johnson

# Chapter 4

## The Essay

### 4.1 Introduction

The essay project has several objectives. First and foremost, it is intended to provide an essentially open-ended framework whereby you may pursue, develop and discover your interests in mathematics unencumbered by syllabus and the prospect of eventual written examination. Basic to this process is the use of the library (for more details see <http://libguides.library.usyd.edu.au/MathsStats>) and communication with others, most especially your supervisor. The writing of the essay is a most valuable part of the project. The very act of writing is an invaluable aid to comprehension. A good essay should be carefully organised, clear, readable by others, laid out well, properly referenced and convey the essential ideas. Attainment of such writing skills is of great benefit whether or not you elect to stay in mathematics.

One point should, perhaps, be emphasised: the essay project is *not* generally intended to be a contribution to original research; however, the essay must clearly demonstrate that you understand and have mastered the material. Originality in presentation or view in the essay is required.

**Writing proficiency.** As mentioned above your essay is also assessed based on the quality of the writing. The university offers several resources that can help you achieve this goal. The [Learning Centre offers workshops](#) for students that need help with extended written work, and a trove of online resources for improving your writing skills is also [available](#). Make sure you make use of these resources as early as possible as writing skills develop slowly over time and with much practice.

### 4.2 Choosing a Supervisor and topic

Choosing a supervisor and topic are the first two things that you should do, and are really not two choices, but one. It is recommended that you begin in the long vacation (preceding your fourth year) by seeking out members of staff and asking them about their interests and topics they would be keen on supervising. (See also Chapter 5 below.) It is a good idea to

ask them about their particular method of supervising and other questions important to you. Do not feel you must settle for the first person you talk to!

All staff members, lecturer and above, are potential supervisors.

There is not necessarily any correlation between supervising style and lecturing style. Also, the subject a lecturer taught you may not be their real area of interest. You should try to decide on a supervisor and topic *before* the start of first semester. Most staff members will be available during the last two weeks of the long vacation; if you have not arranged a topic and supervisor at the beginning of the long vacation, you will probably have to organise your supervisor and topic during these last two weeks.

Changes in supervisor and/or topic are possible during the year (the earlier the better!). If you do change supervisors then you must notify the PM4 Coordinator.

It is a good idea to have a provisional topic and supervisor in mind at the beginning of the long vacation. Your potential supervisor can then suggest some reading for you to do over the long vacation, and, if you have second thoughts about the topic or supervisor, it is then easy to change before the first semester starts.

### 4.3 Essay content and format

The essay must start with an introduction describing the objective and contents of the essay. The essay may end with a summary or conclusion; however, this is optional. Should you wish to make any acknowledgements, they should appear on a separate page, following the introduction.

You should aim at the best scholarly standards in providing bibliographic references. In particular, clear references to cited works should be made, where appropriate, throughout the text. Furthermore, it is not acceptable to base large portions of your essay on the existing literature and whenever part of your essay closely follows one of your sources this must be *explicitly acknowledged* in the text. References should not appear in the bibliography unless they are referred to in the text. For the format of the references see the appendix.

The essay should be clear, coherent, self contained and something that others (your fellow students and other non-specialists in the topic) can read with profit. The essay should not exceed (the equivalent of) 60 pages one and a half spaced type of normal T<sub>E</sub>X font size (that is, as on this page). About 40 to 50 pages would normally be acceptable. Students are asked to try to keep their essays within these limits; overly long essay may be penalised. Supervisors should advise their students accordingly.

Take pains over style, especially clarity, precision and grammar. Aim at readability for the non-specialist. Avoid starting sentences with symbols. Aim for succinct statements of theorems and lemmas. Break up long proofs into lemmas. Cross reference previous results and notation, as this markedly improves readability.

Finally, the essay must be typed or printed and prepared in accordance with the instructions listed in the appendix. You should prepare your essay using a word-processing program such as L<sup>A</sup>T<sub>E</sub>X.

## 4.4 Submission of essay, assessment, corrections

Three bound copies of the essay should be given to the PM4 Course Coordinator for marking no later than noon on Monday of Week 13 in Semester 2 or Semester 1 in case you finish mid year. A PDF of the essay should also be submitted to the relevant Canvas site by this deadline.

Any students submitting their essays past this time can anticipate a penalty of up to 5% per day or part thereof. Essays which are more than one day late may not be accepted. If, during the year, illness or other personal circumstances give a genuine reason for late submission of the essay, such matters should be reported to the Course Coordinator and your supervisor. Such circumstances should be reported as soon as possible, not at the last minute!

Each essay will be read independently by at least two members of the School. (The number of readers will depend on the staff available.) The candidate's supervisor may or may not be one of the readers. The markers may suggest corrections that should be made to the manuscript. If corrections are required, a final corrected copy of the essay should be given to the Course Coordinator for School records. If no corrections are required, one of the markers' copies will normally be kept by the School and the remaining two copies returned to the candidate.

## 4.5 Time management and progress reports

At the end of the first semester you should write a summary (approximately one page in length) of your essay project and progress and give this to the Course Coordinator. This should include a description of the project, the progress made in Semester 1, and what will be achieved in Semester 2. This must be approved by your supervisor before submission to the Course Coordinator.

Here are some rough guidelines and deadlines:

- Select supervisor and topic – before the beginning of first semester
- Reading, discussion and understanding – first semester
- Start work on first draft – by the end of first semester
- Final proofreading – mid-semester break

*Do not underestimate the time it takes you to do the actual writing.* Often it is not until you start writing that you will settle on a final view, or realise that you have misunderstood a particular part of the theory. Allow yourself sufficient time both for the typing and proof reading of the manuscript.

It is strongly advised that you provide your supervisor with drafts of your essay as soon as possible so that he/she may provide constructive feedback. In any case a complete draft must be submitted to your supervisor by 2 weeks before the final essay is due, namely Monday of Week 11.

The essay should be submitted by Monday of Week 13.

## 4.6 Your supervisor

To get the most benefit from the course, you should work closely with your supervisor. To this end, you may set up a regular hour each week to meet and discuss progress and problems with your essay project. Alternatively, you might come to some more informal arrangement.

You can expect your supervisor to:

- Help you select or modify your topic;
- Direct you to useful sources on your topic;
- Explain difficult points;
- Provide feedback on whether you are going in the right direction;
- Advise you on other course matters.

The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact.

A. Whitehead, *Science and the Modern World*

## Chapter 5

### Sample Essay Topics

Here are some topics or areas of interest suggested by members of the Department for 2022. Please note that this list is *not* intended to be complete, the topics suggested are perhaps best regarded as a guide to the likely interests of the proposer, and other staff members are willing to act as essay supervisors. The topics are grouped according to the Research Group to which each staff member belongs.

#### 5.1 Algebra Research Group

##### Dr Nathan Brownlowe – Carslaw 532

Operator Algebra is the study of continuous linear operators on topological vector spaces. A large class of examples comes from  $C^*$ -algebras, which are closed, self-adjoint collections of bounded linear operators on Hilbert space. I study the structure of  $C^*$ -algebras associated to a number of mathematical objects, including algebraic objects like semigroups, groups, and groupoids; combinatorial objects like graphs, and their higher-dimensional analogues; and other mathematical objects, including graphs of groups, and Hilbert bimodules. I am happy to supervise projects in any of these areas.

##### Dr Kevin Coulembier – Carslaw 717

I would like to suggest two topics, as attached below.

i) Algebraic K-theory

$K$ -theory is a branch in mathematics connected to geometry, topology, representation theory, physics and number theory.  $K$ -theory assigns invariants, called  $K$ -groups ( $K_i$  for  $i$  a non-negative integer), to certain geometric and algebraic objects. These  $K$ -groups contain a lot of information about the object, but are difficult to compute. Even the  $K$ -theory of the ring of integers is unknown!

One possible project here aims at understanding  $K_1$ . Historically, the first definition of  $K_1$  of a ring is due to Bass and Schanuel (using the general linear group). Quillen defined  $K$ -groups for a ring using the ‘plus’-construction. Later he improved this

definition by introducing the ‘Q’-construction for any exact category (using simplicial sets). A possible project is to understand some of these constructions, show they yield the same abelian group  $K_1$ , and/or compute  $K_1$  in some explicit examples.

ii) Bruhat orders and affine Weyl groups

This project is concerned with an explicit combinatorial problem with applications in representation theory of affine Kac-Moody algebras. To a Coxeter group  $G$  and a parabolic subgroup  $P$ , there exists a partial order on  $G/P$  known as the Bruhat order. For finite Weyl groups  $G$ , it was recently proved that one can almost always recover  $G$  and  $P$  from the Bruhat order, which had surprising ramifications for BGG category  $\mathcal{O}$  associated to simple Lie algebras. The main aim of the project is to verify to which extent one can recover  $G$  and  $P$  for affine Weyl groups  $G$ .

### **A/Prof David Easdown – Carslaw 619**

A/Prof David Easdown will not be available for supervision in 2022.

### **Professor Anthony Henderson – Quad L4, 44**

Prof Anthony Henderson will not be available for supervision in 2022.

### **Professor Gus Lehrer – Carslaw 813**

There are several possible topics, which come from two basic themes.

- i) The first theme is that of *symmetries of algebraic varieties*; the solution sets of polynomial equations often have symmetries (for example when the defining polynomials are permuted by a group). This leads to actions on the homology of interesting topological spaces, such as discriminant varieties and toric varieties, the latter being defined by monomials associated with polyhedral cones. Topics available here include: computing actions via rational points; geometry of certain classical algebraic varieties; topological and geometric problems arising from reflection groups.
- ii) The second theme is the *representation theory of associative algebras*. There are many semisimple algebras (with easily described representations) which may be deformed into non-semisimple ones by variation of parameters. These occur in algebra (Brauer, Hecke algebras), in mathematical physics (Temperley-Lieb algebras) and topology (BMW algebras, and such like). “Cellular theory” allows one to reduce deep questions about these deformations to (usually hard) problems in linear algebra. There are several possibilities for essay topics in this area.

### **Professor Andrew Mathas – Carslaw 718**

I would be happy to supervise a fourth year essay on any topic in representations theory, or combinatorics. My main research interests are the representation theory of the symmetric groups and related algebras (such as Hecke algebra, Ariki Koike algebras, Schur algebras,



general linear groups, Brauer algebras, Solomon's descent algebras, . . . ), with an emphasis of the non-semisimple case—which is where things start to get interesting, and more difficult!

Possible topics include:

- i) *The modular representation theory of finite groups.* In characteristic zero every representation of a finite group can be decomposed, in a unique way, as a direct sum of irreducible representations. For fields of positive characteristic this is no longer the case, but nevertheless the number of times that a given irreducible module can arise as a composition factor of a representation is uniquely determined. Possible projects in this area range from classifying the number of irreducible representations of a finite group, to studying the Brauer and Green correspondences.
- ii) *Representations of symmetric groups.* The representation theory of the symmetric group is a rich and beautiful subject that involves a lot of algebra and combinatorics. Possible projects here include character formulae, classifying homomorphisms, computing decomposition matrices, Murphy operators, the Jantzen sum formula, . . . .
- iii) *Brauer algebras.* The Brauer algebras arise naturally from the representation theory of the symplectic and orthogonal groups, but they can also be understood from a purely combinatorial viewpoint in terms of a “diagram calculus”. Possible topics include character formulae, classifying semisimplicity, branching theorems, . . . .
- iv) *Seminormal forms.* For many algebras it is possible to give “nice” generating matrices for the irreducible representations in the semisimple case. These explicit matrix representations are called seminormal forms. The study of the seminormal forms, and the resulting character formulae, for one or more algebras would make an interesting essay topic.

### **Professor Alex Molev – Carslaw 707**

- i) *Sugawara operators* Vacuum modules over the affine Kac-Moody algebras are principal examples of vertex algebras which have profound connections in geometry and mathematical physics. The vertex algebra structure on the vacuum module brings up a few bridges connecting its center with the representation theory of the affine Kac-Moody algebra. Elements of the center give rise to Sugawara operators which act as scalars in the Wakimoto modules thus providing an affine analogue of the Harish-Chandra isomorphism and leading to a description of the associated classical W-algebras. The goal of the project is to work out explicit constructions of Sugawara operators which are based on the Schur-Weyl duality and involve special matrix techniques originated in mathematical physics.
- ii) *Yangians and quantum groups* The study of quantum groups has occupied a central stage in mathematics research for the past few decades. The groundwork for this field was laid in the mid-80s. ‘Quantum groups’ refer to a range of Hopf algebras that are deformations (quantisations) of either algebras of functions on groups, or universal enveloping algebras. The aim of the project is to study families of infinite-dimensional quantum groups known as Yangians, which are associated with classical Lie algebras and superalgebras.

### **A/Prof Oded Yacobi – Carslaw 724**

My broad interests are in representation theory, especially in problems of geometric and combinatorial flavour. Most recently, I have been interested in categorical representation theory, which is the study of actions of groups or Lie algebras on categories rather than vector spaces. This is a very new subject, with amazing applications and much yet to be discovered. If you are interested in representation theory and/or category theory, then please feel free to contact me about possible Honours topics.

### **Professor Ruibin Zhang – Carslaw 722**

I can supervise Honours projects in various areas of Lie theory and mathematical physics. Some possible topics are:

- i) *Quantum groups*. Quantum groups are ‘quantized versions’ of universal enveloping algebras of Lie algebras. They originated from the study of the Yang-Baxter equation in physics in the 1980s, and have had very significant impact on many branches of mathematics and physics in recent years. Research on quantum groups is very active, and the subject is rapidly developing.
- ii) *Infinite dimensional Lie algebras*. Infinite dimensional Lie algebras play a key role in conformal field theory and the theory of strings. Typically the Hilbert space of a physical system forms a positive energy module over the Virasoro or a Kac-Moody algebra, where the energy operator is some special element of the algebra. The study of the states and the energy spectrum of the physical system thus may be treated algebraically within the representation theory of these infinite dimensional Lie algebras. Thesis topics in this area involve studying such representations of the Virasoro and Kac-Moody algebras that are most commonly used in physics.
- iii) *Lie superalgebras and supersymmetry*. Supersymmetry is a basic principle that ensures that the fundamental laws of physics are the same for bosons and for fermions. It has permeated many areas of pure mathematics in recent years, leading to deep results such as the Seiberg-Witten theory and mirror symmetry. The algebraic perspective of supersymmetry is the theory of Lie superalgebras, which came to existence in the late 1970s, and is still actively studied today. We shall develop the structure and representations of Lie superalgebras such as the general linear superalgebra and its subalgebras.
- iv) *Quantum field theory and gravity*. Quantum field theory is the conceptual framework for formulating fundamental laws of physics and answering questions about the structure of the physical world. It provides the mathematical means for studying quantum systems of infinitely many degrees of freedom, and making definite predictions that can be tested at experimental facilities like the LHC at CERN. A very active area is the development of a quantum theory of gravity, which is necessary for understanding the structure of spacetime at the Planck scale. Thesis topics in this area involve the study of proposals of quantum theories of gravity using noncommutative generalisations of Riemannian geometry.

## 5.2 Computational Algebra Research Group

### Professor John Cannon – Carlaw 618

i) *Computational Number Theory*. For example:

- Primality testing and factorization,
- Constructive algebraic number theory,
- Computation of Galois groups.

ii) *Computational Group Theory*. For example:

- Algorithmic methods for finitely presented groups,
- Algorithmic methods for permutation groups,
- Computational representation theory,
- Constructive invariant theory.

iii) *Computational Differential Algebra*. For example:

- The Risch algorithm for indefinite integration.

## 5.3 Geometry, Topology and Analysis Research Group

### A/Prof Dmitry Badziahin — Carlaw 634

I do research in the area of Diophantine approximation, a branch of analytic number theory. At its core, it investigates, how well are real numbers (or some other objects) approximated by rationals. Possible topics for essays include:

- *Winning sets and their applications to Diophantine approximation*. Winning sets were firstly invented by Schmidt in 1960's. They have a surprising property: despite of being very small (basically, of zero Lebesgue measure), any countable intersection of winning sets is still winning and hence is uncountable. It appears that many sets of numbers or points in  $\mathbb{R}^n$  sharing certain approximal properties are winning. Projects on this topic will involve the study of the classical Schmidt winning sets and their analogues. Then it will end up with checking a winning property for some set from the area of Diophantine approximation.
- *Approximational properties of Mahler numbers*. Within this topic we will start with the study of continued fractions in the space of Laurent series. Then we will move to Mahler functions — the class of functions which satisfy certain functional equations. We will see that in many cases their continued fraction can be computed by some recurrent formulae. Finally we will investigate, based on that information, how well can values of Mahler functions be approximated by rational numbers.

- *Modern factorisation algorithms.* This topic stays a bit outside of my current research but it may be interesting for some students. We will consider one of the fastest modern factorisation algorithms: elliptic curve method or number field sieve. We investigate their computational complexity, their strengths and weaknesses.

### Dr Emma Carberry – Carslaw 723

My primary research areas are differential geometry and integrable systems, although I also use methods from complex algebraic geometry in my work. I have listed some specific areas below in which I would be happy to supervise an essay but this list is far from exhaustive; if you have other geometric interests please feel free to contact me for further ideas.

- i) *Curves and their Jacobians.* Algebraic curves (smooth algebraic curves are also called compact Riemann surfaces) and line bundles on them are utilised in many areas of mathematics. A project here could go in many directions, a good basic reference is Philip Griffiths' book "An Introduction to Algebraic Curves".
- ii) *Spectral Curves and Integrable Systems.* There is an important class of differential equations that can be written in a particularly simple form, called Lax form. For example, the equations describing a minimal surface in a compact Lie group or symmetric space can be written in this form, as can equations governing Higgs bundles. It is a beautiful fact that solutions to such differential equations on the complex plane (satisfying a finiteness condition) are in one-to-one correspondence with purely algebro-geometric data, consisting of an algebraic curve and a line bundle. The curve is called a spectral curve, and this correspondence gives one powerful new tools with which to attack the original geometric problem. This project is geometric/algebraic in flavour, although the problem originates with a differential equation.

References include Philip Griffiths' article "Linearising flows and a cohomological interpretation of Lax equations" American Journal of mathematics, 107 (1985), no 6, 1445–1484 (1986). and the section by Hitchin in the book "Integrable Systems – Twistors, Loop Groups, and Riemann Surfaces" by Hitchin, Segal and Ward. This area has wide applications in both differential geometry and mathematical physics.

- iii) *Calibrations.* The notion of a calibrated geometry was introduced in a seminal paper by Harvey and Lawson. In these geometries, one studies special submanifolds that are globally area minimising (this is much stronger than the local condition that characterises minimal surfaces). The first non-classical example is special Lagrangian geometry, which plays an important part in mirror symmetry and is currently a hot research area. One can also use the octonions to define three more calibrated geometries, termed exceptional geometries due to their relationship with exceptional Lie groups. This area requires some background in differential geometry, such as that provided by MATH 3968.
- iv) *Quaternionic Holomorphic Geometry.*

When studying the geometry of surfaces, one usually works locally as there are few global tools available. A couple of years ago it was observed that surfaces in  $S^4$  could be studied more globally, using quaternionic analogs of standard complex analytic results.

The quaternions enter the picture since  $S^4$  is isomorphic to the quaternionic projective line, and one can use these tools to study surfaces in  $\mathbb{R}^3$  simply by embedding  $\mathbb{R}^3$  in  $S^4$ . This new theory is being used to study conformal immersions of surfaces, and in particular to attack the Willmore conjecture. A good reference is the book “Conformal geometry of surfaces in  $S^4$  and quaternions” by Burstall, Ferus, Leschke, Pedit and Pinkall, available online at [front.math.ucdavis.edu](http://front.math.ucdavis.edu).

- v) *Minimal Surfaces*. Physically, minimal surfaces model soap films: they locally solve the problem of finding the least area surface with a given boundary. They have been extensively studied and have a rich theory, with many interesting examples and generalisations. They are an active area of current research. There are various possibilities here for a project; ranging from the very explicit (some of the most exciting research here involves finding new examples), to the more theoretical. David Hoffman’s expository article “The computer-aided discovery of new embedded minimal surfaces” in *The Mathematical Intelligencer* **9** no. 3 (1987), and Robert Osserman’s book *A Survey of Minimal Surfaces* are good places in which to get a feel for this area.

## A/Prof Alexander Fish – Carlaw 712

I am doing research in two fields – additive combinatorics and algebraic methods in wireless communication. I will be happy to supervise an honours thesis in any related topic. Some of the possible topics include:

- i) Polynomial method in arithmetic combinatorics

Description: In the last decade it was a surge of breakthroughs in the problems on finding a maximal (or minimal) cardinality of a subset  $A$  of  $F_q^n$  which necessarily avoids a certain structure (or contains a lot of structure). It started with a breakthrough work of Dvir in 2008 on the Kakeya problem in vector spaces over finite fields (Finding a good lower bound on cardinality of a set  $A$  in  $F_q^n$  which contains a line in every direction) and includes the work of Croot-Lev-Pach-Ellenberg-Gijswijt capes bounds and Guth-Katz resolution of Erdos distance problem in the plane. The project will combine studying this new exciting tool (the polynomial method) and working on some new problems which should be amenable to the method.

- ii) Resolution of Erdos conjecture for 3-term arithmetic conjectures

Description: In 2020 it was huge breakthrough in additive combinatorics. Bloom and Sisask posted a preprint on arxiv (<https://arxiv.org/pdf/2007.03528.pdf>) proves the first non-trivial case of the famous Erdos conjecture: Any set  $a_1 < a_2 < a_3 < \dots$  in the positive integers satisfying that the sum of reciprocals diverges contains an arithmetic progression of any finite length. Bloom and Sisask proved the conjecture for three term arithmetic progressions. The conjecture of Erdos can be thought as a generalisation of famous Szemerédi’s theorem which states that any set  $A$  of integers of positive density contains arbitrary long (finite) arithmetic progressions. The case for three term arithmetic progressions was proved by Roth and called Roth theorem. In the project you will study the work of Bloom and Sisask. A necessary background is Measure theory and we will study exciting methods of Harmonic Analysis.

## iii) Ergodic theory in Combinatorial Number Theory

Description: You will study basics of Ergodic Theory, and subsequently, the work of Furstenberg on Szemerédi theorem (every set of integers of positive density contains arbitrary long arithmetic progressions) and if time will permit the Green-Tao theorem that primes contain arbitrary long arithmetic progressions.

## iv) Additive Combinatorics and Freiman Theorem

Description: This project will focus on one of the most important theorems of additive combinatorics - Freiman theorem. It roughly says that if a finite set of integers  $A$  has small doubling, i.e. the number of elements in  $A+A$  is bounded by a constant times the number of elements in  $A$  (constant is independent of  $A$ ), then  $A$  has a structure similar to the structure of arithmetic progression. If time will permit the thesis will include the modern generalisation of Freiman's theorem, namely, the theorem of Breiðlind-Green-Tao on approximate groups (which are non-commutative generalisations of sets with small doubling).

## A/Prof James Parkinson – Carlaw 614

My main research interests lie at the confluence of algebra and geometry. My preference is for combinatorial techniques, attempting to understand complicated mathematical objects using simple combinatorial gadgets and tools. Please come and see me to discuss possible essay topics, with possible starting points including:

- *The incidence geometries of classical groups.* In this project you would discover how the (ubiquitous!) classical matrix groups over arbitrary fields act on beautiful 'point-line geometries' called *incidence geometries*.
- *Projective geometry: The Veblen-Young Theorem (and beyond).* The celebrated Veblen-Young Theorem classifies projective spaces when dimension is greater than 2. In your essay you could prove this beautiful and important theorem, and also investigate the zoo of non-classical examples in the critical dimension 2 case.
- *Random walks on Coxeter groups and hyperplane arrangements.* Representation Theory can be used to study probability theory on groups, with a famous example being the Bayer-Diaconis Theorem stating that 7 shuffles are required to adequately randomise a deck of cards. In this project you would discover the beauty of Coxeter groups and associated geometric configurations, and investigate how representation theory can be used to understand the long term behaviour of random processes on these objects.
- *Automata for Coxeter groups and surface groups.* The concept of an 'automatic structure' for a group has had a profound influence on computational group theory. This project would involve a detailed study of the beautiful class of 'Coxeter groups' and 'surface groups', with a focus on understanding and computing their automata. Applications include computing growth rates in these groups.

**Professor Laurentiu Paunescu – Carslaw 721**

I am interested in the applications of singularity theory to differential equations, and in using the combinatorics of Toric Modifications in investigating the equisingularity problem.

My main research interests are:

- i) *Singularities of complex and real analytic functions.*
- ii) *Stratified Morse theory.*
- iii) *Toric resolution of singularities.*

**A/Prof Anne Thomas – Carslaw 716**

I would be happy to supervise an Honours essay on any topic related to my research interests, which include geometric group theory, rigidity and lattices in locally compact groups. Some sample topics are as follows, but these should be taken only as a guide.

- i) *Hyperbolic groups.* One of the fundamental ideas in geometric group theory is to study finitely generated groups via the geometry of their associated Cayley graphs. A hyperbolic group is a group whose Cayley graph is “negatively curved” in an appropriate sense. Hyperbolic groups share many properties with fundamental groups of compact surfaces of genus  $\geq 2$ . Possible directions include algorithmic properties, boundaries, connections with cube complexes, separability properties and the surface subgroup conjecture. Suggested reading: Bridson and Haefliger, *Metric Spaces of Non-Positive Curvature*.
- ii) *Bass-Serre theory.* Group actions on trees may be encoded using the combinatorial data of a graph of groups. This beautiful theory has many applications to the study of group splittings, low-dimensional topology and lattices. Suggested reading: Serre, *Trees*.
- iii) *Coxeter groups and nonpositive curvature.* Coxeter groups, which are generated by reflections, are ubiquitous in mathematics. Infinite Coxeter groups may be studied via their action on a contractible cell complex, the Davis complex, which satisfies an important combinatorial nonpositive curvature condition called the CAT(0) inequality. Suggested reading: Davis, *The Geometry and Topology of Coxeter Groups*.
- iv) *Buildings.* Buildings are highly symmetric simplicial complexes that are associated to certain groups, and are used to study these groups. Various classes of buildings have been classified and there are important results relating their local to their global structure. Suggested reading: Ronan, *Lectures on Buildings*.

**Professor Stephan Tillmann – Carslaw 710**

I am available to supervise projects in geometry and topology, with an emphasis on low-dimensional objects, such as knots, surfaces, 3-dimensional and 4-dimensional spaces. I tend to use algebraic or combinatorial techniques, as well as synthetic geometric arguments. One of the aims of my research is to understand the structure of low-dimensional spaces and to obtain new invariants of these spaces.

Specific projects will be designed depending on your interests and experience. The aim of a thesis topics could be to understand a beautiful piece of theory; produce new examples; design or implement new algorithms to compute invariants; work out a discretisation of a smooth theory.

## Dr Haotian Wu – Quad L4, 45

My research area is geometric analysis, which employs analytic tools such as partial differential equations to study problems in geometry and topology. I have worked on geometric flows such as Ricci flow and mean curvature flow, (moduli) space of Riemannian metrics, and geometry problems that arise in mathematical general relativity.

I look forward to supervising honours student(s). Your honours essay topic could include:

- i) **Existence of “nice” metrics.** A fundamental question in differential geometry is that if a given manifold  $M$  carries a “nice” Riemannian metric. For example, an  $n$ -sphere equipped with the standard round metric is nice in the sense that its curvatures are all the same positive constant. Two research topics in this direction are:
  - **Ricci flow<sup>1</sup>:** Here one deforms a metric  $g$  by  $\partial_t g = -2\text{Ric}(g)$ , where  $\text{Ric}(g)$  is the Ricci curvature of  $g$ . Ricci flow is an efficient way to deform the metric into a “nice” one known as the Einstein metric. However, Ricci flow is nonlinear and tends to develop singularities in finite time, and singularity analysis poses a fundamental problem in the theory.
  - **Yamabe problem:** Here one seeks the existence of metric with constant scalar curvature in a given conformal class on a given manifold, e.g. the round metric on an  $n$ -sphere. This question can be formulated as an elliptic PDE, or can be approached by parabolic methods.
- ii) **(Moduli) space of metrics.** Given a manifold  $M$ , denote by  $\mathcal{M}$  the space of all Riemannian metrics on  $M$ . Then we can study the properties of the space  $\mathcal{M}$ . For example, is there a tangent space to  $\mathcal{M}$  at a point (i.e. a metric)? If so, what is it? Or, is the space  $\mathcal{M}$  path-connected? What about its higher homotopy groups? If two metrics  $g, g'$  are diffeomorphic, then  $(M, g)$  and  $(M, g')$  are geometrically indistinguishable. Thus, denote by  $\mathcal{D}$  the space of all diffeomorphisms of  $M$ , we also study the moduli space  $\mathcal{M}/\mathcal{D}$ .
- iii) **Zoo of submanifolds.** Classical differential geometry studies curves and surfaces which are submanifolds of the flat Euclidean  $\mathbb{R}^3$  (cf. MATH3968). One can generalise to other ambient space  $(M, g)$  and study submanifolds of special properties such as *minimal* (zero mean curvature) and *constant mean curvature* (CMC) submanifolds. For example, Which CMC submanifolds exist in a given  $(M, g)$ ? How many are there? Can we classify them? In this context, one can also study *mean curvature flow*.

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<sup>1</sup>Ricci flow has been successful in proving Thurston’s Geometrisation Conjecture of three manifolds and the Poincaré Conjecture, one of the “million-dollar maths problems”.



- iv) **Mathematical general relativity.** The space-time of our universe is studied using Riemannian geometry (space) and Lorentzian geometry (space-time) and contains many fascinating topics. To name a few: the mathematics of black holes is related to that of minimal surfaces in a space-time; the Positive Mass Theorem and its connection to scalar curvature; the Riemannian Penrose Inequality can be proved using the inverse mean curvature flow.

If you have other topics related to geometry, topology, or analysis in mind, please do not hesitate to email me so we could discuss the possibilities.

### A/Prof Zhou Zhang – Carslaw 620

My primary research interest lies in differential geometry. Techniques from the theories of partial differential equation and several complex variables are frequently called for. The problems being considered often have strong algebraic geometry background. In the following, a few topics suitable for an honours essay are listed. If you already have your own topic(s) in mind, I'll be more than happy to discuss with you and provide suggestions if you wish.

- i) *Futaki Invariant.* The goal is to look at some basic facts regarding this very important invariant in the study of Einstein metric. The understanding and justification of definition itself would already serve as a good way to introduce basics in complex differential geometry.
- ii) *Ricci Flow and Maximum Principle.* Ricci flow and the complex version of it, Kähler-Ricci flow are topics of great interest in the recent years. Maximum Principle is, in principle, a very simple-minded tool in the study of differential equations. We could focus on Hamilton's Tensor Maximum Principle and provide some taste of how something so intuitive can go such a long way. In this topic, we would start with some introduction to Riemannian geometry if needed.
- iii) *Introduction to Algebraic Curves.* Here I am using the title of the book by Griffiths. It provides a few very good topics to work on for beginners in algebraic geometry, for example, the Riemann-Roch Theorem in Chapter III. The discussion of these low dimensional objects naturally serves as a motivation in the study for general dimension.
- iv) *De Rham Theorem and Hodge Theorem.* This topic serves as an introduction to differential geometry for manifolds. For the most part, we could follow the nice textbook "Foundations of differentiable manifolds and Lie groups" by Warner, and add some more in the Kähler setting, if time allows, where things come together in a very nice way.
- v) *Characteristic Classes.* Here I am using the title of the famous book by Milnor. From one point of view, it provides a topological (and intuitive) way to describe objects of great interests in differential geometry.

## 5.4 Non-Linear Analysis Research Group

### A/Prof Florica Cîrstea – Carslaw 719

My main research interests concern nonlinear partial differential equations (PDEs). In this area there are many important topics that can be treated using various modern approaches. I would be happy to supervise such topics from the theory of both linear and nonlinear PDEs. Some specific projects are provided below, but students are encouraged to negotiate the topic for a best match.

- i) The theory of linear PDEs relies on functional analysis and relatively easy energy estimates to prove the existence of weak solutions to various linear equations. The proper setting for the study of many linear and nonlinear PDEs via energy methods is provided by the so-called Sobolev spaces. If we require the solutions of a given PDE to be very regular, we would usually have a very hard time to find them. A more profitable approach is to consider the issue of existence of solutions separately from the smoothness (or regularity) problems. The idea is to introduce a new concept of solution (weak solution) that does not have too much smoothness so that we could establish its existence, uniqueness and continuous dependence of the given data. Various PDEs could be treated in this way and this is, possibly, the best we can do in many cases. For others, we could hope to prove that our weak solution becomes smooth enough to be deemed as a classical solution. This leads to the issue of regularity of weak solutions, which relies on many intricate estimates.

*Possible topics.* Investigating the solvability of uniformly elliptic, second-order PDEs, subject to prescribed boundary conditions using two essentially different techniques: energy methods within Sobolev spaces and maximum principle methods. The energy methods can be expanded to a variety of linear PDEs characterising evolutions in time. This broadens the class of PDEs to include the heat equation and more general second-order parabolic PDEs, as well as the wave equation (and general second-order hyperbolic PDE).

- ii) The theory of nonlinear PDEs is far less unified in its approach compared with the linear one. Variational methods provide one of the most useful and accessible of the approaches for nonlinear PDEs. Other techniques are also available for nonlinear elliptic and parabolic PDEs such as the monotonicity and fixed-points methods, as well as other devices involving the maximum principle. The study of such techniques would make interesting essay topics.

### A/Prof Daniel Daners – Carslaw 715

Areas of interest:

- i) *Partial differential equations (linear or nonlinear).*
- ii) *Ordinary differential equations (linear or nonlinear).*
- iii) *Bifurcation theory.*

- iv) *Analytic semigroup theory and abstract evolution equations.* (This is a theory of “ordinary differential equations” in infinite dimensional spaces with applications to partial differential equations.)

Please see me to negotiate a topic of your interest or for suggestions for specific projects related to the above areas.

### **A/Prof Leo Tzou – Carslaw 613**

Probing the Universe with Nonlinear Echoes

We see because electromagnetic waves stimulate our eyes. Some animals, however, "see" other types of waves - bats generate sound waves, using the echoes to pinpoint the size and location of their prey.

Although human ingenuity has also given us some tools to see beyond our eyes, present tomography methods nevertheless face limitations - seismology often suffers from a scarcity of detectors; telescopes can collectively receive only a tiny fraction of the signals propagating through the universe. These problems effectively require us to deduce large-scaled information from a single measurement point, inspiring challenging mathematical models that we have yet to overcome.

In this project the student will learn the necessary mathematics to potentially participate on my current work on single detector gravitational wave astronomy.

### **Prof Eduardo Altmann – Carslaw 523**

Monte Carlo methods in triangulation problems

The goal of this project is to investigate how Markov Chain Monte Carlo methods can be used to optimise triangulations of manifolds with a range of different applications in mind. After reviewing the known results for simple configurations (in low dimensions), we will focus on computational methods to efficiently find triangulations with good properties. Within a Monte Carlo framework, we will investigate the efficiency of different proposal steps such as moves that merge and subdivide triangles or flip edges (so-called bistellar moves). This project involves programming, it lies in the intersection between Applied and Pure mathematics, and will be co-supervised by Prof Eduardo Altmann and Dr. Jonathan Spreer.

Related work:

[1] Aste, Gramatica, and Di Matteo,

"Random and Frozen States in Complex Triangulations." *Philosophical Magazine*, 92:1-3, 246-254, (201) DOI: 10.1080/14786435.2011.613861

[2] Björner and Lutz,

"Simplicial manifolds, bistellar flips and a 16-vertex triangulation of the Poincaré homology 3-sphere", *Experiment. Math.* Volume 9, Issue 2 (2000), 275-289.

**Prof Ben Goldys – Carslaw 709**

## **5.5 PROJECT 1:**

### **Strict topologies and nonlinear semigroups on spaces of continuous functions**

The aim of this project is to study applications of the topological measure theory to analysis of nonlinear partial differential equations in spaces of continuous functions. First, we will consider some classes of linear operators in such spaces, including some differential operators. It turns out, that even in the linear case the norm topology is not suitable for the analysis of solutions and associated semigroups of linear operators. We will learn that certain natural topologies, weaker than the norm topology, are ideally suited for this purpose. In the next step we will use this machinery to study viscosity solutions to a large class of nonlinear PDEs.

The topic of this project is closely related to recent developments in the theory of nonlinear Markov semigroups and associated nonlinear Markov processes. However, knowledge of stochastic analysis is not needed for this project.

The project will require tools from functional analysis and theory of measures on topological spaces, especially on locally convex spaces. This is mainly a learning project, but some new developments will be also possible.

Our starting point will be the paper:

*Michael Röckner and Max Nendel:*

*Upper envelopes of families of Feller semigroups and viscosity solutions to a class of nonlinear Cauchy problems, <https://arxiv.org/pdf/1906.04430.pdf>*

## **5.6 PROJECT 2: Renormalisation and invariant measures for wave equations**

Some fundamental equations of quantum field theory and statistical physics require a special procedure known as renormalisation. A rigorous theory of renormalisation developed in mathematics requires arguments from algebra, measure theory and analysis. In this project we will apply the renormalisation procedure to certain nonlinear wave equations of mathematical physics. Our aim will be to show that such equations have well defined solutions if the initial condition belongs to the support of an invariant measure. We will start with studying the paper

*Oh, Tadahiro and Thomann, Laurent: Invariant Gibbs measures for the 2-d defocusing nonlinear wave equations. Ann. Fac. Sci. Toulouse Math. (6) 29 (2020), no. 1, 1-26*

## Chapter 6

### The Talk

#### 6.1 General remarks

Before the essay is submitted at the end of Second Semester, each student gives a talk on their essay project. The talks will usually take place about the mid-semester break or student's final semester.

The aim of the talk is to provide training in the explanation to others of the purpose and nature of a project, within definite time limits; twenty minutes for each talk, plus five minutes for questions.

All members of the Department, Fourth Year and postgraduate students are invited to the Fourth Year talks.

*The talk is worth 5% of your essay mark.*

#### 6.2 Preparing the talk

The purpose of your talk is to convey to your *fellow students* (and the academic staff) what you are working on. They probably know very little about your essay topic; this comment may also apply to the academic staff. Do not make the talk too long or ambitious. Aim to convey the essence of your project to the audience rather than trying to impress the audience; after all, it is unlikely that you can cover the whole of your project in 20 minutes!.

The key to giving a successful mathematical talk is: "Keep it simple!" *One* idea, illustrated by one or two examples, is enough for a good talk. A special case often conveys more than a general, all-encompassing theorem. For example, to give the flavour of general fields, a detailed study of a simple, but unfamiliar field, such as  $GF(9)$ , might be appropriate.

Keep in mind that the audience is swept along with you and that they cannot go back to earlier stages of your talk like when they are reading an article. You are not giving a lecture, so although some definitions may be appropriate, lengthy technical proofs should be avoided. It is also not a good idea to over-develop the theory at the expense of examples: a well-chosen example is worth ten thousand theorems. Finally, try and relate your content to other areas of mathematics or applications; this can make the talk much more interesting for the general audience.

You should aim your talk at a general mathematical audience and *avoid* directing it at the odd specialist in your topic in the audience. Thus a good talk is judged by one criterion: you have given the audience, especially your fellow fourth year students, a good idea of your project and its significance.

Discuss the talk with your supervisor.

Having chosen the topic for your talk, prepare a written outline. Some people write their talk out in full, while others prefer to use only a written outline and allow improvisations. As it is probably your first talk of this kind, it is advisable to do a full dress rehearsal the previous evening; so find a blackboard or a projector and go through the complete talk. This will help you in judging the timing of your talk properly: it takes much longer to say things than you probably realise. If you can, find a sympathetic listener to give you feedback. Your listener does not have to be mathematically literate: a good talk is almost as much about theatre and presentation as it is about mathematics.

### 6.3 Slide Talks

Decide if the use of Beamer, PowerPoint or an overhead projector is appropriate. This allows preparation of complicated figures or tables ahead of time, or the inclusion of photocopies of published material in your exposition. Beware, however, that although the speaker can by this means pass a vast amount of information before the eyes of the audience very quickly, the audience will probably not take it all in. It is important either to write clearly and in large letters and to refer explicitly to each line (say by gradually revealing line-by-line) or, in the case of a diagram or complicated formula, to allow your audience time to absorb its detail.

If you are going to use  $\text{\LaTeX}$  to create slides then the use of the Beamer package is recommended.

## Chapter 7

# Your Future and Mathematics

As a fourth year student you are a member of the mathematics department and you should take advantage of the facilities it offers. The University of Sydney has one of the top mathematics research departments in the country, and it ranks very highly internationally in several areas. There are also a number of prominent international (short and long term) visitors to the department who give seminar talks within the department. It pays to keep an eye on scnews (the School's web based bulletin board), for upcoming seminar announcements.

The academic staff, the many postdocs and the visitors to the department are all usually very happy to talk mathematics talk with interested students: all you have to do is find the courage to ask!

Fourth year students are also very welcome to join the staff and postgraduates in the use of the tea room; this can be a good place to meet other people in the department.

### 7.1 Seminars

Students are welcome to any seminar run in the Department. For a schedule of upcoming seminars, see scnews and the seminar websites that are linked to from the main school webpage.

### 7.2 After fourth year, what then?

Recent graduates have found employment in a wide variety of occupations: computer related jobs, teaching (University or School), positions in insurance and finance. To find out more about where maths can take you:

<https://sydney.edu.au/careers/>

<http://www.amsi.org.au>

<http://www.austms.org.au/Jobs>

Here we shall just outline briefly the postgraduate degree options. For more information consult the departments web pages.

### 7.3 Higher degrees

A result of II-2 or better is the minimum requirement for entry into a higher degree at Sydney. However it should be noted that one should not normally contemplate continuing without a result of at least II-1. Anyone intending to undertake a higher degree should consult with the Mathematics Postgraduate Coordinator (Dr Pengyi Yang), as soon as possible. The usual practice is to enrol for an M.Sc. in the first instance and later to convert to a Ph.D. if it is desired to continue.

Information on scholarships for postgraduate study can be found at

<http://www.maths.usyd.edu.au/math/s/u/PG/>.

### 7.4 Scholarships and other support

Scholarships, prizes and travel grants are available both for study at Sydney and for study elsewhere. Full details can be found in the University Calendar and from the Scholarships Office (Administration Building). Intending applicants should obtain application forms from the Scholarships Office as soon as possible. *The closing dates for some scholarships can be as early as September.*

If you are considering further study at an Australian University, you should apply for an Australian Postgraduate Research Award (even for an M.Sc. by coursework). For study at a university in Britain or Canada, apply for a University of Sydney travelling Scholarship and also apply to the chosen university for employment as a Graduate Assistant.

### 7.5 Further study in another subject

As mentioned in the introduction to this booklet, it is quite possible to do Fourth Year Pure Mathematics and then continue with a higher degree in another subject. Within Australia, prerequisites vary from university to university and department to department, and for those intending to follow this path it is advisable to consult with the department concerned to determine an appropriate choice of fourth year topics. If you are intending to continue with postgraduate studies in another field *outside* Australia, do check prerequisites. Provided you have done third year courses in the subject at Sydney, you will *probably* not encounter significant problems over prerequisites.



It has long been an axiom of mine that the little things are infinitely the most important.

Sherlock Holmes, *A Case of Identity*

## Appendix A

# Instructions on Preparing the Manuscript

Essays must be typed using  $\text{\LaTeX}$  (or  $\text{\TeX}$ ), or a commercial word processing program such as word. Amongst professional mathematicians  $\text{\LaTeX}$  has become the standard; it produces better quality output than any word processing programs program—at least when it comes to mathematics. The downside to  $\text{\LaTeX}$  is that it takes some time to learn.

The fourth year coordinator will give an introduction to using  $\text{\TeX}$  and  $\text{\LaTeX}$  before the beginning of second semester. For those wishing to use  $\text{\LaTeX}$  Prof Mathas has written a  $\text{\LaTeX}$  class file that takes care of the basic layout of the essay; for information, as well as some basic tips on how to use  $\text{\LaTeX}$ , see <http://www.maths.usyd.edu.au/u/mathas/courses/pm4/>.

See <http://www.maths.usyd.edu.au/u/SMS/texintheschool.html> for links to  $\text{\TeX}$  and  $\text{\LaTeX}$  documentation available on the School's website. The  $\text{\LaTeX}$  package `amsmath` and `BiBTeX` are perhaps the most important. (Note that Prof Mathas' class file preloads the `amsmath` package, which is essential for adequately typesetting mathematics in a  $\text{\LaTeX}$  document. `BiBTeX` is used for automatically including a bibliography in a  $\text{\LaTeX}$  document.)

If you decide not to use this  $\text{\LaTeX}$  class file, then your document must nevertheless satisfy the following requirements.

1. A margin of at least 2.5cm must be left at the top, bottom, left- and right-hand side of each page. The margin is determined by the last letter or character in the longest line on the page.
2. All pages must be numbered (in a consistent way), except for the title page.
3. Avoid excessive use of footnotes. They are rarely necessary in mathematics.
4. Diagrams should be created using appropriate software; check with your supervisor first if you intend to use hand drawn diagrams.
5. Theorems, Propositions, and such like, should be labelled consistently throughout the document.
6. The font size must be 12pt.