One period

$S = S_{stock}$

$S(0) = u, \ S_0$

$t = 0$

$S_1(t) = d \ S_0$

\[
p \leq u
\]

One $\$ invested in money market at $t=0$ gives

\[
1 + r \quad \text{at} \quad t=1
\]

\[
r > -1
\]

\[
0 < d < 1 + r < u
\]

Usually we assume

\[
d = \frac{1}{u}
\]
Why \( d < 1 + r \)

If \( d > 1 + r \)

\( t = 0 \) start with 0\$ borrow \( \frac{s_0}{d} \) money buy stock

\( t = 1 \)

\[ d \cdot s_0 \geq (1 + r) s_0 \]

you always make profit
Two periods
\[ d = \frac{1}{2} w \text{ - recombining tree} \]

\[ s_0 \]
\[ s_0 u \]
\[ s_0 u^2 \]

\[ s_0 d \]
\[ s_0 u d = s_0 \]
\[ d u = s_0 \]

\[ s_0 d^2 \]
European Call Option

Owner has the right but not obligation to buy one share of stock at \( t=1 \) for the strike price \( K \)

\[ S_1(T) < K < S_1(H) \]

\[ S_0 \quad \downarrow \quad S_1(T) \quad \downarrow \quad S_1(H) \]

If \( H \) you earn \( S_1(H) - K \); so you buy.

If \( T \) you do not buy.
At time \( t=1 \) the option is worth
\[
(\mathbf{S}_1 - K)^+ \quad \text{if } \mathbf{S}_1 < K
\]
\[
0 \quad \text{if } \mathbf{S}_1 \geq K
\]

\[\alpha^+ = \begin{cases} 
\alpha & \alpha \geq 0 \\
0 & \alpha \leq 0
\end{cases}\]

How much would you pay for such a contract at time \( t=0 \)?
Initial wealth: $X_0 = 1.20$

Buy $D_0 = 2$ shares

We need to borrow $0.80$

Cash position: $X_0 - D_0 S_0 = -0.80$

$t = 1$

Cash position: $(1+r)(X_0 - D_0 S_0) = (1.25)(-0.80) = -1$

$H$: $\frac{1}{2} S_t(H) = 2$

$T$: $\frac{1}{2} S_t(T) = 1$
\[
X_1(H) = \frac{1}{2} S_1(H) + (1+\Gamma) (X_0 - \Delta_0 S_0) = 4 - 1 = 3
\]
\[
X_1(T) = \frac{1}{2} S_1(T) + (1+\Gamma) (X_0 - \Delta_0 S_0) = 1 - 1 = 0
\]
\[
X_1(H) = (S_1(H) - 5)^+ = 3
\]
\[
X_1(T) = (S_1(T) - 5)^+ = 0
\]

Replicating portfolio:
\[
\Delta = \frac{1}{2}
\]
\[
X_0 = 1.20 \quad \text{fair price}
\]

Riskless portfolio

Buy one option for 1.20

same

no risk
Wealth at $t=0$, $t=1$

\[ X_0, X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) \]

\[ X_1 = (1+r)X_0 + \Delta_0 (S_1 - (1+r)S_0) \]

We want to choose $X_0, \Delta_0$ so that

\[ X_1(H) = (S_1(H) - K)^+ \]
\[ X_1(T) = (S_1(T) - K)^+ \]

\[ \frac{X_1(H)}{1+r} = \frac{(S_1(H) - K)^+}{1+r} \]
\[ \frac{X_1(T)}{1+r} = \frac{(S_1(T) - K)^+}{1+r} \]

\[ X_0 + \Delta_0 \left[ \frac{S_1(H)}{1+r} - S_0 \right] = \frac{(S_1(H) - K)^+}{1+r} \]
\[ X_0 + \Delta_0 \left[ \frac{S_1(T)}{1+r} - S_0 \right] = \frac{(S_1(T) - K)^+}{1+r} \]
Put
\[ p = \frac{1+r-d}{u-d}, \quad q = 1-p = \frac{u-d-1+r+d}{u-d} = \frac{u-1-r}{u-d}, \]

\[ 0 < p < 1 \quad \text{because} \quad 0 < d < 1+r < u. \]

\[ X_0 + A_0 \left[ \frac{1}{1+r} \left( p S_1(H) + q S_1(T) \right) - S_0 \right] \]

\[ = \frac{1}{1+r} \left[ p \left( S_1(H) - k \right)^+ + q \left( S_1(T) - k \right)^+ \right] \]

\[ s_0 = \frac{1}{1+r} \quad \text{if} \quad s_1, \quad \text{boxed} \quad u d = 1. \]

\[ s_1 = p s_1(H) + q s_1(T) = p s_0 u + q s_0 d \]

\[ = \left[ \frac{1+r-d}{u-d} u + \frac{u-1-r}{u-d} d \right] s_0 = s_0 \]

\[ X_0 = \frac{1}{1+r} \quad \text{if} \quad (s_1 - k)^+ \]

\[ A_0 = \frac{(s_1(H)-k)^+ - (s_1(T)-k)^+}{s_1(H) - s_1(T)} \]