(a)(i) \[ x^2 \]

Equilibrium populations are \( x = 0 \) (unstable) and \( x = M \) (stable).

(ii) \( \lambda \) has dimensions \( \text{L} \cdot \text{T}^{-1} \).
\[ M \] has dimensions \( \text{L} \cdot \text{X} \).

\( M \) is the carrying capacity of the environment for \( x \).
\( \lambda \) is the linear growth rate of the population.

(b)(i) \[ \frac{dx}{dt} = r x (1 - \frac{x}{M}) - E x \]

\[ = (r - E) x - \frac{r x^2}{M} \]

\[ = (r - E) x \left( 1 - \frac{r x}{(r-E)M} \right) \]

Hence there are two steady states: \( x = 0 \) and \( x = \frac{M(r-E)}{r} \).
\[
F = (r-E)x \left(1 - \frac{ra}{(r-E)x}\right)
\]

\[
\frac{dF}{dx} = (r-E) - \frac{2r}{M} x.
\]

When \( x = 0 \), \( F(0) = r-E \)

\[
\begin{cases} 
> 0 & \text{if } r > E \\
< 0 & \text{if } E > r
\end{cases}
\]

so \( x = 0 \) is

\[
\begin{cases} 
\text{stable if } r < E \\
\text{unstable if } r > E
\end{cases}
\]

When \( 1 - \frac{ra}{(r-E)x} = 0 \), \( x = \frac{r-E}{r} M \)

\[
F'(\frac{r-E}{r} M) = r-E - 2(r-E)
\]

\[
= -(r-E)
\]

so \( F'(\frac{r-E}{r} M) \)

\[
\begin{cases} 
> 0 & \text{if } r < E \\
< 0 & \text{if } E < r
\end{cases}
\]

so \( x = \frac{r-E}{r} M \) is

\[
\begin{cases} 
\text{stable if } r > E \\
\text{unstable if } r < E
\end{cases}
\]
(b)(i) **Bifurcation diagram**

\[ x^* = \frac{M}{r} (r-E) E. \]

**Maximum sustainable yield** is:

\[ \text{max of } E x^* \]

\[ E x^* = \frac{M}{r} (r-E) E. \]

Maximum at this occurs when:

\[ \frac{d}{dE} \left( \frac{M}{r} (r-E) E \right) = 0 \]

i.e. \[ \frac{M}{r} (r-E - E) = 0 \]

so \[ E = \frac{r}{2}. \]

**Hence max. sustainable yield**

\[ = \frac{M}{r} (r - \frac{r}{2}) \frac{r}{2} \]

\[ = \frac{Mr}{2}. \]
Return time to a stable equilibrium $x_t = x^*$ is

$$\frac{-1}{F'(x^*)} = \frac{-1}{F'(\frac{Mr}{2})}$$

$$F'(\frac{Mr}{2}) = (r - E) - 2x \frac{Mr}{2}$$ at MSY.

$$= (r - r/2) = \frac{r}{2}$$

Substituting $E = r/2$.

$$= \frac{r}{2} - r^2$$

So return time is $\frac{1}{r(r - r/2)}$.

(1b)(iii) If the effort is slowly increased from zero then the equilibrium population will slowly decrease until it reaches 0 at $E = r$. When $E > r$ then the population remains at $x^* = 0$.

As $E$ increases from zero, the yield will increase until it reaches maximum sustainable yield at $E = r/2$. As $E$ continues to increase, the yield decreases until it is zero at $E = r$. For $E > r$ the yield remains zero.
2(a) in matrix form
\[
\begin{pmatrix}
  x' \\
  y'
\end{pmatrix} = \begin{pmatrix}
  2 & -1 \\
  -1 & 2
\end{pmatrix} \begin{pmatrix}
  x \\
  y
\end{pmatrix} \Rightarrow \begin{pmatrix}
  x' \\
  y'
\end{pmatrix} = A \begin{pmatrix}
  x \\
  y
\end{pmatrix}.
\]

Characteristic eqn of \( A \) is \( \lambda^2 - 4\lambda + 3 = 0 \)
\[
(\lambda - 1)(\lambda - 3) = 0 \quad \text{so} \quad \lambda = 1, 3.
\]

Eigenvalues are \( \lambda = 1 \) and \( \lambda = 3 \).

Finding eigenvectors \( \begin{pmatrix}
  u \\
  v
\end{pmatrix} \).

1. \( \begin{pmatrix}
  u \\
  v
\end{pmatrix} = \begin{pmatrix}
  2 & -1 \\
  -1 & 2
\end{pmatrix} \begin{pmatrix}
  u \\
  v
\end{pmatrix} \).

\[ u = 2u - v \quad \text{and} \quad v = -u + 2v. \]

So eigenvector corresponding to \( \lambda = 1 \) is \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

2. \( \begin{pmatrix}
  u \\
  v
\end{pmatrix} = \begin{pmatrix}
  2 & -1 \\
  -1 & 2
\end{pmatrix} \begin{pmatrix}
  u \\
  v
\end{pmatrix} \).

\[ 3u = 2u - v \quad \text{so} \quad u = -v. \]

\[ 3v = -u + 2v \]

So eigenvector corresponding to \( \lambda = 3 \) is \( \begin{pmatrix} -1 \\ 1 \end{pmatrix} \).

(ii) Phase plane.

Nullclines \( \dot{x} = 0 \) \quad \begin{pmatrix} x \\ y \end{pmatrix} = 2x, \quad y = \frac{\sqrt{2}}{2}. \)
Unstable node at \((0,0)\).

\[
\begin{align*}
\vec{x} &= \frac{\partial F}{\partial x} \\
\vec{y} &= -\frac{\partial F}{\partial y}
\end{align*}
\]

(2007 - 0)

(b)(i) For a gradient system

\[
\dot{x} = f(x, y) = -\frac{\partial F}{\partial x} \quad \text{and} \quad \dot{y} = g(x, y) = -\frac{\partial F}{\partial y}
\]

so \(\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}\).

Here \(\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (y - x^5) = 1\).

\[
\frac{\partial g}{\partial x} = \frac{\partial}{\partial x} (-x - y^5) = -1 \neq \frac{\partial f}{\partial y}
\]

So this is not a gradient system.
(b)(iii) \( V(x,y) \) and both its first order partial
derivatives are continuous.
\[ V(x, y) = x^2 + y^2 > 0 \text{ for all neighbourhoods of } (0, 0). \]

\[ V = \frac{\partial V}{\partial x} \frac{dx}{dt} + \frac{\partial V}{\partial y} \frac{dy}{dt} \]

\[ = 2x(y - x^5) + 2y(-x - y^5). \]

\[ = 2xy - 2x^6 - 2xy^2 - 2y^6. \]

\[ = -2x^6 - 2y^6 < 0 \text{ in every neighbourhood of } (0, 0). \]

So \( V(x,y) \) is a strict Liapunov function for
\( (0,0). \)

(c). At equilibrium \( \dot{x} = \dot{y} = 0 \)
so \( y'' = 0 \) and.

\[ x + x^2 - y = 0 \text{ so } x(1 + x) = 0, x = 0, -1. \]

Equilibria are \((0, 0)\) and \((-1, 0)\).

\[ J = \begin{pmatrix} 0 & 1 \\ 1 + 2x & -1 \end{pmatrix} \]
At $(0,0)$

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

\[ \text{Det } J = -1 < 0 \text{ so } (0,0) \text{ is a saddle.} \]

At $(0, -1)$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

\[ \text{Det } J = 1 > 0 \]

\[ \text{Tr } J = -1. < 0 \]

\[ (\text{Tr } J)^2 - 4 \text{Det } J = 1 - 4 = -3 < 0 \]

So $(0, -1)$ is a stable focus.
3(a). A limit cycle is an isolated periodic solution. By "isolated" we mean that there exists a region around the solution that contains no other periodic solution.

(b). (i). At equilibrium

\[ y - \left( \frac{x^3}{3} - x \right) = 0 \]
\[ -x = 0 \]

So steady state is \( x = 0, y = 0 \).

\[ J = \begin{pmatrix} -x^2 + 1 & 1 \\ -1 & 0 \end{pmatrix} \]

\[ J(0,0) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \]

\[ \text{Det} J = 1 > 0 \] and \( (0, 0) \) is a source.
\[ \text{Tr} J = 1 > 0 \] (unstable node or focus).

(ii) Nullclines \( x = 0 \), \( y = \frac{x^3}{3} - x \).
\[ \dot{y} = 0 \quad x = 0. \]

Flow \( x \geq 0 \) if \( y \geq \frac{x^3}{3} - x \),
\( \dot{y} \geq 0 \) if \( x \leq 0 \).
(c). The Bendixson-Dulac Theorem is not part of this course in 2009.
4. (a).

\[
\frac{dx}{dt} = r x (1 - \frac{x}{K}) + ax y
\]

When \( y = 0 \), then the popn of x grows logistically.

When \( xy \neq 0 \) then the growth of x increases with y, at a rate proportional to the number of x and the number of y.

\[
\frac{dy}{dt} = s y (1 - \frac{y}{L}) + b x y
\]

When \( x = 0 \) then y grows logistically.

When \( x \neq 0 \) then the growth of y increases. The increased (additional) growth rate is proportional to \( x \) and proportional to \( y \).

(b). At equilibrium

\[
X((1-x) + a y) = 0
\]

\[
X y ((-y) + b x) = 0
\]

so \( x \neq 0 \) and \( y \neq 0 \):

\[
(1-x) + a y = 0 \quad y = \frac{1}{a}(1-x)
\]

\[
(1-y) + b x = 0
\]
4(c) chd. Sub in for $Y$ into second eqn.

\[
(1 - \frac{1}{\alpha}(X-1)) + \beta X = 0.
\]

\[
(1 + \frac{1}{\alpha}) + (\beta - \frac{1}{\alpha})X = 0.
\]

So, provided $\frac{1}{\alpha} > \beta$, \( X = \frac{(1 + \frac{1}{\alpha})}{\alpha - \beta} \)

\[
Y = \frac{1}{\alpha} \left( \frac{1 + \frac{1}{\alpha}}{1/\alpha - \beta} - 1 \right)
\]

\[
= \frac{1}{\alpha} \left( \frac{1 + 1/\alpha - (\alpha - \beta)}{1/\alpha - \beta} \right)
\]

\[
= \frac{1}{\alpha} \left( \frac{1 + \beta}{1/\alpha - \beta} \right), \text{ again, provided } \frac{1}{\alpha} > \beta.
\]

\[\text{or } \alpha < 1.\]

\[c). \quad J = \begin{pmatrix}
(1-\alpha X + \alpha Y) & -X & \alpha X \\
\lambda \beta Y & \lambda (1-\gamma + \beta X - Y)
\end{pmatrix}
\]

\[J(0,0) = \begin{pmatrix}
1 & 0 \\
0 & \alpha
\end{pmatrix}
\]

Characteristic eqn in $\mu$.

\[(1-\mu)(\alpha - \mu) = 0 \quad \text{so } \mu = 1, \lambda,
\]

both +ve.

So $(0,0)$ is an unstable node.
\[ J(1, 0) = \begin{pmatrix} -1 & 1 \\ 0 & +\beta \lambda \end{pmatrix} \]

Characteristic eqn \( r^2 + \mu = 0 \)

\[ (-1 - \mu)(+\beta \lambda - \mu) = 0 \text{ so } \mu = -1, +\beta \lambda \]

So \((1, 0)\) is a saddle.

\[ J(0, 1) = \begin{pmatrix} +\alpha & 0 \\ \lambda \beta & -\gamma \end{pmatrix} \]

Characteristic eqn \( r^2 + \mu = 0 \)

\[ (-\alpha - \mu)(-\lambda - \mu) = 0 \text{ so } \mu = +\alpha, -\lambda. \]

So \((0, 1)\) is a saddle.

\[ J\left( \frac{1 + \frac{\gamma}{\alpha - \beta}}{\lambda \beta}, \frac{\beta(1 + \beta)}{\gamma \alpha - \beta} \right) = \begin{pmatrix} -\left(1 + \frac{\gamma}{\alpha - \beta}\right) & \frac{1 + \alpha}{\lambda \beta - \beta} \\ \frac{\lambda \beta (1 + \beta)}{\gamma \alpha - \beta} & -\frac{\beta (1 + \beta)}{\gamma \alpha - \beta} \end{pmatrix} \]

or \(J(X^*, Y^*) = \begin{pmatrix} -X^* & \alpha X^* \\ \lambda \beta Y^* & -\gamma Y^* \end{pmatrix} \)
\[ \text{Det } J = \lambda x^p y^q - \lambda \alpha \beta x^p y^q \]
\[ = \lambda x^p y^q (1 - \alpha \beta) > 0 \text{ if } \alpha \beta < 1. \]
\[ T \text{r } J = -x^p - \lambda y^q < 0. \]
So \((x^*, y^*)\) is either a stable node or stable focus.

\[ (T \text{r } J)^2 - 4 \text{Det } J = (-x^* - \lambda y^*)^2 - 4 \lambda x^p y^q (1 - \alpha \beta) \]
\[ = x^{*2} + 2 \lambda x^p y^q + \lambda^2 y^{*2} - 4 \lambda x^p y^q \]
\[ + 4 \alpha \lambda \beta x^p y^q \]
\[ = x^{*2} - 2 \lambda y^{*2} + \lambda^2 y^{*2} + 4 \alpha \lambda \beta x^p y^q \]
\[ = (x^* - \lambda y^*)^2 + 4 \alpha \beta \lambda x^p y^q \]
\[ > 0. \]

So \((x^*, y^*)\) is a stable node.

(i). Nullclines \( \dot{x} = 0 \) \( x = 0 \) or \( y = \frac{1}{\alpha} (x - 1) \)
\[ \dot{y} = 0 \] \( y = 0 \) or \( y = 1 + \beta x. \)

Flow \( \dot{x} \geq 0 \) if \( y \geq \frac{1}{\alpha} (x - 1) \)
\[ \dot{y} \geq 0 \] if \( y \leq 1 + \beta x \)
(d) If the effect of mutualism is smaller \( (\alpha \beta < 1) \) then the pops go to a steady state \( (x^*, y^*) \) where \( x^* > 1 \) and \( y^* > 1 \) so at a higher value than each popn would achieve an its own. If the effect of mutualism is strong \( (\alpha \beta > 1) \) then both pops go to infinity as \( t \to \infty \).