Introduction

representation = homomorphism
—from an “abstract” group $G$ (that we want to study)
to a “concrete” group (that we think we understand).

permutation representation = homomorphism $G \rightarrow \text{Sym}(n)$
(as in Galois Theory, where groups of field automorphisms are represented by permutations of the roots of polynomials)

linear representation = homomorphism $G \rightarrow GL(V)$
(where elements of $G$ are represented by invertible linear transformations on the vector space $V$).

“Representation” (unqualified) means “linear representation”.

matrix representation = homomorphism $G \rightarrow GL_n(F)$, where
Definition: $GL_n(F)$ is the group of all $n \times n$ invertible matrices over the field $F$.

Some terminology

Definition: If $G$ is a group and $F$ a field then a matrix representation of $G$ over $F$ of degree $n$ is a group homomorphism $M: G \rightarrow GL_n(F)$.

In other words it is a way of assigning to each $g \in G$ an invertible $n \times n$ matrix $M(g)$ subject to the requirement that $M(gh) = M(g)M(h)$ for all $g, h \in G$.

This automatically implies that $M(1) = 1$ since group homomorphisms always take identity elements to identity elements.

(n.b. the equation above means $M(1_G) = 1_{GL(V)}$.)

And $M(g^{-1}) = M(g)^{-1}$ (also a general property of group homomorphisms).

The matrices $M(g)$ are called the representing matrices.

Matrices and linear transformations

are just about the same thing.

$GL_n(F)$ is just about the same as $GL(F^n)$.

For every linear map $\phi: F^n \rightarrow F^n$ there is a matrix $M$ such that $\phi(\mathbf{v}) = M\mathbf{v}$ for all $\mathbf{v} \in F^n$.

And vice versa.

$\phi(\mathbf{v}) = M\mathbf{v}$ means that $M$ determines $\phi$.

$\phi$ determines $M$ since $j$th column of $M$ is $M\mathbf{e}_j = \phi(\mathbf{e}_j)$.

So we have a 1-1 correspondence:

$\{\text{linear maps } F^n \rightarrow F^n\} \leftrightarrow \{n \times n \text{ matrices}\}$.

This preserves addition and multiplication: it’s a ring isomorphism. So invertible maps $\leftrightarrow$ invertible matrices.

So we have a multiplication-preserving bijection

$\gamma: GL(F^n) \rightarrow GL_n(F)$. (That is, $\gamma$ is an isomorphism.)
Bases in vector spaces

Every finitely generated vector space $V$ over $F$ has a basis. A basis $(v_1, v_2, \ldots, v_n)$ yields an isomorphism $\psi: F^n \rightarrow V$

$$\psi: \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} \mapsto \sum_{i=1}^n \lambda_i v_i.$$ 

The isomorphism $\psi: F^n \rightarrow V$ gives an isomorphism $\tilde{\psi}: GL(F^n) \rightarrow GL(V)$:

$$\tilde{\psi}: \tau \mapsto \psi \tau \psi^{-1}$$

(If $v \in V$ then $\psi^{-1}v \in F^n$, and $(\psi \tau \psi^{-1})v \in F^n$, and $(\psi \tau \psi^{-1})v \in V$.)

The composite $GL(V) \xrightarrow{\psi} GL(F^n) \xrightarrow{\tau} GL_n(F)$ takes $\tau \in GL(V)$ to $[\tau]$, its matrix relative to the given basis.

Being explicit about it

Using matrices relative to the standard basis gives

$1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $x \mapsto \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$, $x^2 \mapsto \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$

$y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $xy \mapsto \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$, $x^2y \mapsto \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

where $\theta = 2\pi/3$.

Alternatively, use the basis $\{v_1, v_2\}$ shown here:

Then $x \mapsto \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, $y \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$x^2 \mapsto \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, $xy \mapsto \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$, $x^2y \mapsto \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$.

We denote these two matrix representations of $D_3$ by $M$ and $M'$.

An example

Let $D_3 = \langle x, y \mid x^3 = 1, y^2 = 1, xyx = x^{-1} \rangle$, the abstract dihedral group of order 6. Its elements are $1, x, x^2, y, xy, x^2y$.

(Note that you can use $yx = x^{-1}y$ to "move a $y$ past an $x$".

e.g. $y(x^2y) = (yx)y = (x^{-1}y)xy = x^{-1}(y(xy))y = x$.

In this way you can show that the set of elements listed above is closed under multiplication and inversion — and hence they are all the elements of $D_3$, given that $x$ and $y$ generate $D_3$.)

There is a linear representation of $D_3$ given by

$x \mapsto (\text{rotation through } 2\pi/3 \text{ about } (0, 0))$, $y \mapsto (\text{reflection in the } X\text{-axis}).$

We get a matrix representation by

$x \mapsto (\text{matrix of the rotation through } 2\pi/3)$, $y \mapsto (\text{matrix of the reflection in the } X\text{-axis}).$

Equivalence for matrix representations

Let $B = (v_1, \ldots, v_n)$ and $B' = (v'_1, \ldots, v'_n)$ be two ordered bases of $V$. There is a change-of-basis matrix $A$ such that for all linear maps $\tau: V \rightarrow V$, if $P$ is the matrix of $\tau$ relative to $B$ then the matrix of $\tau$ relative to $B'$ is $APA^{-1}$.

The representations $M$, $M'$ on the previous page are related by

$$M'(g) = AM(g)A^{-1} \quad \text{for all } g \in D_3$$

where $A$ is a change-of-basis matrix.

We make the following general definition.

**Definition:** Matrix representations $M$, $M'$ of a group $G$ are said to be equivalent if there is an invertible matrix $A$ such that $M'(g) = AM(g)A^{-1}$ for all $g \in G$.

(Equivalence of linear representations was defined similarly.)
Equivalence classes of representations

Let $V$ be a vector space over the field $F$.

If $\dim V = n$ then a linear representation $T : G \to GL(V)$ determines a matrix representation $[T] : G \to GL_n(F)$ as soon as a basis of $V$ is chosen.

Changing the basis will usually change $[T]$, but only to an equivalent matrix representation.

Moreover, equivalent linear representations also give equivalent matrix representations.

Proposition: If $\dim V = n$ then there is a bijection between the set of equivalence classes of linear representations of $G$ on $V$ and the set of equivalence classes of matrix representations of $G$ of degree $n$ over $F$, induced by $T \mapsto [T]$ for all $T$, where $[T]$ is calculated using any basis.

(Check that equivalence of representations is an equivalence relation!)

Proof

Given a representation $T$, let $\mathcal{T}$ be the equivalence class containing $T$.

Fix a basis of $V$. Then $T \mapsto [T]$ maps linear representations to matrix representations. We must first check that $T \mapsto [T]$ is well-defined: we must show that if $T = \mathcal{T}$ then $[T] = [\mathcal{T}]$.

If $T = \mathcal{U}$ there exists $\sigma \in GL(V)$ such that $U(g) = \sigma T(g) \sigma^{-1}$ for all $g \in G$. Taking matrices gives $[U(g)] = [\sigma][T(g)][\sigma]^{-1}$, which shows that $[T] = [\mathcal{U}]$. So the map is well-defined.

Injectivity: Suppose $[T] = [\mathcal{U}]$. There is a matrix $A$ such that $[U(g)] = [A][T(g)][A]^{-1}$ for all $g \in G$. There is $\sigma \in GL(V)$ with $A = [\sigma]$. This gives $U(g) = \sigma T(g) \sigma^{-1}$ for all $g \in G$.

So $\mathcal{T} = \mathcal{U}$ whenever $[T] = [\mathcal{U}]$.

Surjectivity: Let $\mathcal{C}$ be an equivalence class of matrix representations. Choose $M \in \mathcal{C}$. For each $g \in G$ define $T(g)$ to be the linear map such that $[T(g)] = M(g)$.

Then $T$ is a linear representation with $T \mapsto \mathcal{M} = \mathcal{C}$.

Our classification problem

Note that the bijection defined in the proposition doesn’t depend on the basis of $V$ that you choose, because $[T]_{\mathcal{B}}$ is always equivalent to $[T]_{\mathcal{B}}$.

Given a groups $G$ and a vector space $F$, we would like to be able to solve the following problem:

**classify the linear representations of $G$ on $V$ up to equivalence.**

The proposition tells us that the following matrix version is essentially the same:

**classify degree $n$ matrix representations of $G$ up to equivalence.**

Matrices are naturally identifiable with linear maps $F^n \to F^n$.

Any $n$-dimensional $V$ is isomorphic to $F^n$; hence we can convert a representation of $G$ on $V$ to a matrix representation.

But equally we can convert a representation on $V$ to a representation on any $V'$ of the same dimension.

Equivalence generalized

**Definition:** Linear representations $T : G \to GL(V)$ and $T' : G \to GL(V')$ are *equivalent* if there is a vector space isomorphism $\sigma : V \to V'$ such that $\sigma T(g) \sigma^{-1} = T'(g)$ for all $g \in G$.

The condition can also be stated as follows: $\sigma T(g) = T'(g) \sigma$ for all $g \in G$.

In turn, this is the same as saying that the following diagram commutes:

$$
\begin{array}{ccc}
V & \xrightarrow{\sigma} & V' \\
T(g) \downarrow & & \downarrow \sigma \\
V' & \xrightarrow{\sigma} & V'
\end{array}
$$

Note that all the maps are vector space isomorphisms.
More comments on equivalence

If we are given a representation $T: G \to \text{GL}(V)$ and a vector space isomorphism $\sigma: V \to V'$ then we can define $T': G \to \text{GL}(V')$ by $T'(g) = \sigma T(g) \sigma^{-1}$.

This clearly gives a representation on $V'$, since

$$T'(g) T'(h) = \sigma T(g) \sigma^{-1} \sigma T(h) \sigma^{-1} = \sigma T(g) T(h) \sigma^{-1} = \sigma T(gh) \sigma^{-1} = T'(gh),$$

and of course $T'$ is equivalent to $T$.

We also have

**Proposition:** Let $T: G \to \text{GL}(V)$ and $T': G \to \text{GL}(V')$ be representations. Then $T$, $T'$ are equivalent iff we can choose bases so that $[T] = [T']$.

Proof of this

**Proof:** Let $\sigma: V \to V'$ be an isomorphism with $\sigma T(g) \sigma^{-1} = T'(g)$.

Let $B = (v_1, v_2, \ldots, v_n)$ be any basis of $V$, and put $B' = (\sigma v_1, \sigma v_2, \ldots, \sigma v_n)$.

Then $B'$ is a basis of $V'$ such that the matrix of $\sigma$ relative to $B'$ and $B$ is the identity.

This means that $[T'(g)] = [\sigma T(g) \sigma^{-1}] = [\sigma][T(g)][\sigma]^{-1} = T(g)$ for all $g$, as required.

Conversely, suppose there exist bases $B = (v_1, \ldots, v_n)$ and $B' = (v'_1, \ldots, v'_n)$ such that $[T'(g)] = [T(g)]$ for all $g \in G$.

Define $\sigma: V \to V'$ by $\sigma v_i = v'_i$ for all $i$.

Then $[\sigma] = 1$, whence $T'(g) = [\sigma][T(g)][\sigma]^{-1}$, and $T'(g) = \sigma T(g) \sigma^{-1}$ for all $g$. 