Chapter 7: Portfolio Theory

Introduction 1

1. A financial *portfolio* is an investment consisting of any collection of securities or assets eg stocks and bonds.
2. We will be studying the so called Mean-Variance Portfolio Theory (MVPT).

Introduction 2

The assumptions of the MVPT are:

1. The market provides perfect information and perfect competition
2. Asset returns refer to a single fixed time-period
3. Asset returns are normally distributed with known means and covariance matrix
4. Investors act rationally and are risk averse
5. Investors base their decisions solely on the means and covariances of asset returns

Introduction 3

1. By asset return, R, we mean
$$R = \frac{X_1 - X_0}{X_0},$$
where X_0, X_1 are the asset prices at the start and end of the period.
2. Since X_1 is random (but X_0 is known) R is a random variable
3. There is no universally accepted definition of risk for a portfolio. We use the standard deviation of its return.
4. One important feature of a portfolio is that a smart choice of assets within the portfolio can lead to reduced risk; even lower than that of the individual assets in the portfolio.

Introduction 4

Example 1.1

Portfolio Basics 1

1. Let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be n risky securities with random returns $R_1, \ldots, R_n$.
2. Their expected returns and covariances are assumed known.
3. Let $r_i = \mathbb{E}(R_i)$, expected return of $\mathcal{S}_i$
4. Recall that $\mathrm{cov}(R_i, R_j) = \mathbb{E}(R_i R_j) - r_i r_j$

Portfolio Basics 2

The covariance matrix also contains the correlation matrix with components ρ_{ij} through the relation $s_{ij} = \rho_{ij}\sigma_i\sigma_j$.

Therefore if we define the diagonal matrix D by $D^2 = \mathrm{diag}(S)$ where S is the covariance matrix, then the correlation matrix $P = \rho_{ij}$ satisfies $S = DPD$ or equivalently $P = D^{-1}SD^{-1}$.

Portfolio Basics 3

1. A *portfolio* is an investment containing a selection of the securities $\mathcal{S}_i$ in some proportion.
2. W_0 is the amount of total wealth that is initally available for investment.
3. x_i is the proportion of W_0 invested in $\mathcal{S}_i$
4. All of W_0 must be invested in $\mathcal{P}$ (a.k.a. the budget constraint)
5. This budget constraint defines the decision domain $\mathcal{D}$ for portfolio selection as a hyperplane in an n-dimensional vector space:

$$\mathcal{D} = \{x_i | \sum_{i=1}^{n} x_i = 1\}$$

Portfolio Basics 4

1. Another important constraint is the *no short selling constraint*:

$$x_i \geq 0, \qquad \text{if no short selling of asset } S_i$$

2. If short-selling is allowed, then the corresponding x_i may be negative and also have absolute value exceeding 1. The budget constraint is not violated. Basically, short-selling raises extra capital to invest in other assets in the portfolio.

Portfolio Basics 5

Short selling is the practice of selling assets, usually securities, that have been borrowed from a third party (usually a broker) with the intention of buying identical assets back at a later date to return to the lender.

In our setup the short-sold item will have a negative value since you are receiving cash rather than paying out (positive)

Portfolio Basics 6

1. The return on the entire portfolio is

$$R = \sum_{i=1}^{n} x_i r_i,$$

   and the mean and variance of the portfolio return is

$$\mu = \mathbb{E}(R) = \sum_{i=1}^{n} x_i r_i = \underline{x}' \underline{r}$$

$$\sigma^2 = \mathbb{V}(R) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j s_{ij} = \underline{x}' S \underline{x}$$

2. The previous two equations are called the *Portfolio Equations*

The Feasible Set 1

1. Definition: The *feasible set* $\mathcal{F}$ is the set of $x_i \in \mathcal{D}$ which also satisfy all other contraints.
2. If there are no such constraints (other than the budget condition) then the portfolio is termed *unrestricted*.
3. If we forbid short short-selling of asset i then the constraint is

   $$0 \leq x_i < \infty$$

   With $x_i > 1$ we will need a negative x_j to ensure that the budget constraint is satisfied.
4. If we forbid short selling of all assets then for all i we have

   $$0 \leq x_i \leq 1$$

The Feasible Set 2

1. Another common constraint is the *sector constraint*
2. Let $\underline{b}$ be a vector of maximum asset allocations within each sector. Then a sector constraint has the (linear) form

$$A \leq \underline{b}; \quad A_{ij} = \begin{cases} 1, & \text{if asset } j \text{ belongs to sector } i \\ 0, & \text{otherwise} \end{cases}$$

Example

The Feasible Set 2

1. The feasible set $\mathcal{F}$ for a given portfolio problem, when mapped to the $\mu\sigma$ plane may turn out to be empty, a single point, a curve, a closed region or an open region

Portfolio Selection Rules 1

We now assume that the portfolio variance σ^2 measures its risk level. The following two portfolio selection rules apply to all rational risk-averse investors.

1. If $\mu_1 = \mu_2$, then select the portfolio with smaller σ (ie lower risk)
2. If $\sigma_1 = \sigma_2$ then select the portfolio with larger μ (i.e. with higher return)

Combining the two rules we get *North-West Rule*: Given two feasible portfolios $\mathcal{P}_1 = (\mu_1, \sigma_1)$ and $\mathcal{P}_2 = (\mu_2, \sigma_2)$ in the $\mu\sigma$-plane, select the one lying in any region North West of the other.

Portfolio Selection Rules 2

North-West Rule ct'd

1. Portfolios P which are NW of portfolios Q have both higher return and lower risk. Any rational risk-averse investor would prefer P to Q.
2. If neither portfolio lies NW of the other, then the NW rule makes no preferential selection.

The Efficient Frontier 1

[Blackboard: Batwing with points A,B,C]

1. Point A is the max return, max risk portfolio; point B is the min risk portfolio; and point C is the min return portfolio
2. Points on curve ABC represent min risk portfolios for any given feasible return. curve ABC is hence called the minimum variance frontier or MVF.

The Efficient Frontier 2

Each point on the upper branch AB of the MVF has the special property that no other point of the feasible set lies NW of it. Ie. points on AB are optimal portfolios in the sense that no other can have a higher return and a lower risk. The curve AB, the upper branch of the MVF, is called the *efficient frontier* or EF for short.

The Efficient Frontier 3

1. Proposition: All rational risk-averse investors will select portfolios on the efficient frontier.
2. Exactly which portfolio on the EF is chosen depends on the investor's risk-return preferences. The risk averse investor will choose B to ensure the least possible risk regardless of return. The risk-lover chooses A and obtains the highest possible return regardless of risk.

The Efficient Frontier 4

Markowitz Criterion: All rational risk averse investors have risk-return preferences derived from a utility function $U = U(\mu, \sigma)$, which depends only on the portfolio return μ and portfolio variance σ^2 This is called the mean-variance criterion. It is a model of investor behaviour.

Indifference Curves 1

1. According to the Markowitz criterion, the degree of risk required for a given level of return depends on the investor's utility function $U(\mu, \sigma)$

Indifference Curves 2

In the following, we use a utility function of the form

$$U(\mu, \sigma) = F(t\mu - \sigma^2/2)$$

where

1. t is a non-negative parameter
2. F is a concave, monotonic, increasing function of its argument

NOTE: this is just one choice among many

Indifference Curves 3

Let the negative of the argument of U:

$$Z(\mu, \sigma) = -t\mu + \sigma^2/2$$

denote the investor's *objective function*. It will be useful that Z can be expressed as

$$Z(\mu, \sigma) = -t(\underline{r}'\underline{x}) + \frac{1}{2}\underline{x}' S \underline{x}.$$

Indifference Curves 4

1. Note that U is *maximized* with respect to x_is, when Z is minimized with respect to x_is
2. Therefore the basic portfolio problem is to find the x_i which minimizes the investor's objective function $Z(\underline{x})$

Indifference Curves 5

1. The parameter t in Z (for $t \geq 0$) is a *risk aversion parameter*.
2. When $t = 0$, the problem reduces to $\min Z = \sigma^2/2$. The interpretation is that the investor is interested only in minimizing risk without regard to portfolio return.
3. When $t \to \infty$ the problem becomes $\max Z = \mu$. The interpretation is that the investor wishes to maximize portfolio return regardless of risk,
4. Other investors have risk aversion parameters in $(0, \infty)$.

[Blackboard: Sketch batwing + $t = 0$ vertical lines and $t = \infty$ horizontal lines]

Indifference Curves 6

[Blackboard Diagrams]

1. Curves of constant Z in the $\mu\sigma$-plane are called *indifference curves*, because an investor has no preference for any portfolio lying on a given indifference curve.
2. There are two degenerate cases: $t = 0$ the indifference curves are vertical lines; and $t \to \infty$ then indifference curves become horizontal lines.
3. When $t \in (0, \infty)$ the optimum portfolio P^* occurs on the efficient frontier where it is tangential to the indifference curve. Since the EF is concave and the indifference curveis convex, there will always be a unique optimum portfolio P^*

Indifference Curves 7

With Z constant, const$= -t\mu + \sigma^2/2$, so

$$\mu = \frac{1}{t}(\sigma^2/2 - const),$$

So the curves $Z(\mu, \sigma) = Z_0$ are a family of parallel convex parabolas.

[Blackboard: Sketch batwing +indifference curves (p93)]

[Blackboard: Example 1.3]

The Two-Asset Portfolio 1

1. With 2 assets we can set $x_1 = x$ and $x_2 = 1 - x$ and then the budget constraint $x_1 + x_2 = 1$ is automatically satisfied.
2. Portfolio return and variance are therefore functions of the single variable x and are given by:

$$\mu = r_1 x + r_2(1-x) \quad \sigma^2 = \sigma_1^2 x^2 + \sigma_2^2 (1-x)^2 + 2\rho\sigma_1\sigma_2 x(1-x),$$

where $\rho = \mathrm{corr}(R_1, R_2)$.

1. One can write the equation in the form

$$A\mu^2 + B\mu\sigma + C\sigma^2 + D\mu + E\sigma + F = 0$$

   and check that it satisfies

$$\det \begin{pmatrix} A & B \\ B & C \end{pmatrix} < 0$$

2. Therefore the pair of equations of the previous slides are the equations of a hyperbola.
3. If there are are no other constraints this hyperbola is both the feasible set $\mathcal{F}$ and the Minimum Variance Frontier (MVF).

The Two-Asset Portfolio 2

1. Before finding its explicit equation, we consider three degenerate cases.

The case $\rho = 1$ When the asset returns R_1 and R_2 are perfectly correlated ($\rho = 1$), the portfolio equations previous reduces to [Blackboard:working]

The Two-Asset Portfolio 3

1. The feasible set consists of all points (portfolios) which lie on this pair of lines
2. P_1 is the portfolio ($x = 1$) in which all wealth is invested in asset $\mathcal{S}_1$ and none in set $\mathcal{S}_2$.
3. P_2 corresponds to the case in which all wealth is invested in $\mathcal{S}_2$.
4. The segment joining P_1 and P_2 corresponds to $0 \leq x \leq 1$, namely no short selling of either asset.
5. The upper and lower dashed segments correspond respectively to the cases $x > 1$ $x > 1$ ($\mathcal{S}_2$ sold short) and $x < 0$ ($\mathcal{S}_1$)

The Two-Asset Portfolio 4

In this case ($\rho = 1$), it is possible to obtain a portfolio with zero risk, represented by the point P_0:

$$x_0 = \frac{-\sigma_2}{\sigma_1 - \sigma_2}; \quad \mu_0 = \frac{\sigma_1 r_2 - \sigma_2 r_1}{\sigma_0 = 0}$$

[Blackboard: Working]

The Two-Asset Portfolio 6

The General Two Asset Case

When none of the preceeding 3 cases happens, the feasible set is the hyperbola described parametrically by

$$\mu = r_1 x + r_2(1-x)$$

$$\sigma^2 = \sigma_1^2 x^2 + \sigma_2^2 (1-x)^2 + 2\rho\sigma_1\sigma_2 x(1-x)$$

[Blackboard: Working]

Unrestricted n-Asset Portfolios 1

We consider the optimization problem:

$$\text{Minimize} \quad Z(\underline{x}) = -t(\underline{r}'\underline{x}) + \frac{1}{2}\underline{x}'S\underline{x} \quad \text{s.t.} \;\; \underline{x}'\underline{e} = 1$$

where $\underline{e} = (1, 1, \ldots, 1)'$ so that $\underline{x}'\underline{e}$ is equivalent to the budget constraint $\sum_i x_i = 1$.

We want a parametric solution of the form $\underline{x} = \underline{x}(t)$ for all allowed $t \geq 0$. This solution can then be mapped to the $\mu\sigma$-plane through the portfolio equations

$$\mu = \underline{x}'\underline{r} = \mu(t) \quad \text{and} \quad \sigma^2 = \underline{x}'S\underline{x} = \sigma^2(t)$$

Hence we want to find $\underline{x}(t), \mu(t), \sigma^2(t)$ for $t \geq 0$

Unrestricted n-Asset Portfolios 2

Solution using Lagrange Parameters The method states that the constrained optimization of Z is equivalent to the unconstrained optimization of $\mathcal{L}$ defined by

$$\underline{\mathcal{L}} = Z((\underline{x}) - \lambda_1 G_1((\underline{x}))(+\lambda_2 G_2(x) + \ldots)$$

where the λs are called Lagrange parameters and $\mathcal{L}$ is called the Lagrangian function.

Here the optimal solution satisfies the functional equations:

$$\frac{\partial \mathcal{L}}{\partial \underline{x}} = 0; \quad G_1((\underline{x})) = 0; \ldots$$

Unrestricted n-Asset Portfolios 3

In the current problem we have

$$\underline{\mathcal{L}} = -t(\underline{x}'\underline{r}) + \frac{1}{2}x'Sx - \lambda(\underline{x}'\underline{e} - 1)$$

and the optimal solution satisfies the $(n+1)$-linear equations:

$$S\underline{x} = t\underline{r} + \lambda\underline{e}$$

$$\underline{x}'\underline{e} = 1$$

Unrestricted n-Asset Portfolios 4

1. Provided that S is non-singular we have

$$\underline{x} = S^{-1}(t\underline{r} + \lambda\underline{e})$$

2. Since $\underline{x}'\underline{e} = \underline{e}'\underline{x} = 1$

$$\underline{e}'\underline{x} = \underline{e}'S^{-1}(t\underline{r} + \lambda\underline{e}) = 1,$$

equivalently,

$$\underline{e}'\underline{x} = t\underline{e}'S^{-1}\underline{r} + \lambda\underline{e}'S^{-1}\underline{e}$$

or

$$a\lambda + bt = 1$$

where

$$a = \underline{e}'S^{-1}\underline{e} \text{ and } b = \underline{e}'S^{-1}\underline{r}$$

Unrestricted n-Asset Portfolios 5

We get that $\lambda(t) = \dfrac{1 - bt}{a}$

1. Introduce also

$$c = \underline{r}' S^{-1} \underline{r} \text{ and } d = ac - b^2$$

2. We have that a, c and d are positive.

The Critical Line 1

Since λ (from previous section) is linear in the parameter t, so is $\underline{x}$.

Hence

$$\underline{x}(t) = \underline{\alpha} + \underline{\beta} t,$$

where

$$\underline{\alpha} = \frac{1}{a} S^{-1} \underline{e}; \quad \text{and} \quad \underline{\beta} = S^{-1}(\underline{r} - \frac{b}{a}\underline{e})$$

This is the equation of a straight line in n-dimensional x-space.

This is termed the *critical line*. Note that the budget constraint

$\underline{e}'\underline{x} = 1$ implies

$$\underline{e}'\underline{\alpha} = \sum_i \alpha_i \quad \text{and} \quad \underline{e}'\underline{\beta} = \sum_i \beta_i = 0$$

The Critical Line 2

The Efficient Frontier The image of the critical line when it is mapped to the $\mu\sigma$-plane defines the EF for this portfolio problem. We get that

$$\mu = \frac{b + dt}{a} \quad \text{and} \quad \sigma^2 = \frac{1 + d^2}{a}$$

BLACKBOARD: Do calculations

The Critical Line 3

Eliminating the risk aversion parameter t from this pair of equations now gives

$$t = \frac{a}{d}(\mu - b/a) \quad \text{and } \sigma^2 = \frac{1}{a} + \frac{a}{d}(\mu - \frac{b}{a})^2$$

From which the MVF is given by either of the equations

$$\mu = \frac{b + dt}{a} \text{ and } \sigma^2 = \frac{1 + dt^2}{a}$$

or

$$\sigma^2 - \frac{a}{d}(\mu - \frac{b}{a})^2 = \frac{1}{a}$$

BLACKBOARD: Do calculations

The Critical Line 4

THEOREM: In the $\mu\sigma$-plane:

1. The minimum variance frontier is an hyperbola and is the image of the Critical Line CL
2. The EF is the upper branch of this hyperbola ($\mu > b/a$) and is paramterised by the C.L. with $t \geq 0$.
3. The minimum risk portfolio is the vertex????? of this hyperbola corresponding to $t = 0$, $\mu_0 = b/a$, $\sigma_0^2 = 1/a$ and $\underline{x}_0 = \underline{\alpha}$.
4. The lower branch of the hyperbola ($\mu < b/a$, the inefficient frontier) corresponds to all values of the C.L. with $t < 0$
5. Provided that $n \geq 3$, the feasible set is the interior of the

Two-Fund Theorem 1

Two-Fund Theorem Let $\mathcal{P}$ and $\mathcal{Q}$ be any two distinct efficient portfolios or funds.

Then the two-fund theorem states

All investors seeking efficient portfolios need only invest in combinations of funds $\mathcal{P}$ and $\mathcal{Q}$.

The funds are assumed to have no constraints other than the budget condition.

Two-Fund Theorem 2

Proof of the Two-Fund Theorem

Let $\underline{x}^P$ and $\underline{x}^Q$ be the allocation vectors for $\mathcal{P}$ and $\mathcal{Q}$ respectively.

Then since these are efficient portfolios there exists vectors $\underline{\alpha}$ and $\underline{\beta}$ and non-negative parameters t^P and t^Q such that

$$\underline{x}^P = \underline{\alpha} + t^P \underline{\beta} \quad \text{and} \quad \underline{x}^Q = \underline{\alpha} + t^Q \underline{\beta}$$

Now consider a new portfolio containing a proportion $(1 - \theta)$ of $\mathcal{P}$ and a proportion θ of $\mathcal{Q}$.

Two-Fund Theorem 3

This portfolio will have allocation vector

$$\begin{aligned}\underline{x} &= (1-\theta)\underline{x}^P + \theta \underline{x}^Q \\ &= \underline{\alpha} + t\underline{\beta}\end{aligned}$$

if $t = (1-\theta)t^P + \theta t^Q$.

Hence the combined portfolio is efficient provided we choose θ such that the effective risk aversion parameter $t \geq 0$.

BLACKBOARD: Implications of 2 fund theorem

Restricted n-Asset Portfolios 1

We will now look at what happens when we include additional constraints of the form

$$L_i \leq x_i \leq U_i, \text{ for each} i = 1, 2, \ldots, n$$

The U_i may be ∞ (so no upper constraint), or L_i may be $-\infty$ (so no lower constraint).

Restricted n-Asset Portfolios 2

The efficient frontier (EF) is then obtained by solving the constrained optimisation problem:

$$\begin{aligned} \text{Minimise} \quad & Z = -t(\underline{x}'\underline{r}) + \frac{1}{2}\underline{x}'S\underline{x} \\ \text{s.t.} \quad & \underline{x}'\underline{e} = 1 \\ \text{and} \quad & \underline{L} \le \underline{x} \le \underline{U} \end{aligned}$$

This problem is to be solved for all $t \geq 0$ for which the solution is feasible. For restricted problems, not all values of t in $(0, \infty)$ may lead to feasible solutions. Therefore part of the problem is to find the allowed values of the parameter t.

Restricted n-Asset Portfolios 3