PHD PROJECTS

NALINI JOSHI

The projects listed below lie in the field of *integrable systems*, which has had a deep impact on mathematics and physics. Integrable systems are non-linear differential or difference equations that arise in a wide variety of fields, including ion channel transport in biology, random matrix theory, string theory, plasma physics, optics, laser physics and fluid dynamics. These systems may be partial differential equations, ordinary differential equations or their discrete versions, which are partial or ordinary difference (or discrete) equations. Although nonlinear, they arise as compatibility conditions for an associated system of linear equations, which give unexpected insights into the solutions of the nonlinear system.

In one dimension, the systems of interest are called the Painlevé or discrete Painlevé equations. Many of their solutions turn out to be universal in the sense that they describe behaviours in large classes of applications. The projects in this area are concerned with new, often very deep mathematical descriptions of such solutions and will involve complex analysis, algebra and geometry.

(i) *Elusive, unique Solutions of Integrable Equations.* The solutions of integrable equations are highly transcendental functions, with beautiful mathematical properties. In the space of all solutions, there are certain solutions that are well behaved over an extended interval on the real line or region in the complex plane. These often turn out to be solutions of great physical interest in applications. However, their definitions rely on boundary conditions in certain singular limits that make them difficult to capture. In this project, we aim to develop new mathematical and asymptotic methods to define and describe such solutions in the complex plane.

(ii) *Reductions of Integrable Equations.* From the theory of Lie groups, we know how to find all possible symmetries of a given partial differential equation and thereby find all its reductions to an ordinary differential equation. But the corresponding theory for partial difference equations contains many gaps. There is a classification of integrable partial difference equations on the one hand and a different classification of integrable ordinary difference equations on the other, which appear to be unrelated to each other. In this project, we aim to fill this gap by studying all possible reductions of integrable partial difference equations (in two or more dimensions). In particular, we will use the theory of reflection groups and tiling patterns to understand and classify the reductions to discrete Painlevé equations.

(iii) *q-difference calculus.* There are three primary types of integrable ordinary difference equations. The additive discrete equations are iterated on lines, multiplicative or q-discrete equations are iterated on spirals, and elliptic-difference equations are iterated on elliptic curves in the complex plane. In this project, we will study methods of solutions for q-difference equations. In particular, we will use the theory of linear q-difference equations to tackle the compatible linear problems associated with integrable discrete equations and use it to extend our knowledge of their solutions.

(iv) *Initial-value spaces.* Integrable non-linear differential or discrete equations have limiting forms that are associated with families of algebraic curves, i.e., curves given by the zero sets of polynomials in two variables. In this setting, the solutions parametrize such curves in initial-value space or phase space. But the time evolution of the solutions becomes ambiguous at places called *base points* where all the curves intersect. By resolving the flow of the system through such points, we find a geometric description of the initial-value space of each integrable system. The project will use this connection to geometry to search for new equations, and to study the behaviours of their solutions.

School of Mathematics and Statistics F07, The University of Sydney, NSW 2006, Australia

E-mail address: nalini.joshi@sydney.edu.au