UNWINDING SPIRALS I

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Abstract. We show that there is no bi-Lipschitz homeomorphism of $\mathbb{R}^2$ that maps a spiral with a sub-exponential decay of winding radii to an unwound arc. This result is sharp as shows an example of a logarithmic spiral.

1. introduction

In this paper we consider a natural question whether it is possible to find a bi-Lipschitz homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$ that will map a given continuous curve C_1 onto another continuous curve C_2. Recall that $h : \mathbb{R}^d \to \mathbb{R}^d$ is called a bi-Lipschitz map with constant $L > 0$, if it satisfies for all $x, y \in \mathbb{R}^d$:

$$\frac{1}{L}\|x - y\| \leq \|h(x) - h(y)\| \leq L\|x - y\|.$$

Let $\phi : [0, \infty) \to (0, 1]$ be a continuous function monotonically decreasing to zero. Then we correspond to ϕ the spiral C_ϕ:

$$C_\phi(t) = \phi(t)e^{it},\ t \in [0, \infty],$$

where $C_\phi(\infty) = \{(0, 0)\}$, see Figure 1.

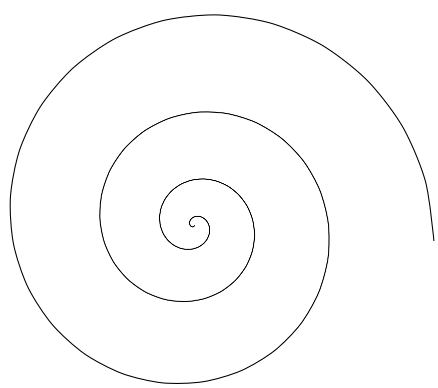

Figure 1. A spiral.

C_ϕ is the *logarithmic spiral* if $\phi(t) = e^{-at}$, for $a > 0$. We will define the notion of an unwound curve around a point in $\mathbb{R}^2$, after introducing the concept of asymptotic directions for a set $A \in \mathbb{R}^d$ around a point $p \in \overline{A}$.

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Definition 1.1 (asymptotic directions at p)**.** The set of *asymptotic directions* in A at p is defined by

$$D_p(A) = \left\{ u \in S^{d-1} \,|\, \exists (a_n) \in A \text{ with } a_n \to p, \text{ and } \frac{a_n - p}{\|a_n - p\|} \to u \right\}.$$

We will denote by $D(A)$ the set of asymptotic directions in A at the origin of $\mathbb{R}^d$.

Definition 1.2 (unwound curve)**.** We will say that a curve $C \subset \mathbb{R}^2$ is *unwound* at p if $D_p(C) \neq S^1$.

In this paper we will address the question of the existence of a bi-Lipschitz homeomorphism of $\mathbb{R}^2$ that maps the spiral C_ϕ of a finite length onto an unwound curve. Spirals and spiral maps are very important object in geometry and have been used extensively to provide counterexamples to many natural conjectures, see for instance [G], [FH]. It is well known fact that it is possible to 'unwind' the logarithmic spiral, see [KNS]:

Theorem 1.1 (unwinding of the logarithmic spiral)**.** *There exists a bi-Lipschitz homeomorphism of* $\mathbb{R}^2$ *which maps the logarithmic spiral onto* $\{(t,0) \,|\, 0 \leq t \leq 1\}$.

However, it seems to be unknown, whether it is possible to 'unwind' a spiral with a sub-exponential decay of the radius-vector. We recall the notion of the sub-exponential decay.

Definition 1.3 (sub-exponential decay)**.** Let $\phi : [0,\infty) \to (0,1]$ be a function monotonically decreasing to zero. We say that ϕ *decays sub-exponentially fast* if $\frac{\log(\phi(n))}{n} \to 0$ as $n \to \infty$.

Notice that the rate of decay of the function ϕ corresponding to the logarithmic spiral is exponential. Our main result is the following:

Theorem 1.2. *Let* $\phi : [0,\infty) \to (0,1]$ *be a function monotonically sub-exponentially decaying to zero. Then there is no bi-Lipschitz homeomorphism of* $\mathbb{R}^2$ *which maps* C_ϕ *into an unwound curve. I.e., for every* $h : \mathbb{R}^2 \to \mathbb{R}^2$ *a bi-Lipschitz homeomorphism we have*

$$D_{h(0_{\mathbb{R}^2})}(h(C_\phi)) = S^1.$$

Let C_ϕ be a spiral with a sub-exponential decay of the radius-vector. It is not hard to see by use of the 'length' of the curves and the triangle inequality, see Section 4, that there is no bi-Lipschitz homeomorphism of $\mathbb{R}^2$ that sends C_ϕ to the segment $\{(t,0) \,|\, 0 \leq t \leq 1\}$. However, the impossibility of unwinding in general case requires a substantially harder argument. We will use the notion of the cone at $p \in \overline{A} \subset \mathbb{R}^d$.

Definition 1.4 (cone at p)**.** The cone at p of A is

$$LD_p(A) = \{tu \,|\, u \in D_p(A), t \geq 0\}.$$

In the case of $p = 0_{\mathbb{R}^d}$ we will denote the cone at zero by $LD(A)$.

Theorem 1.2 follows easily from the following claim.

Proposition 1.1. *Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a bi-Lipschitz homeomorphism with $h(0_{\mathbb{R}^2}) = 0_{\mathbb{R}^2}$, and let $\phi : [0, \infty) \to (0, 1]$ be a function monotonically sub-exponentially decaying to zero. Then there exists a bi-Lipschitz homeomorphism $\overline{h}$ of $\mathbb{R}^2$, such that $\overline{h}(S^1) \subset LD(h(C_\phi))$, and $\overline{h}(0_{\mathbb{R}^2}) = 0_{\mathbb{R}^2}$.*

Proof of Theorem 1.2. Let us assume that $\phi : [0, \infty) \to [0, 1]$ is a function of a sub-exponential decay monotonically decreasing to zero. If there exists a bi-Lipschitz homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ that sends the spiral C_ϕ to an unwound curve then for the bi-Lipschitz map $\tilde{h}(x) = h(x) - h(0_{\mathbb{R}^2})$ we will have

$$D(\tilde{h}(C_\phi)) \neq S^1.$$

By Proposition 1.1 there exists a bi-Lipschitz map $\overline{h}$ of $\mathbb{R}^2$ with $\overline{h}\left(S^1\right) \subset LD(\tilde{h}(C_\phi))$. By the assumption $\tilde{h}$ 'unwinds' C_ϕ, therefore $LD(\tilde{h}(C_\phi))$ is a proper cone in $\mathbb{R}^2$. It follows from the Jordan curve theorem that the image of S^1 under a Lipschitz homeomorphism of $\mathbb{R}^2$ mapping zero into zero is not inside a proper cone. Assume, on the contrary, it is mapped inside a proper cone. By Jordan curve theorem the image of S^1 divides the plane into two connected components. Since the map is open it follows that the connected component of the image of the disk $\{(x, y) \,|\, x^2 + y^2 \leq 1\}$ is in unbounded connected component, since it contains the complement of the cone. But the map is a homeomorphism, and we get a contradiction.

□

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2. Proof of Proposition 1.1

Given a spiral C_ϕ, we will define the sequence (r_n) of *winding radii* corresponding to it as follows:

$$r_1 = \phi(0),\ r_2 = \phi(2\pi)\,, \ldots,\ r_n = \phi(2\pi(n-1)), \ldots.$$

In Section 3 we will show:

Lemma 2.1. *Let (r_n) be a sequence of a sub-exponential decay monotonically decreasing to zero. Then there exists a subsequence of indices (n_k) (of density one), such that*

$$\frac{r_{n_k+1}}{r_{n_k}} \to 1, \text{ as } k \to \infty.$$

In view of Lemma 2.1 it is natural to define the following notion.

Definition 2.1 (regular sequence)**.** We say that a positive monotonically decreasing to zero sequence (r_n) is *regular* if there is a subsequence of indices (n_k) such that (r_{n_k}) satisfies the property:

$$\lim_{k\to\infty} \frac{r_{n_k+1}}{r_{n_k}} = 1.$$

Next, we define a key property used in this paper.

Definition 2.2 ($\widetilde{SSP}$ condition)**.** We say that $p \in \overline{A}$ satisfies a *sub-sequence selection property* ($\widetilde{SSP}$) if the following holds:

- $\exists (r_n)$ regular sequence of radii (around p).
- For every $u \in D_p(A)$ there exists a sequence $(a_n) \subset A$ such that $r_{n+1} \leq \|a_n - p\| \leq r_n$ for n large enough, and $\frac{a_n - p}{\|a_n - p\|} \to u$ as $n \to \infty$.

As an immediate consequence of Lemma 2.1 we obtain the following implication.

Corollary 2.1. *Let $\phi : [0, \infty) \to (0, 1]$ be a function monotonically sub-exponentially decreasing to zero. Let (r_n) be the winding radii corresponding to the spiral C_ϕ. Then $p = \{(0, 0)\} \in \overline{C_\phi}$ satisfies the $(\widetilde{SSP})$ condition.*

Proposition 1.1 follows immediately from the following claim and Corollary 2.1.

Proposition 2.1. *Let $A \subset \mathbb{R}^d$ and let $h : \mathbb{R}^d \to \mathbb{R}^d$ be a bi-Lipschitz homeomorphism which satisfies $h(0_{\mathbb{R}^d}) = 0_{\mathbb{R}^d}$. Assume that $0_{\mathbb{R}^d} \in \overline{A}$ satisfies the condition $(\widetilde{SSP})$. Then there exists a bi-Lipschitz homeomorphism $\overline{h}$ of $\mathbb{R}^d$ such that $\overline{h}(0_{\mathbb{R}^d}) = 0_{\mathbb{R}^d}$ and*

$$\overline{h}\left(D(A)\right) \subset LD\left(h(A)\right).$$

Proof of Proposition 2.1. Let $0_{\mathbb{R}^d} \in \overline{A}$ be satisfying condition $(\widetilde{SSP})$. Assume that $h : \mathbb{R}^d \to \mathbb{R}^d$ is a bi-Lipschitz homeomorphism with $h(0_{\mathbb{R}^d}) = 0_{\mathbb{R}^d}$. Let (r_n) be a regular sequence of radii around $0_{\mathbb{R}^d}$ such that for every $u \in D(A)$ there exists a sequence $(a_n) \subset A$ such that $r_{n+1} \leq \|a_n\| \leq r_n$ for n large enough, and $\frac{a_n}{\|a_n\|} \to u$ as $n \to \infty$. Then there exists (n_k) subsequence of indices such that

$$\frac{r_{n_k+1}}{r_{n_k}} \to 1 \text{ as } k \to \infty.$$

Denote by (h_k) a sequence of bi-Lipschitz homeomorphisms defined by $h_k(x) = \frac{1}{r_{n_k}} h(r_{n_k} x)$. By Arzela-Ascoli theorem, since all these maps have the same Lipschitz constant, there is a subsequence along which the limit exists. Denote the limit by $\overline{h}$, and without loss of generality, assume that $h_k \to \overline{h}$. It is clear that $\overline{h}$ is a bi-Lipschitz homeomorphism of $\mathbb{R}^d$ (having the same Lipschitz constant as h).

Next, take any $u \in D(A)$. Then for $k \in \mathbb{N}$ large, $a_{n_k} \in A$ satisfies

$$\|a_{n_k} - r_{n_k} u\| \ll r_{n_k}, \tag{1}$$

by the triangle inequality and the identity

$$\|a_{n_k} - \|a_{n_k}\| u\| \ll \|a_{n_k}\|.$$

Now we apply h on the inequality (1) and use the Lipschitz property of h, to obtain

$$\left\| \frac{1}{r_{n_k}} h(a_{n_k}) - \frac{1}{r_{n_k}} h(r_{n_k} u) \right\| \to 0, \text{ as } k \to \infty.$$

Therefore, we have $\frac{1}{r_{n_k}} h(a_{n_k}) \to \overline{h}(u)$, which shows that there exists $s > 0$ with $s\overline{h}(u) \in D(h(A))$. In other words, we have shown that $\overline{h}(u) \subset LD(h(A))$. □

3. Proof of Lemma 2.1

Let $a_n = \frac{1}{r_n}$. Then a_n is an increasing sequence, and $a_n \leq g(n)$ satisfying $\frac{\log(g(n))}{n} \to 0$, as $n \to \infty$. It is enough to show that there exists a sparse set of indices $R \subset \mathbb{N}$, such that the sequence $b_n = \frac{a_{n+1}}{a_n}$ converges to one as $n \to \infty$ and $n \in \mathbb{N} \setminus R$. For every $m \in \mathbb{N}$, denote by

$$R_m = \left\{ n \in \mathbb{N} \,|\, \frac{a_{n+1}}{a_n} \geq 1 + \frac{1}{m} \right\}.$$

Our first claim is the following.

Claim 3.1. *For N large enough (independent of m) we have*

$$|R_m \cap [1, N]| \leq 3m \ln(g(N)).$$

Proof. For any fixed (large) N, denote by $c = |R_m \cap [1, N]|$. Since the largest possible index contained in $R_m \cap [1, N]$ is N, and (a_n) is an increasing sequence, we get the bound

$$a_N \geq \left(1 + \frac{1}{m}\right)^{c - n_o},$$

where n_o is the smallest n for which $a_n \geq 1$. On other hand, we also have $a_N \leq g(N)$. The last two inequalities imply that

$$(c - n_o) \log(1 + \frac{1}{m}) \leq \ln(g(N)).$$

This implies $c - n_o \leq \frac{\ln(g(N))}{\ln(1+\frac{1}{m})} \leq 2m \ln(g(N))$, where in the last inequality we used the easy fact that $\frac{1}{\ln(1+x)} \leq \frac{2}{x}$, for $0 < x \leq 1$.

Thus, we have for N large enough (independent of m) that $c \leq 3m \ln(g(N))$, which finishes the proof of the claim. □

Next, we define the sets $R'_m \subset \mathbb{N}$ of indices in the following way:

$$R'_m = R_m \cap [f_1(m), \infty),$$

where $f_1 : \mathbb{N} \to \mathbb{N}$ is very fast increasing function satisfying that $\ln(g(n)) f_2^2(n) \ll n$, where

$$f_2(n) = \min\{m \in \mathbb{N} \,|\, f_1(m) \geq n\}.$$

And, finally, let

$$R = \cup_{m \geq 1} R'_m.$$

We will show that R is a set of density zero. It is clear that $\frac{a_{n+1}}{a_n} \to 1$, as $n \to \infty$ and $n \in \mathbb{N} \setminus R$. Our final claim shows the sparsity of R.

Claim 3.2. *For N large enough we have $|R \cap [1, N]| \leq 3 \ln(g(N)) f_2^2(N)$.*

Proof. First, notice that for any given N, the only contributors to $R \cap [1, N]$ are from the sets $R'_1, R'_2, \ldots, R'_L$, where $L = f_2(N)$. Therefore, by use of Claim 3.1, we estimate

$$|R \cap [1, N]| \leq \sum_{m=1}^{L} |R'_m \cap [1, N]| \leq \sum_{m=1}^{L} |R_m \cap [1, N]|$$

$$\leq \sum_{m=1}^{L} 3m \ln(g(N)) \leq 3L^2 \ln(g(N)) = 3 \ln(g(N)) f_2^2(N),$$

for N large enough. $\square$

4. Unwinding spirals to straight segments

We will give an elementary argument for the following special case of Theorem 1.2.

Proposition 4.1. *Let $\phi : [0,\infty) \to (0,1]$ be a function monotonically sub-exponentially decaying to zero. Then there is no bi-Lipschitz homeomorphism of $\mathbb{R}^2$ which maps C_ϕ into the line segment $\{(t,0)\,|\,0 \le t \le 1\}$.*

We will use the notion of the length for a curve. Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a bi-Lipschitz map. Let $I = [a,b]$ be a subinterval of $\{(t,0)\,|\,0 \le t \le 1\}$. By a finite partition P of I we mean $t_0 = a < t_1 < \ldots < t_k = b$. To the partition P we correspond the approximate length of $h(I)$ along P as follows:

$$\mathcal{L}_P(h(I)) = \sum_{j=1}^{k} \|h(t_j) - h(t_{j-1}).\|$$

Then the length of $h(I)$ is defined by

$$\mathcal{L}(h(I)) = \sup_{P \text{ is a finite partition of } I} \mathcal{L}_P(h(I)).$$

Notice that if $h(I)$ is a rectifiable curve, then the usual length coincides with the one that we just defined. If h has Lipschitz constant L, then we have $\frac{|I|}{L} \le \mathcal{L}(h(I)) \le L|I|$, where $|I|$ denotes the length of I equal to $b-a$.

Proof of Proposition 4.1. Let us assume that there exists a bi-Lipschitz homeomorphism h of $\mathbb{R}^2$ with Lipschitz constant L that sends a spiral C_ϕ, satisfying the sub-exponential decay condition on winding radii, to the line segment $\{(t,0)\,|\,0 \le t \le 1\}$. Without loss of generality, we can assume that $h(C_\phi) = \{(t,0)\,|\,0 \le t \le 1\}$. Then by Lemma 2.1 from the sequence of winding radii (r_n) corresponding to the spiral C_ϕ we can extract a subsequence (r_{n_k}) such that

$$\frac{r_{n_k+1}}{r_{n_k}} \to 1, \text{ as } k \to \infty. \tag{2}$$

Then we look at the closed curve Γ_k in $\mathbb{R}^2$ comprising the part of the spiral Γ'_k between the radii r_{n_k} and r_{n_k+1}, and the line segment I_k connecting r_{n_k+1} and r_{n_k}, see Figure 2.

Since the length of Γ'_k is greater or equal than r_{n_k+1}, it follows that $|h(\Gamma'_k)| \ge \frac{r_{n_k+1}}{L}$. On the other hand, $|I_k| = r_{n_k} - r_{n_k+1} \ll r_{n_k+1}$ by (2). Therefore $\mathcal{L}(h(I_k)) \le L|I_k| \ll r_{n_k+1}$. Notice that $h(\Gamma'_k)$ is mapped into the straight segment which is the geodesic connecting the points $h((r_{n_k},0))$ and $h((r_{n_k+1},0))$ in $\mathbb{R}^2$. The endpoints of $h(I_k)$ coincide with the endpoints of $h(\Gamma'_k)$, and therefore we must have

$$\mathcal{L}(h(I_k)) \ge |h(\Gamma'_k)|,$$

which is impossible. This finishes the proof of the proposition. $\square$

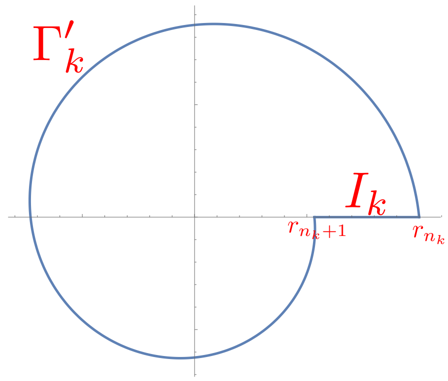

FIGURE 2. The curve $\Gamma_k = \Gamma'_k \cup I_k$.

5. Appendix – Remarks on the (SSP) condition

The sequence selection property (SSP) condition for the set of directions at a point $p \in \overline{A} \subset \mathbb{R}^d$ was introduced by Koike and the second author in [KP1], and has been studied in [KP1], [KP2] and [KP3].

Definition 5.1 ((SSP) condition)**.** The set of directions at $p \in \overline{A} \subset \mathbb{R}^d$ satisfies (SSP) condition if for every $u \in D_p(A)$ and every positive sequence (t_n) decreasing to zero, there exists a sequence $(a_n) \in A$ such that

$$\|(a_n - p) - t_n u\| \ll \max(\|a_n - p\|, t_n).$$

It is proved in [KP3] the following.

Proposition 5.1 (Lemma 2.9, [KP3])**.** *Let $h : \mathbb{R}^d \to \mathbb{R}^d$ be a bi-Lipschitz homeomorphism with $h(p) = p$, for $p \in \overline{A} \subset \mathbb{R}^d$, and $p \in \overline{A}$ satisfies (SSP) condition. Then there exists a bi-Lipschitz homeomorphism $\overline{h}$ of $\mathbb{R}^d$ such that $\overline{h}(LD_p(A)) \subset LD_p(h(A))$, and $\overline{h}(p) = p$.*

The main consequence of Proposition 5.1 is that for any reasonable (satisfying the monotonicity property along Lipschitz maps) notion of the dimension in $\mathbb{R}^d$, we will have the following.

Corollary 5.1. *Let $h : \mathbb{R}^d \to \mathbb{R}^d$ be a bi-Lipschitz homeomorphism with $h(p) = p$, and let $A \subset \mathbb{R}^d$. If $p \in \overline{A}$ satisfies the (SSP) condition, then*

$$Dim(LD_p(A)) \leq Dim(LD_p(h(A))).$$

The sub-exponential decay condition on ϕ does not guarantee that C_ϕ satisfies the (SSP) condition. Notice that the conclusion in Proposition 2.1 is much weaker than in Proposition 5.1. We can obtain almost the same conclusion as in Proposition 5.1, if we will require slightly more from a regular sequence of radii (r_n) appearing in the definition of the $\widetilde{SSP}$ condition. It is not hard to prove the following statement by use of the similar techniques as in the proof of Proposition 2.1.

Proposition 5.2. *Let $h : \mathbb{R}^d \to \mathbb{R}^d$ be a bi-Lipschitz homeomorphism with $h(p) = p$. Assume that $p \in \overline{A}$ satisfies $(\widetilde{SSP})$ condition. In addition, assume that the* *regular family of radii* (r_n) *has the property that there exists* $M > 0$ *with*

$$\sup_{n>m} \frac{r_n - r_{n+1}}{r_m - r_{m+1}} < M, \text{ for all } m \in \mathbb{N}.$$

Then there exists a bi-Lipschitz homeomorphism $\overline{h}$ *of* $\mathbb{R}^d$ *with* $\overline{h}(p) = p$ *such that* $\overline{h}(L_{\leq 1} D_p(A)) \subset LD_p(h(A))$, *where* $L_{\leq 1} D_p(A) = \{tu \,|\, 0 < t \leq 1, u \in D_p(A)\}$.

As a corollary of Proposition 5.2 we obtain the following.

Corollary 5.2. *Let* $h : \mathbb{R}^d \to \mathbb{R}^d$ *be a bi-Lipschitz homeomorphism with* $h(p) = p$, *and let* $A \subset \mathbb{R}^d$. *If* $p \in \overline{A}$ *satisfies the* $\widetilde{(SSP)}$ *condition with the sequence of radii* (r_n) *satisfying*

$$\sup_{n>m} \frac{r_n - r_{n+1}}{r_m - r_{m+1}} < M, \text{ for all } m \in \mathbb{N}.$$

Then

$$Dim(LD_p(A)) \leq Dim(LD_p(h(A))).$$

The following natural question will be addressed in [FP2].

Question 1. *Is it true that the conclusion in Corollary 5.2 still holds true under a weaker assumption that* $p \in \overline{A}$ *satisfies the* $\widetilde{(SSP)}$ *condition?*

We finish the Appendix with another natural question related to Definition 1.2.

Question 2. *Can we find a spiral* $C \subset \mathbb{R}^2$ *around the zero which can be unwound by a bi-Lipschitz homemorphism of* $\mathbb{R}^2$, *but cannot be unwound into a line segment?*

The answer to Question 2 is affirmative, and it will be presented in [FP2].

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